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SECONDARY INVARIANTS AND CHIRAL ANOMALIES OF BASIC DIRAC FAMILIES

BY

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Abstract. The theory of characteristic classes of foliated bundles is applied to the study of a class of geometric Dirac operators. For a foliation of even codimension with minimally immersed leaves on the base, the nature of chiral anomalies is examined in view of the cohomology of the truncated Weil algebra. A foliated Wess-Zumino term is defined and for certain cases the homotopy groups of the gauge group of the foliation are determined.

§1. Introduction

Amongst several bi-complexes suited to the study of various anomalies and BRST cohomology are those relative to foliated G-bundles $(P, \tilde{\mathcal{F}})$ over smooth foliated manifolds $(M, \mathcal{F})$ **[11]**, **[14]**. Here the leafwise operator is the Chevalley–Eilenberg differential and the operator on the transversal complex can be the basic DeRham differential. This paper presents an initial advance to showing how these complexes may be suitably adapted to studying the cohomology of restricted gauge groups and that of the parameter spaces of pseudodifferential operators on M. In the setting described here, we consider a Riemannian foliation $(M, \mathcal{F})$ furnished with a transversal spin structure thus permitting classes of transversal Dirac-type operators to be defined. Then one may consider a family of such operators restricted to the basic sections relative to $\mathcal{F}$ and parametrized by the orbit spaces Y of basic connections on $(P, \tilde{\mathcal{F}})$ by the gauge group of the foliation $\mathcal{G}(\tilde{\mathcal{F}})$. The restricted operators are the *basic Dirac operators* studied in **[4]**, **[7]** and the importance of these operators lies in their exceptional regularity properties on the basic sections.

In the spirit of **[11]**, we construct several chain maps from the truncated Weil algebra relative to $\mathcal{F}$ to forms on $\mathcal{G}(\tilde{\mathcal{F}})$ and consequently we obtain induced maps in cohomology, that is secondary characteristic classes. The construction of these maps may be regarded as providing some initial steps towards a transversal families index theorem cast in terms of the secondary characteristic classes of the foliation. In the absolute case where the (even) codimension of $\mathcal{F}$ equals the dimension of M, our formulism is seen to reduce to the gauge anomalies described in **[2]**, **[3]**, **[17]**, **[18]**.

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As one might expect, we confirm that the geometric structure of the foliation plays an important role. Analogous cocycles representative of the anomalies are those determined when the foliation has minimal leaves on M. A foliated version of the Wess–Zumino term can be studied in view of the theory of generalized Cheeger–Chern–Simons classes evolving from **[6]** (cf. **[5]**). In the case $P \cong M \times G$, the homotopy groups of $\mathcal{G}(\tilde{\mathcal{F}})$ can be calculated in terms of the K-theory of the Molino space of $\mathcal{F}$.

As regards the analysis on infinite dimensional manifolds we refer to **[15]**.

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§2. Parametrization of the basic Dirac families

Let M be an oriented compact Riemannian manifold of dimension m and $(M, \mathcal{F}, g_M)$ an oriented Riemannian foliation of codimension q with a bundle–like metric g_M. Taking $L(\mathcal{F})$ to denote the tangent bundle along the leaves of $\mathcal{F}$, we have the exact sequence

$$0 \to L(\mathcal{F}) \to TM \to Q \to 0 \tag{2.1}$$

where Q is the normal bundle to $\mathcal{F}$. A bundle–like metric g_M on M gives the identification $Q \cong (L(\mathcal{F}))^\perp$ with g_Q holonomy invariant.

Consider a Hermitian foliated bundle E on M with foliation lifted from $\mathcal{F}$, endowed with the structure of a Clifford module over the transversal (complex) Clifford algebra of $\mathcal{F}$. The *basic* sections of E are defined by

$$\Gamma_b(E) = \{s \in C^\infty(E) : \nabla_X s = 0, X \in C^\infty(L(\mathcal{F}))\}. \tag{2.2}$$

In **[4]**, **[7]** a symmetric transversally elliptic differential operator $\not{D}_{tr}$ was defined on $C^\infty(E)$. The restriction of this operator to the basic section defines the *basic Dirac operator*

$$\not{D}_b = \not{D}_{tr}\,|_{\Gamma_b(E)} \tag{2.3}$$

which maps $\Gamma_b(E)$ to itself. Little generality is lost by assuming that Q is endowed with a $\mathrm{Spin}(q)$ structure which we elect to do henceforth. Thus we have the exact sequence of principal bundles

$$\mathbb{Z}_2 \ \to \ F_{\mathrm{Spin}}(Q) \ \to \ F_{SO}(Q).$$

We take $E = S \otimes V$, where $S = F_{\mathrm{Spin}}(Q) \times_{\mathrm{Spin}(q)} \triangle_S$ denotes the spin bundle associated to Q, $\triangle_S$ the spin representation and V a complex coefficient bundle. We view V as the vector bundle associated to a *foliated* principal G-bundle $P \overset{\pi}{\longrightarrow} M$, where G is a connected, compact Lie group. Here, the G-action permutes the leaves of the foliation $\bar{\mathcal{F}}$ on E and the differential π_* pointwise maps $L(\bar{\mathcal{F}})$ to $L(\mathcal{F})$ isomorphically. For instance, V is often associated to a $SO(q)$-reduction of the principal frame bundle $F(Q)$.

Let $\rho : G \to U(r)$ be a representation of G on $\mathbb{C}^r$ and let $V = P \times_\rho \mathbb{C}^r$ be the resulting complex vector bundle endowed with a Hermitian structure. We call the resulting bundle E as above a *foliated twisted spin bundle.* In **[7]** we defined a restricted gauge group $\mathcal{G}(\tilde{\mathcal{F}})$, preserving the G-invariant foliation $\tilde{\mathcal{F}}$ on the principal bundle P and acting freely on the (convex) subset $\mathcal{A}_b$ of *basic* connections in $\mathcal{A}$, the set of all connections defined on P. Recall that $A \in \mathcal{A}_b$ is *basic* if for $\tilde{X} \in C^\infty(L(\tilde{\mathcal{F}}))$

$$i_{\tilde{X}} A = 0 \text{ (i.e. } A \text{ is adapted) and } L_{\tilde{X}} A = 0 \ . \tag{2.4}$$

For a choice of connection $A \in \mathcal{A}_b$, we have then a basic Dirac operator $(\not{D}_b)_A$ depending on A, satisfying the *covariance* condition **[2]**, **[7]**

$$(\not{D}_b)_{\phi \cdot A} = \hat{\phi}^{-1} (\not{D}_b)_A \, \hat{\phi} \ , \tag{2.5}$$

where $\hat{\phi}$ is the lift of $\phi \in \mathcal{G}(\tilde{\mathcal{F}})$ to E. We remark that all metrics in question remain fixed throughout. We proceed by defining a principal bundle

$$\Big(\mathcal{A}_b \times P, \mathcal{G}(\tilde{\mathcal{F}}), \mathcal{A}_b \times_{\mathcal{G}(\tilde{\mathcal{F}})} P = \tilde{P}\Big) \ , \tag{2.6}$$

where $\phi \in \mathcal{G}(\tilde{\mathcal{F}})$ acts on $\mathcal{A}_b \times P$ by $(A, u) \to \big(\phi^* A, \phi^{-1}(u)\big)$. Now with our choice of $\mathcal{G}(\tilde{\mathcal{F}})$ which commutes with G, $\tilde{P}$ is itself a foliated principal G-bundle

$$G \to \tilde{P} \to \mathcal{A}_b / \mathcal{G}(\tilde{\mathcal{F}}) \times M \ . \tag{2.7}$$

For later purposes it will be convenient to denote $\mathcal{A}_b/\mathcal{G}(\tilde{\mathcal{F}})$ by Y and then let $Z = Y \times M$. As in **[7]**, the space Y adopts the role of a parameter space for families of the basic Dirac operators $\{(\not{D}_b)_A\}$. In other words, the operators are parametrized by orbit classes of basic connections under the gauge group $\mathcal{G}(\tilde{\mathcal{F}})$ (see §4).

In **[10]** we showed how to define a connection $\tilde{\omega}$ on $\tilde{P}$ in terms of the connections on P and $\mathcal{A}_b \to Y$, and explicity described its curvature $\tilde{\Omega}$.

PROPOSITION 2.1 **[10]**
The connection $\tilde{\omega}$ on $\tilde{P}$ is a basic connection with respect to the foliation $\tilde{\mathcal{F}}$ on $\tilde{P}$ induced from P.

In the following section we will investigate the characteristic classes of the G-bundle $\tilde{P}$ and compute their integrated suspensions in terms of the evaluation map.

§3. Characteristic classes and suspension of the gauge group

Henceforth we take $\mathcal{F}$ to be a transversally oriented foliation of *even* codimension $q = 2k$ and denote by $\chi_{\mathcal{F}}$ the characteristic form of $\mathcal{F}$ of degree $(m - q)$. We shall also assume that $\mathcal{F}$ is *minimal*, in other words, the mean curvature form κ of $\mathcal{F}$ satisfies $\kappa = 0$. Then the following operator

$$z = \int_M \wedge \chi_{\mathcal{F}} \tag{3.1}$$

induces an isomorphism $H^q(\Omega_b(\mathcal{F})) \cong \mathbb{R}$ (see **[13]**, **[14]**).

Let $W = W(G)$ be the Weil algebra of G and $FW = F^{2(k+1)}W(G)$ the filtration ideal defined by symmetric polynomials of degree $> 2k$ and define the *truncated Weil algebra* by $W_k = W/FW$. As W is contractible, we have a canonical isomorphism

$$\delta : H^{\bullet-1}(W_k) \xrightarrow{\cong} H^\bullet(FW) \ . \tag{3.2}$$

We observe that the canonical inclusion $j : FI(G) \to FW$ induces a map $j_* : FI(G)^\bullet \to H^\bullet(FW) \cong H^{\bullet-1}(W_k)$ with kernel given by the *decomposable* elements in $FI(G)$ by **[6]**, **[11]** , namely

$$I(G)^+ \cdot FI(G) \subset FI(G) \ .$$

Since we have shown that the connection $\tilde{\omega}$, and hence the curvature $\tilde{\Omega}$ on $\tilde{P}$ are $\mathcal{F}$–basic, there exists a homomorphism of differential algebras

$$k(\tilde{\omega}) : W \to \Omega_b(\tilde{P}, \tilde{\mathcal{F}}). \tag{3.3}$$

On G–basic forms, this induces the Chern-Weil homomorphism

$$h(\tilde{\omega}) : I(G) \to \Omega_b(\tilde{P}, \tilde{\mathcal{F}})_G \cong \Omega_b(Z, \mathcal{F}) \ . \tag{3.4}$$

Note that the codimension of $\tilde{\mathcal{F}}$ on $\tilde{P}$ is infinite and the powers of $\tilde{\Omega}$ do not in general satisfy a vanishing condition.

The operator z in (3.1) defines a map

$$z : \Omega_b^{\bullet+2k}(Z, \mathcal{F}) \to \Omega^\bullet(Y) \tag{3.5}$$

as follows. For $w \in \Omega_b^{2k+j}(Y \times M, \mathcal{F})$ we use the bigrading on $\Omega_b^{\bullet+2k}(Y \times M, \mathcal{F})$ to write

$$w = \sum_{j \le \ell \le 2k+j} w_{\ell, 2k+j-\ell}$$

and denote by $(\partial \pm d)$ the total differential on $\Omega_b(Y \times M, \mathcal{F})$. We then define

$$z(w) = \int_M w_{j,2k} \wedge \chi_{\mathcal{F}} \ . \tag{3.6}$$

LEMMA 3.1
The map z defines a cycle, that is

$$\partial z = z(\partial \pm d) \ . \tag{3.7}$$

Proof. Rummler's formula **[13]**, **[14]** states that modulo $\mathcal{F}$-trivial forms

$$d\chi_{\mathcal{F}} + \kappa \wedge \chi_{\mathcal{F}} \equiv 0 \ . \tag{3.8}$$

Since we have assumed $\kappa = 0$, Green's theorem yields

$$\begin{aligned}\partial z(w) &= \partial \int_M w_{j,2k} \wedge \chi_{\mathcal{F}} \\ &= \int_M (\partial w_{j,2k} \pm dw_{j+1,2k-1}) \wedge \chi_{\mathcal{F}} \pm \int_M w_{j+1,2k-1} \wedge d\chi_{\mathcal{F}} \\ &= \int_M ((\partial \pm d)w)_{j+1,2k} \wedge \chi_{\mathcal{F}} \\ &= z((\partial \pm d)w) \ . \quad \square\end{aligned}$$

Thus we obtain a composite chain map of degree $-2k$

$$k = z \circ h(\tilde{\omega}) : FI(G) \to \Omega(Y) \tag{3.9}$$

and we denote by $k_* = z_* \circ h_*$ the induced map into the DeRham cohomology $H_{DR}(Y)$.

Letting $\mathcal{E}^\circ = \tilde{P} \times_\rho \mathbb{C}^r$ be the associated vector bundle to $\tilde{P}$ in (2.7), we observe that

$$\int_M ch(\mathcal{E}^\circ) \wedge \chi_{\mathcal{F}} = k_*(\rho^* ch) \ , \tag{3.10}$$

where ch on $I(U(r))$ is the Chern character.

In **[10]** it was noted that terms involving the Green's operator figured in certain components of the curvature $\tilde{\Omega}$ with the implication that the integrated characteristic forms of $\tilde{P}$ are not local on $Y = B\mathcal{G}(\tilde{\mathcal{F}})$ (cf. **[2]**). In order to circumvent this problem and obtain local forms on $\mathcal{G}(\tilde{\mathcal{F}})$, we will determine the suspension $\Sigma\mathcal{G}(\tilde{\mathcal{F}}) \to B\mathcal{G}(\tilde{\mathcal{F}})$ by the following construction suggested to us by J. L. Dupont. In particular, this will relate the characteristic classes discussed in **[3]** using the difference construction, with the construction in **[2]**.

Setting $\tilde{\mathcal{G}} = \mathcal{G}(\tilde{\mathcal{F}})$ for brevity, the evaluation map $\mu : \tilde{\mathcal{G}} \times P \to P$ defined by

$$\mu(\phi, u) = \phi(u)$$

satisfies (left) $\tilde{\mathcal{G}}$ and (right) G–equivariance. We proceed to define a map $\bar{\mu} : \tilde{\mathcal{G}} \times P \to \tilde{\mathcal{G}} \times P$ given by

$$\bar{\mu}(\phi, u) = (\phi, \mu(\phi, u)) = (\phi, \phi(u)).$$

Effectively, we see that

$$\psi\bar{\mu}(\phi, u) = \bar{\mu}(\psi(\phi, u)) = \bar{\mu}(\psi\phi, (\psi\phi)(u)),$$

$$\bar{\mu}(\phi, u)g = \bar{\mu}((\phi, u)g) = \bar{\mu}(\phi, ug)$$

and

$$\bar{\mu}(\phi, u) = (\phi, \phi(u)) = \phi\bar{\mu}(id, u) = \phi(id, u),$$

thus showing that $\bar{\mu}$ defines an diffeomorphism. Also, $\bar{\mu} = (id, \mu)$ is a gauge transformation on both bundles. Letting $\mathbb{Z}$ here denote $\mathbb{Z}(\bar{\mu})$ we may suspend the entire gauge group via the suspension of $\bar{\mu}$ to obtain the bundle

$$\bar{P} = \mathbb{R} \times_{\bar{\mu}} (\tilde{\mathcal{G}} \times P) = \mathbb{R} \times_{\mathbb{Z}} (\tilde{\mathcal{G}} \times P) \tag{3.11}$$

over $\mathbb{R} \times_{\bar{\mu}} (\tilde{\mathcal{G}} \times P/G) \cong S^1 \times (\tilde{\mathcal{G}} \times M)$. In terms of the orbits of the groups in question, the following diagram summarizes the situation:

$$\begin{array}{ccc} \bar{P} = \mathbb{R} \times_{\mathbb{Z}} (\tilde{\mathcal{G}} \times P) & \xrightarrow{\tilde{\mathcal{G}}} & S^1 \times P = \mathbb{R} \times_{\mathbb{Z}} P \\ \big\downarrow G & & \big\downarrow G \\ S^1 \times (\tilde{\mathcal{G}} \times M) & \xrightarrow{\tilde{\mathcal{G}}} & S^1 \times M. \end{array}$$

The point here is that $\bar{P}$ is the pullback of $\tilde{P}$ by a map $f = s \times id$ where

$$s : S^1 \times \tilde{\mathcal{G}} \to Y = B\tilde{\mathcal{G}}$$

realizes the suspension map. Explicitly, for any $A \in \mathcal{A}_b$, we define for $t \in [0, 1]$

$$A_{\phi,t} = (1 - t)A + t\phi^* A \tag{3.12}$$

and $A_{\phi,t+n} = (\phi^n)^* A_{\phi,t}$, for $n \in \mathbb{Z}$. Consider the map

$$\bar{f} : \mathbb{R} \times \tilde{\mathcal{G}} \times P \to \mathcal{A}_b \times P \tag{3.13}$$

defined by $\bar{f}(t, \phi, u) = (A_{\phi,t}, u)$. Then

$$\bar{f}(t + 1, \phi, u) = (A_{\phi,t+1}, u) = (\phi^* A_{\phi,t}, u),$$

$$\bar{f}(t, \bar{\mu}(\phi, u)) = \bar{f}(t, \phi, \phi(u)) = (A_{\phi,t}, \phi(u))$$

and further, as a G-map

$$\bar{f}(t, \phi, ug) = \bar{f}(t, \phi, u)g.$$

Thus $\bar{f}$ induces a bundle map $\tilde{f} : \bar{P} \to \tilde{P}$ of foliated G–bundles covering $f = s \times id$ where $s(t, \phi) = [A_{\phi,t}]$. In the following, we view A also as a connection on $\tilde{\mathcal{G}} \times P$ and on $\bar{P}$, pulled back via projection.

Taking $\bar{\omega} = \tilde{f}^* \tilde{\omega}$, we observe that at level $t \in [0, 1]$, we have $\bar{\omega}_t = (\bar{\mu}^* A)_t = (1 - t)A + t\bar{\mu}^* A$ on $\tilde{\mathcal{G}} \times P$ and $\bar{\omega}_t(\frac{\partial}{\partial t}) = 0$. Furthermore, $(\bar{\mu}^* A)_t$ is given at $[t, \phi, u]$ by

$$(\bar{\mu}^* A)_t(X, Y) = A_{\phi,t}(Y_u) + i_{X_0^*} A_{\phi,t} \tag{3.14}$$

where $Y_u \in T_u P$, $X = L_{\phi *} X_0$ and $X_0 \in \mathrm{Lie}(\tilde{\mathcal{G}})$ (cf. **[3]**). The curvature $\bar{\Omega}$ of $\bar{\omega}$ is given by

$$F_{(\bar{\mu}^* A)_t} + (\bar{\mu}^* A - A) \wedge dt \ .$$

We observe that $(F_A)^{k+1} = 0$ and $(F_{\bar{\mu}^* A})^{k+1} = (\bar{\mu}^* F_A)^{k+1} = 0$, whereas this is generally not the case for arbitrary values of t.

So for $\Phi \in FI(G)$, the difference construction ([**11**], §5), applied to $(\bar{\mu}^* A, A)$ yields a well defined chain map

$$\Delta(\bar{\mu}^* A, A) : W_k \to \Omega_b(\tilde{\mathcal{G}} \times M, \mathcal{F}) \cong \Omega_b(\tilde{\mathcal{G}} \times P, \mathcal{F})_G \tag{3.15}$$

which via the map j, is given on indecomposable elements $\Phi \in FI(G)$ by

$$\begin{aligned} \Delta(\bar{\mu}^* A, A)(j\Phi) &= \int_{S^1} h(\bar{\omega})(\Phi) \\ &= \int_{S^1} \Phi(\bar{\mu}^* A - A, F_{(\bar{\mu}^* A)_t}, \ldots, F_{(\bar{\mu}^* A)_t}) \wedge dt \ . \end{aligned} \tag{3.16}$$

Observing that the cycle z is also defined on $\Omega_b(\tilde{\mathcal{G}} \times M, \mathcal{F})$, we can now formulate the main result of this section.

THEOREM 3.2
For $\Phi \in FI(G)^{2k+j+1}$ indecomposable and j odd, the form

$$z\Delta(\bar{\mu}^* A, A)(j\Phi) \tag{3.17}$$

in $\Omega^j(\tilde{\mathcal{G}})$ represents the suspension of $(z \circ h)(\tilde{\omega})\Phi \in \Omega^{j+1}(Y)$. In particular, the following diagram is commutative

$$\begin{array}{ccc} FI(G)^{2k+j+1} & \xrightarrow{k_*} & H^{j+1}_{DR}(Y) \\ \big\downarrow \delta^{-1} j_* & & \big\downarrow \sigma \\ H^{2k+j}(W_k) & \xrightarrow{z_* \Delta_*} & H^j_{DR}(\tilde{\mathcal{G}}) \ . \end{array} \tag{3.18}$$

Proof. This follows from the following calculation:

$$\begin{aligned} z\Delta(\bar{\mu}^* A, A)(j\Phi) &= \int_M \left[\int_{S^1} h(\bar{\omega})(\Phi) \right] \wedge \chi_{\mathcal{F}} \\ &= \int_M \left[\int_{S^1} f^* h(\tilde{\omega})(\Phi) \right] \wedge \chi_{\mathcal{F}} \\ &= \int_{S^1} \left[\int_M f^* h(\tilde{\omega})(\Phi) \wedge \chi_{\mathcal{F}} \right] \\ &= \int_{S^1} s^* \left[\int_M h(\tilde{\omega})(\Phi) \wedge \chi_{\mathcal{F}} \right] \\ &= \int_{S^1} s^* z(h(\tilde{\omega})(\Phi)) \ , \end{aligned}$$

and $\int_{S^1} s^*$ realizes the suspension $H^\bullet(B\tilde{\mathcal{G}}) \to H^\bullet(\Sigma\tilde{\mathcal{G}})$. □

By Theorem 3.2, the mappings

$$\Psi = \sigma \circ k_* : FI(G)^{2k+j+1} \longrightarrow H^{j+1}_{DR}(Y) \longrightarrow H^j_{DR}(\tilde{\mathcal{G}}). \tag{3.19}$$

and

$$\Psi_1 = z_* \circ \Delta_* : \delta^{-1} j_* FI(G) \to H^j_{DR}(\tilde{\mathcal{G}}) \tag{3.20}$$

are related by the boundary isomorphism δ and the canonical map j_* (3.2). The composition

$$\begin{array}{c} FI(G)^{j+1} \xrightarrow{j_*} H^{j+1}(FW(G)) \xrightarrow[\cong]{\delta^{-1}} H^j(W_k) \\ \qquad\qquad\qquad [d\Phi] \longleftarrow [\Phi] \end{array} \tag{3.21}$$

can be determined explicitly. For admissible cocycles **[6]** , we have

$$\begin{aligned} z_{(I|J)} &= y_I \otimes c_J \quad (|c_J| \leq 2k, |c_{i_1} c_J| > 2k) \ , \\ \delta z_{(I|J)} &= [y_{i_2} \wedge \cdots \wedge y_{i_s} \otimes c_{i_1} c_J \\ &\quad \pm \sum_{k>1} y_{i_1} \wedge \cdots \wedge \hat{y}_{i_k} \cdots \wedge y_{i_s} \otimes c_{i_k} c_J] \ . \end{aligned}$$

Thus for the generators $c_i c_J \in FI(G) \mod I^+(G) \cdot FI(G)$ we find

$$\delta^{-1}(c_i c_J) = z_{(i|J)} \ .$$

For $J = \varnothing$, $|c_i| > 2k$, $z_{(i)} = y_i \otimes 1$ is a cocycle with $\delta z_{(i)} = 1 \otimes c_i$. For $I = (1)$ we have classes of Godbillon–Vey type $z_{(1|J)}$, $\deg y_1 = 1$, $|c_J| = 2k$ corresponding to the generators $c_1 c_J$.

We may now use the above formulas to evaluate the mapping $\Psi = \sigma \circ k_*$ in (3.19) in yet another way (cf. **[2]**). We consider the generators $\Phi = c_i c_J \in FI(G)$ corresponding to the set of classes $\{z_{(i|J)}\} \subset H(W_k(G))^{2k+j}$. In the notation of **[11]**, **[12]**, the difference construction yields

$$\Delta(\tilde{\omega}, A) z_{(i|J)} = \lambda^1(\tilde{\omega}, A)\Phi$$

with $d\lambda^1\Phi = h(\tilde{\omega})\Phi$ (observe that $h(A)\Phi = 0$ by the vanishing theorem). The sequence

$$\begin{array}{ccccc} W(G)_k & \xrightarrow{\Delta(\tilde{\omega},A)} \Omega_b^{2k+j}(\tilde{P}, \tilde{\mathcal{F}})_G & \longrightarrow & \Omega_b^{2k+j}(\mathcal{A}_b \times M, \mathcal{F}) & \\ & & & \Big\downarrow {\scriptstyle \int_M \wedge \chi_{\mathcal{F}}} & \\ & & & \Omega^j(\mathcal{A}_b) \qquad \longrightarrow & \Omega^j(\tilde{\mathcal{G}}) \end{array} \tag{3.22}$$

induces a map

$$\Psi_2 : \delta^{-1} j_* FI(G)^{2k+j} \longrightarrow H^j_{DR}(\tilde{\mathcal{G}}) \tag{3.23}$$

such that the following diagram is commutative

$$\begin{array}{ccccc} H(W(G)_k)^{2k+j} & \supseteq & \delta^{-1} j_* FI(G)^{2k+j} & \xrightarrow{\Psi_2} & H^j_{DR}(\tilde{\mathcal{G}}) \\ \cong \Big\downarrow \delta & & \Big\downarrow \delta & & \Big\| \\ H(FW(G)) & \supseteq & j_* FI(G)^{2k+j+1} & \xrightarrow{\Psi} & H^j_{DR}(\tilde{\mathcal{G}}) \ , \end{array} \tag{3.24}$$

so that

$$\Psi_1 = \Psi_2 \tag{3.25}$$

by Theorem 3.2.

We find it instructive to remark that the above description of the integrated, suspended characteristic classes Ψ of $\tilde{P}$ in terms of the mappings Ψ_1 and Ψ_2 embraces the situations discussed in **[2]** , **[3]**,II and **[18]** for the absolute case where $q = m$. Indeed, it should be seen that the above construction of secondary foliation classes lends new insight even in the absolute case where such techniques have been used implicitly (loc.cit).

Since q is even we have a splitting of the basic Dirac operator

$$\not{D}_b = \not{D}_b^+ + \not{D}_b^-$$

respecting the $\pm$-eigenspace decomposition. Letting $\overline{\pi} : Z \to M$ be the projection, we define

$$\mathcal{E} = \overline{\pi}^* S \otimes \mathcal{E}^0$$

and extend the family $\left\{ \left(\not{D}^{\pm}{}_b \right)_A \right\}$ to $\mathcal{E}^{\pm}$ as a family of operators

$$\{ (\tilde{\not{D}}^{\pm}{}_b)_A \} \; : \Gamma_b \left(\mathcal{E}^{\pm} \right) \to \Gamma_b \left(\mathcal{E}^{\mp} \right) \tag{3.26}$$

over Z. Alternatively, we may also regard $\{ (\tilde{\not{D}}^{\pm}{}_b)_A \}$ as an operator $\tilde{\not{D}}^{\pm}{}_b$ on the fibers of the Hilbert bundle

$$\mathcal{H}^{\pm} = \mathcal{A}_b \times_{\tilde{\mathcal{G}}} L^2 \left(\Gamma_b \left(E^{\pm} \right) \right) \to Y$$

inheriting the same spectral properties (**[7]**, §5). The construction of the *basic analytic families index* $\mathrm{Ind}(\tilde{\not{D}}_b^+)$ in **[7]** leads to the existence of classes of maps to Fredholm operators of fixed index:

THEOREM 3.3
There exist well-defined homotopy classes of maps $\Psi^+_{(P,\tilde{\mathcal{F}})}$ from $B\tilde{\mathcal{G}}$ to $BU \simeq \mathbf{F}_c$ and $\Omega\Psi^+_{P,(\tilde{\mathcal{F}})}$ from $\tilde{\mathcal{G}}$ to $U \simeq GL_{cpt}$, such that the induced homomorphisms

$$\Psi^{+*}_{(P,\tilde{\mathcal{F}})} : H^\bullet(BU, \mathbb{Z}) \to H^\bullet \left(B\tilde{\mathcal{G}}, \mathbb{Z} \right) \tag{3.27}$$

and

$$\Omega\Psi^{+*}_{(P,\tilde{\mathcal{F}})} : H^\bullet(U, \mathbb{Z}) \to H^\bullet \left(\tilde{\mathcal{G}}, \mathbb{Z} \right) \tag{3.28}$$

are related by transgression in the respective universal bundles. We have moreover

$$\Psi^{+*}_{(P,\tilde{\mathcal{F}})} (\tilde{c}_j) = c_j \left(\mathrm{Ind} \left(\tilde{\not{D}}_b^+ \right) \right) \ , \qquad ((3.29))$$

where $\tilde{c}_j$ denotes the universal Chern class.

We may then interpret the characteristic class

$$c_1\left(\mathrm{Ind}\left(\tilde{\not{D}}_b^+\right)\right) \in H^2\left(B\tilde{\mathcal{G}}, \mathbb{Z}\right) ,$$

resp. its suspension in $H^1\left(\tilde{\mathcal{G}}, \mathbb{Z}\right)$ via the determinant map on $\tilde{\mathcal{G}}$, that is as an obstruction to defining a gauge invariant determinant for the family $\{(\tilde{\not{D}}_b^+)_A\}$. This is a particular case of a *$\mathcal{F}$-relative chiral anomaly.*

It is now possible to express the suspension $[\sigma]$ of the class

$$c_1\left(\mathrm{Ind}\left(\tilde{\not{D}}_b^+\right)\right) = c_1(\mathcal{L}) \in H^2(Y, \mathbb{Z})$$

by the foliation invariants in Theorem 3.2, where $\mathcal{L}$ denotes the determinant line bundle (cf. **[9]**). So we pass to the vector bundle $\mathcal{E}^0$ and consider the composite cohomology map of degree $-2k$

$$H(W(U(r))_k) \xrightarrow{\rho^*} H(W(G)_k) \xrightarrow{z_*\Delta_*} H\left(\tilde{\mathcal{G}}, \mathbb{R}\right) \tag{3.30}$$

defined in (3.18). Now the cohomology of $W(U(r))_k$ is well known (cf. **[11]**, §5). In degree $2k+1$ it is generated by the classes $y_i \otimes c_J$, where c_J are monomials of degree $\leq 2k$ in the Chern polynomials $c_j \in I(U(r))$ and y_i is the suspension of c_i with $\deg(c_i c_J) > 2k$. In particular, it contains the classes of Godbillon–Vey type when $i = 1$, and the classes $y_i \otimes 1$ for c_i of degree $> 2k$, if $r > k$. We have then the following result which could properly be viewed as a special type of transversal families index theorem in the present context.

THEOREM 3.4
The cohomology class

$$[\sigma] = \sigma(c_1(\mathcal{L})) \in H^1\left(\tilde{\mathcal{G}}, \mathbb{R}\right)$$

is contained in the image of $H^{2k+1}(W(U(r))_k)$, that is, the linear space generated by the classes $z_*\Delta_*(\rho^*(y_i \otimes c_J))$.

§4. The Wess–Zumino term in the foliation context and generalised Cheeger–Chern–Simons classes.

Firstly, let us consider the case where $P \cong M \times G$ is trivial as a foliated bundle, that is $P \cong M \times G$ and $L(\tilde{\mathcal{F}}) = L(\mathcal{F})$. We deduce from **[7]** that

$$\tilde{\mathcal{G}} = \mathcal{G}(\tilde{\mathcal{F}}) \simeq \mathrm{Map}_*^{\tilde{\mathcal{F}}}(P, G)^{AdjG} \simeq \mathrm{Map}_*^{\mathcal{F}}(M, G)$$

where the maps are constant along the leaves; as in **[3]**,II , we take $f \in \mathrm{Map}_*^{\mathcal{F}}(M, G)_{f_0}$, that is $f \sim f_0$ for some f_0 .

The evaluation map

$$\mu : \mathrm{Map}_*^{\mathcal{F}}(M, G) \times M \longrightarrow G \tag{4.1}$$

inducing on paths an isomorphism

$$\mathcal{P}_{f_0}(\mathrm{Map}_*^{\mathcal{F}}(M, G)) \xrightarrow{\cong} \mathrm{Map}_*^{\mathcal{F}}(M, \mathcal{P}_e G)$$

where $p \in \mathcal{P}_{f_0}(\mathrm{Map}^{\mathcal{F}}(M,G))$, is given by a path $p : M \times I \to G$ with $\tilde{p} : M \to \mathrm{Map}(I,G)$ defined by $\tilde{p}(x)(t) = (f_0(x))^{-1}p(x,t)$.

The situation is described in the following diagram

$$\begin{array}{ccccc} M \times \mathcal{P}_{f_0}(\mathrm{Map}^{\mathcal{F}}_*(M,G)) & \cong & M \times \mathrm{Map}^{\mathcal{F}}_*(M,\mathcal{P}_eG) & \xrightarrow{\mu=ev} & \mathcal{P}_eG \\ {\scriptstyle\pi_1}\downarrow & & \downarrow & & \downarrow \\ M \times \mathrm{Map}^{\mathcal{F}}_*(M,G) & \xrightarrow{\tilde{\mu}} & M \times G & \xrightarrow{f_0^{-1}} & G \end{array} \tag{4.2}$$

Taking the T to denote transgression, we recall that in the absolute case one obtains an $(q+1)$–form on $M \times \mathrm{Map}^{\mathcal{F}}_*(M,G)_{f_0}$ via $(f_0^{-1}\tilde{\mu})(T\Phi)$. The term of order $(q,1)$ satisfies a consistency condition and may be regarded as a 'transversal anomaly' (cf. **[3]**,II). If we consider the map

$$\hat{\mu} : M \times I \times \mathrm{Map}^{\mathcal{F}}_*(M,\mathcal{P}_eG) \to G$$

derived from (4.2), then in our context *the foliated Wess–Zumino term* $WZ(\tilde{\mathcal{F}})$ is given by

$$WZ(\tilde{\mathcal{F}}) = \int_{M\times I} \hat{\mu}^* T\Phi. \tag{4.3}$$

The pertinent question is :

Does $\exp(2\pi i(WZ(\tilde{\mathcal{F}})))$ descend to a functional on $\mathrm{Map}^{\mathcal{F}}_*(M,G)_{f_0}$? Specifically, under the map

$$\int_{M\times S^1} \hat{\mu}^* T\Phi : \mathrm{Map}^{\mathcal{F}}_m(M,\Omega_eG) \to \mathbb{R} \tag{4.4}$$

does $WZ(\tilde{\mathcal{F}})$ take integer values when restricted to loops?

In order to address these and other questions, we proceed in this section to introduce part of the theory of the generalized Cheeger–Chern–Simons classes as developed in **[6]** (cf. **[5]**).

If G is a Lie group with finitely many components, $K \subseteq G$ a maximal compact subgroup and $\mathfrak{k} \subseteq \mathfrak{g}$ are the corresponding Lie algebras, then *the relative Weil algebra* $W(G,K)$ is defined as

$$W(G,K) = [\Lambda((\mathfrak{g}/\mathfrak{k})^* \otimes S(\mathfrak{g}^*)]^K. \tag{4.5}$$

For k an integer, $k \geq 0$, $FW = F^{2(k+1)}W(G,K) \subseteq W(G,K)$ denotes as before the ideal generated by $S^{\ell}(g^*)$, $\ell \geq k+1$. For a subring $\Lambda \subseteq \mathbb{R}$ (typically $\Lambda = 0$, $\mathbb{Z}$ or $\mathbb{Q}$), the algebra of *generalized Cheeger–Chern–Simons classes* is defined by

$$\mathcal{S}^*(G,K;\Lambda)_k = \{(\Phi,u) \in H^*(FW) \times H^*(BK,\Lambda) : wj_*u = ru\};$$

in other words, as the pull–back in the diagram

$$\begin{array}{ccccc} S^*(G,K;\Lambda)_k & & \longrightarrow & & H^*(BK,\Lambda) \\ \downarrow & & & & \downarrow{\scriptstyle r} \\ H(F^{2(k+1)}W(G,K)) & \xrightarrow{i_*} & H(W(G,K)) & \underset{\cong}{\xrightarrow{w}} & H^*(BK,\mathbb{R}) \end{array} \tag{4.6}$$

where r is induced via the inclusion $\Lambda \subseteq R$. For a principal G–bundle on M with connection θ and curvature F_θ satisfying $(F_\theta)^{k+1} = 0$, one defines secondary characteristic classes

$$S(\Phi, u)(\theta) \in H^{\ell-1}(M, \mathbb{R}/\Lambda)$$

associated to $(\Phi, u) \in S^\ell(G, K; \Lambda)_k$ (modulo an indeterminacy). In our case $G = K$ is compact, $W(G,G) = I(G)$ and $\mathcal{S}^\ell(G, G; \Lambda)_k \cong H^\ell(BG, \Lambda)$, $\ell \geq 2(k+1)$. The suspension construction of the previous section allows us to define these classes on the bundle $\tilde{\mathcal{G}} \times P \to \tilde{\mathcal{G}} \times M$ with connection $\mu^* A$. In particular, we can establish for $\Lambda = \mathbb{Z}$:

PROPOSITION 4.1
For $q = 2k$, there exists a well–defined map

$$\Psi_3 : \mathcal{S}^{2k+j+1}(G, G, \mathbb{Z})_k \to H^j(\tilde{\mathcal{G}}, \mathbb{R}/\mathbb{Z}), \tag{4.7}$$

for j odd, which is determined by the maps Ψ, Ψ_1 and Ψ_2 on $FI(G)$ mod $\mathbb{Z}$.
Proof. For the connection $\mu^* A$ on $\tilde{\mathcal{G}} \times P \to \tilde{\mathcal{G}} \times M$, the relation $F_{\mu^* A} = \mu^* F_A$ implies

$$(F_{\mu^* A})^{k+1} = 0 \quad \text{for} \quad q = 2k.$$

In view of (4.6) with $G = K$, the map Ψ_3 is defined as the composition in the diagram below in which we have recalled (3.1):

$$\begin{array}{ccc} \mathcal{S}^{2k+j+1}(G, G, \mathbb{Z})_k & \xrightarrow{S(\Phi, \mu^* A)} & H^{2k+j}(\tilde{\mathcal{G}} \times M, \mathbb{R}/\mathbb{Z}) \\ & & \Big\downarrow \int_m \wedge \chi_{\mathcal{F}} \square \\ & & H^j(\tilde{\mathcal{G}}, \mathbb{R}/\mathbb{Z}). \end{array}$$

The Bockstein homomorphism

$$H^{2k+j}(\tilde{\mathcal{G}} \times M, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} H^{2k+j+1}(\tilde{\mathcal{G}} \times M, \mathbb{Z}) \tag{4.8}$$

gives

$$\beta S(\mu^* A)(\Phi, u) = -u(\tilde{\mathcal{G}} \times P)$$

where $u(\tilde{\mathcal{G}} \times P)$ is the characteristic class associated to P in $H^{2k+j+1}(\tilde{\mathcal{G}} \times M, \mathbb{Z})$ and also gives the integral class (mod $h(\mu^* A)\Phi = 0$, for $\Phi \in FI(G)$) by the above curvature property.

§5. Homotopy groups of the foliated gauge group.

In order to exemplify matters, we return to the case where $(P, \mathcal{F}) \to (M, \mathcal{F})$ is trivial as a foliated bundle.

Let $\hat{\mathcal{F}}$ be the induced foliation on the transversal (oriented) frame bundle $F_{SO(q)}(Q)$. The Molino space construction **[16]** gives the basic $SO(q)$-invariant fibration

$$\begin{array}{ccccc} (\hat{\mathcal{F}}_0) & & (\hat{\mathcal{F}}) & & \\ X & \longrightarrow & F_{SO(q)}(Q) & \longrightarrow & W \\ & & \Big\downarrow & & \Big\downarrow \\ & & M & \longrightarrow & W/SO(q) \ , \end{array} \tag{5.1}$$

where W here denotes the space of leaf closures of $\hat{\mathcal{F}}$ and the quotient $W/SO(q)$ is that of $\mathcal{F}$. This also implies the homotopy equivalences

$$\tilde{\mathcal{G}} \simeq \mathrm{Map}_*^{\mathcal{F}}(M,G) \simeq \mathrm{Map}_*(W/SO(q),G). \tag{5.2}$$

Letting $W/SO(q)_0$ be the space of principal orbits in $W/SO(q)$ and $q' = \dim(W/SO(q)_0)$, $0 \leq q' \leq q$, we can establish the following homotopy equivalences for $G = U(r)$, $r >> q'$. It will be convenient to denote $W/SO(q)$ by W'.

THEOREM 5.1
For $G = U(r)$, $r >> q'$ in the stable range, we have

$$\pi_j(\tilde{\mathcal{G}}) \simeq \begin{cases} K^1(W')\ , & j \quad \textit{even} \\ K^0(W')\ , & j \quad \textit{odd.} \end{cases} \tag{5.3}$$

Proof. First of all we have

$$\pi_0(\tilde{\mathcal{G}}) \simeq \pi_0\,\mathrm{Map}_*(W',U) \simeq [W',U] \simeq K^1(W')$$

(here $U = \lim_{N\to\infty} U(N)$).
Secondly it follows that

$$\pi_1(\tilde{\mathcal{G}}) \simeq \pi_0(\Omega\tilde{\mathcal{G}}) = \pi_0\Omega\,\mathrm{Map}_*(W',U) \simeq \pi_0\,\mathrm{Map}_*(W',\Omega U) \simeq [W',BU] \simeq K^0(W')$$

and the result then follows by Bott periodicity. $\square$
For general G, one has (cf. **[18]**)

$$\begin{aligned}\pi_j(\tilde{\mathcal{G}}) &\simeq \pi_0(\Omega^j\,\mathrm{Map}_*(W',G)) \simeq \pi_0\,\mathrm{Map}_*(W',\Omega^j G) \\ &\simeq [W',\Omega^j G].\end{aligned} \tag{5.4}$$

Taking ℓ =rank of G, we have by Hopf's theorem for coefficients in $\mathbb{Q}$ or $\mathbb{R}$,

$$H^*(G) \cong \Lambda(y_1\ldots,y_\ell) \cong \Lambda P_G \quad \text{(primitive elements)}$$

where $y_i = \sigma(c_i)$ and $H(BG,\mathbb{Z})\otimes\mathbb{Q} \subset H(BG,\mathbb{R}) \cong I(G) \cong \mathbb{R}[c_1,\ldots,c_\ell]$.

For a simply connected topological space X, the simplicial DeRham complex for the singular complex of X was treated in **[6]**. Taking the minimal model $\mathcal{M}$ over the latter determines the rational homotopy type of X. Here we find $\mathcal{M}(G) \cong \Lambda(y_1,\ldots,y_\ell)$, with $d_{\mathcal{M}} = 0$.

Suppose now that $W' = W/SO(q) \simeq S^q$ and $q' = q = 2k$. Then we have $\tilde{\mathcal{G}} = \mathrm{Map}_*(S^q,G) \simeq \Omega^q G$ and it follows that

$$\pi_j(\tilde{\mathcal{G}}) \cong \pi_{j+q}(G) \tag{5.5}$$

By Sullivan's technique **[19]** , the rational homotopy type of $\tilde{\mathcal{G}}$ is then determined by the minimal algebra with trivial differential

$$\mathcal{M}(\tilde{\mathcal{G}}) \cong \Lambda(\bar{y}_{j_0},\ldots,\bar{y}_l)\ , \tag{5.6}$$

where j_0 is the least such integer with $\deg y_{j_0} > q$ and $\deg \bar{y}_i = \deg y_i - q$. Thus we have proved the following theorem.

THEOREM 5.2
Under the above assumptions, the minimal model of $\tilde{\mathcal{G}}$ is given by

$$\mathcal{M}(\tilde{\mathcal{G}}) \cong \Lambda(\bar{y}_{j_0}, \ldots, \bar{y}_l) \ ,$$

with trivial differential and thus $\tilde{\mathcal{G}}$ has the rational homotopy type of a product of spheres of odd dimension.

We remark that similar considerations apply equally well to the case q odd **[8]** and this will be reported upon elsewhere.

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