Resonance based schemes for dispersive equations via decorated trees

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Dispersive PDEs

We consider nonlinear dispersive equations of the form

$$
\begin{aligned}
&i\partial_t u(t,x) + \mathcal{L}u(t,x) = p\left(u(t,x), \overline{u}(t,x)\right) \\
&u(0,x) = v(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbf{T}^d
\end{aligned}
$$

where $\mathcal{L}$ is a differential operator and p is a polynomial nonlinearity.

- Assume local wellposedness of the problem on the finite time interval $]0, T]$, $T < \infty$ for $v \in H^n$.
- Aim: give a numerical approximation of u at low regularity when n is small.

NLS: $\mathcal{L} = \Delta$ and $p(u, \overline{u}) = |u|^2 u$.

KdV: $\mathcal{L} = i\partial_x^3$ and $p(u, \overline{u}) = i\partial_x(u^2)$.

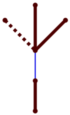
Decorated trees approach

Mild solution given by Duhamel's formula:

$$u(t) = \underbrace{e^{it\mathcal{L}}}_{\text{edge}} v + \underbrace{e^{it\mathcal{L}}}_{\text{edge}} \underbrace{(-i \int_0^t e^{-i\xi\mathcal{L}}}_{\text{edge}} p\left(u(\xi), \overline{u}(\xi)\right) d\xi)$$

Definition of a character Π : *Decorated trees* $\rightarrow$ *Iterated integrals*

1. $e^{it\mathcal{L}} v = (\Pi T_0)(t, v), \quad T_0 =$
2. $-ie^{it\mathcal{L}} \int_0^t e^{-i\xi\mathcal{L}} p\left(e^{i\xi\mathcal{L}} v, e^{-i\xi\mathcal{L}} \overline{v}\right) d\xi = (\Pi T_1)(t, v),$

$T_1 =$ 

B-series type expansion

Solution U^r up to order r can be represented by a series:

$$U^r(t,v) = \sum_{T \in \mathcal{V}^r} \frac{\Upsilon^p(T)}{S(T)} (\Pi T)(t,v),$$

- $\mathcal{V}^r$: decorated trees of order r.
- $S(T)$: symmetry factor.
- Υ^p: elementary differentials.
- Error of order

$$\mathcal{O}\Big(t^{r+1} q(v)\Big)$$

for some polynomial q.

Treatment of oscillations

The principal oscillatory integral takes the form

$$\mathcal{I}_1(t, \mathcal{L}, v, p) = \int_0^t \mathrm{Osc}(\xi, \mathcal{L}, v, p) d\xi$$

with the central oscillations Osc given by

$$\mathrm{Osc}(\xi, \mathcal{L}, v, p) = e^{-i\xi\mathcal{L}} p\left(e^{i\xi\mathcal{L}} v, e^{-i\xi\mathcal{L}} \overline{v}\right).$$

In general it will be

$$\mathrm{Osc}(\xi, \mathcal{L}, v, p) = e^{-i\xi\mathcal{L}} p\left(e^{i\xi(\mathcal{L}+\mathcal{L}_1)} q_1(v), e^{-i\xi(\mathcal{L}+\mathcal{L}_2)} q_2(v)\right).$$

Various approaches

Classical Methods:

- exponential method: $\mathrm{Osc}(\xi, \mathcal{L}, v, p) \approx e^{-i\xi\mathcal{L}} p(v, \overline{v})$
- splitting method: $\mathrm{Osc}(\xi, \mathcal{L}, v, p) \approx p(v, \overline{v})$

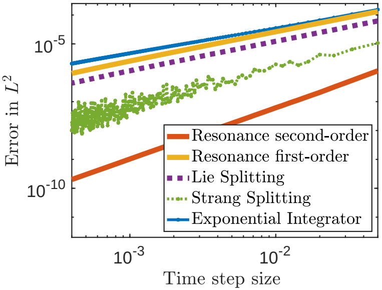
Resonance as a computational tool:

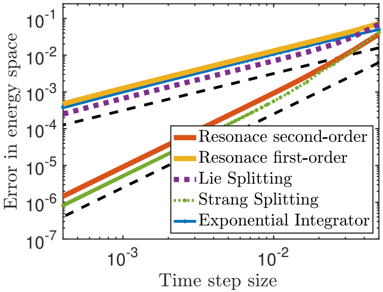
$$\mathrm{Osc}(\xi, \mathcal{L}, v, p) = \left[e^{i\xi\mathcal{L}_{\mathrm{dom}}} p_{\mathrm{dom}}\left(v, \overline{v}\right) \right] p_{\mathrm{low}}(v, \overline{v}) + \mathcal{O}\Big(\xi \mathcal{L}_{\mathrm{low}} v\Big).$$

Here, $\mathcal{L}_{\mathrm{dom}}$ denotes a suitable dominant part of the high frequency interactions and

$$\mathcal{L}_{\mathrm{low}} = \mathcal{L} - \mathcal{L}_{\mathrm{dom}}$$

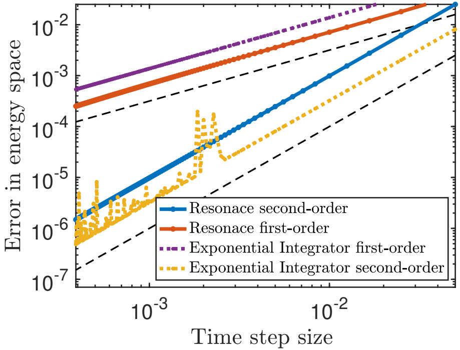
Comparison of classical and resonance based schemes for the Schrödinger equation for smooth ($\mathcal{C}^\infty$ data) and non-smooth (H^2 data) solutions.





Experiments

Comparison of classical and resonance based schemes for the KdV equation with smooth data in $\mathcal{C}^\infty$.



Fourier iterated integrals

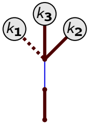
Mild solution given by Duhamel's formula in Fourier ($P(k) \leftrightarrow \mathcal{L}$):

$$\hat{u}_k(t) = \underbrace{e^{itP(k)}}_{\text{edge}} \hat{v}_k + \underbrace{e^{itP(k)}}_{\text{edge}} \underbrace{\left(-i \int_0^t e^{-i\xi P(k)}\right.}_{\text{edge}} p_k\left(u(\xi), \overline{u}(\xi)\right) d\xi)$$

Definition of a character $\hat{\Pi}$: *Decorated trees* $\to$ *Iterated integrals*

1. $e^{itP(k)} = (\hat{\Pi} T_0)(t), \quad T_0 =$ 

2. $-ie^{itP(k)} \int_0^t e^{-i\xi(P(k)-P(-k_1)+P(k_2)+P(k_3))} d\xi = (\hat{\Pi} T_1)(t),$

$T_1 =$ 

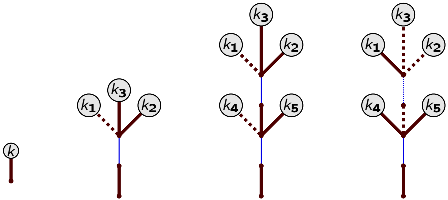
Fourier B-series type expansion

Resonance scheme U_k^r of order r with regularity n (initial data):

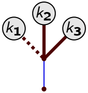
$$U_k^r(\tau, v) = \sum_{T \in \mathcal{V}_k^r} \frac{\Upsilon^p(T)(v)}{S(T)} \left(\hat{\Pi}_n^r T\right)(\tau)$$

- $\mathcal{V}_k^r$: decorated trees of order r with frenquency k.
- Character $\hat{\Pi}_n^r$ resonance approximation of $\hat{\Pi}$.

Examples of decorated trees for NLS ($r = 2$):



A practical example

The iterated integral associated to $T =$ 

is given by:

$$(\hat{\Pi} T)(t) = \int_0^t e^{i\xi\left(-k^2 - k_1^2 + k_2^2 + k_3^2\right)} d\xi, \quad k = -k_1 + k_2 + k_3$$

One has

$$-k^2 - k_1^2 + k_2^2 + k_3^2 = \underbrace{\mathcal{L}_{\text{dom}}}_{-2k_1^2} + \underbrace{\mathcal{L}_{\text{low}}}_{\text{order one}}$$

Taylor expansion of $\mathcal{L}_{\text{low}}$:

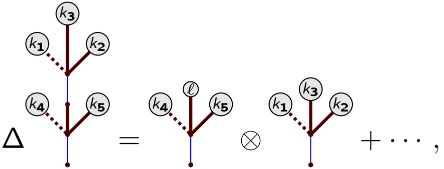
$$(\hat{\Pi} T)(t) = \underbrace{\frac{e^{-2itk_1^2} - 1}{-2ik_1^2}}_{(\hat{\Pi}_n^r T)(t)} + \mathcal{O}\left(t\mathcal{L}_{\text{low}}\right)$$

Local Error Analysis

Main idea is to single out oscillations:

$$\int_0^t e^{i\xi P(k)} d\xi = \frac{e^{itP(k)} - 1}{iP(k)}$$

Butcher-Connes-Kreimer coproduct Δ



$$\ell = -k_1 + k_2 + k_3$$

Integrals $\int_0^t \xi^\ell e^{i\xi P(k)} d\xi \to$ deformed BCK coproduct $\hat{\Delta}$ (SPDEs).

Main result

Birkhoff type factoriation:

$$\tilde{\Pi}_n^r = \left(\hat{\Pi}_n^r \otimes (\mathcal{Q} \circ \hat{\Pi}_n^r \mathcal{A} \cdot)(0) \right) \hat{\Delta}.$$

where $\mathcal{A}$ is an antipode and $\mathcal{Q}$ is a projector.

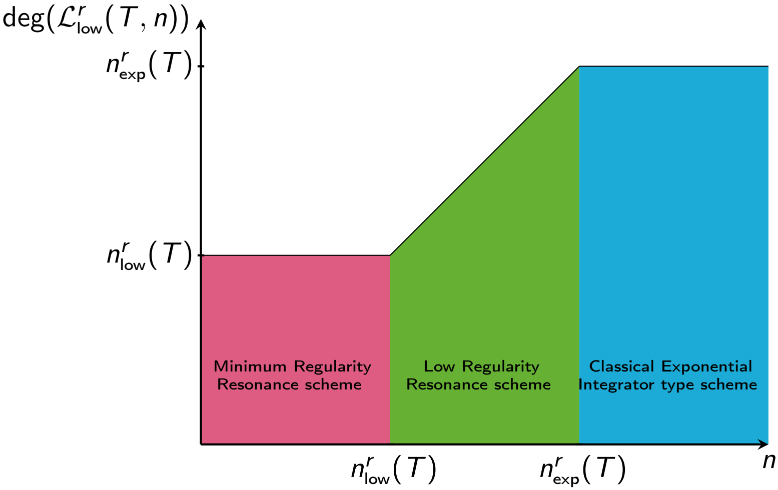
Theorem (B., Schratz 2020)

For every $T \in \mathcal{V}_k^r$

$$\left(\hat{\Pi} T - \hat{\Pi}_n^r T \right)(\tau) = \mathcal{O}\left(\tau^{r+1} \mathcal{L}_{low}^r(T, n) \right).$$

where $\mathcal{L}_{low}^r(T, n)$ *involves all lower order frequency interactions.*

Family of schemes $\hat{\Pi}_n^r$



Perspectives

- B-series $\rightarrow$ Regularity Structures $\rightarrow$ PDEs Numerical Schemes
- New example of a deformation of the BCK coproduct.
- Birkhoff type factorisation as in SPDEs see (B., Ebrahimi-Fard 2020).
- Backward error analysis for the scheme.
- Structure preservation.
- Generalisation to more general domains not only $\mathbf{T}^d$ and wave equations.
- Potential connection with the study of dispersive (S)PDEs.