Tate-Hochschild Cohomology of Hypersurfaces and singularity categories

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I. Tate-Hochschild cohomology (intro)
II. Cohomology of hypersurfaces
III. Deformation theory
IV. $\mathbb{Z}$-graded vs $\mathbb{Z}/2$-graded cohomology

I. Tate-Hochschild cohomology

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Since Tate cohomology is more popular in number theory, I briefly recall it.
Let $k :=$ any commutative ring, $G :=$ finite group, $M =$ fg G-module $(/k)$

$$H^\bullet(G;M) := \mathrm{Ext}^\bullet_{k[G]}(k, M) \cong H^\bullet(\mathrm{Hom}_{k[G]}(P_\bullet, M)),$$

where: $\cdots \to P_1 \to P_0 \twoheadrightarrow k$ fg projective G-module resolution

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<u>Defn</u>: $M^\vee := \mathrm{Hom}_{k[G]}(M, k[G])$ module dual.

- $M \mapsto M^\vee$ is a contravariant functor.
- On G-modules which are fg projective$/k$, it is exact (e.g., $k =$ field, M finite-dimensional, since $k[G]$ is Frobenius $\Rightarrow$ injective over itself)

$$\Rightarrow \text{ get } \quad k \hookrightarrow \underbrace{P_0^\vee \to P_1^\vee \to \cdots}_{\text{fg projective } G\text{-modules}}$$

<u>Put together</u>:

$$\begin{tikzcd}
\hat{P} := \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, phantom, "\tilde{P}_1"] & P_0 \arrow[rr] \arrow[dr, two heads] \arrow[d, phantom, "\tilde{P}_0"] & & P_0^\vee \arrow[r] \arrow[d, phantom, "\tilde{P}_{-1}"] & P_1^\vee \arrow[r] \arrow[d, phantom, "\tilde{P}_0"] & \cdots \\
& & & k \arrow[ur, hook] & & &
\end{tikzcd}$$

is <u>exact</u> complex of projective G-modules,

$$\hat{H}^i(G; M) := H^i(\mathrm{Hom}(\tilde{P}_\cdot, M)) \cong \begin{cases} H^i(G; M), & i > 0 \\ H_{-i-1}(G; M), & i < 0 \\ \mathrm{coker}\left(M \xrightarrow[\text{"Norm"}]{\sum_{g \in G} g \cdot -} M\right), & i = 0. \end{cases}$$

Then $\hat{H}$ encodes <u>positive</u>-degree cohomology + homology, <u>and</u>

$$\hat{H}^\cdot(G; \underset{\text{"}\mathrm{Ind}_{\{e\}}^G V\text{"}}{V[G]}) = 0, \quad \forall\ k\text{-modules } V.$$

I. Tate-Hochschild cohomology:

Recall: $HH^\cdot(A;M) := \mathrm{Ext}^\cdot_{A^e}(A, M)$ bimod

A = associative algebra$/k$

$A^e = A \otimes_k A^{op}$: A^e-modules $=:$ A-bimodules $/k$

$$HH^\cdot(A;\underbrace{M}_{A\text{-bimodule}}) = H^\cdot(\mathrm{Hom}(P_\cdot, M)), \qquad \underbrace{\cdots \to P_1 \to P_0}_{\text{projective } A\text{-bimodule resolution}} \twoheadrightarrow A$$

Definition: $M^\vee := \mathrm{Hom}_{A^e}(M, \overset{A^e}{\overset{\shortparallel}{A \otimes A}})$, bimodule dual. (bimodule via inner action)

$\boxed{\text{Case } A = k[G]\text{: } M \mapsto M^\vee \text{ is anti-equivalence}}$ $\Rightarrow$ can construct:

k field, G finite

$$\hat{P} := \underbrace{\cdots \to P_1 \to P_0}_{\text{projective bimodule res.}} \longrightarrow \underbrace{P_0^\vee \to P_1^\vee \to \cdots}_{\text{bimodule dual of resolution}} \quad \text{exact} \quad \boxed{(\star)}$$

acyclic

$P_0 \twoheadrightarrow A \hookrightarrow P_0^\vee$

Defn $\hat{HH}^i(A) := H^i(\mathrm{Hom}_{A^e}(\hat{P}_\cdot, M)) \cong \begin{cases} HH^i(A;M), & i>0 \\ HH_{-i-1}(A;M), & i<0 \\ \mathrm{coker}\left(M \xrightarrow{\text{Norm}} M,\ m \mapsto \sum_{g\in G} gmg^{-1}\right) \end{cases}$ Hochschild cohomology + homology

'Analogue of $\hat{H}^i(G;N)$ "Tate Cohomology"', N = G-module. \puts together group homology + cohomology.

Note: This all works more generally if A is "Frobenius$/k$"
(A is fg projective k-module, with $A \times A \longrightarrow k$ = perfect invariant pairing.)

Generalise further: if $\exists$ some exact complex ($\star$), | (4/)

Defn. $\widehat{HH}^{\cdot}(A;M) := H^{\cdot}(\mathrm{Hom}(\tilde{P}, M))$, $\widehat{HH}_{\cdot}(A;M) := H_{\cdot}(\tilde{P} \otimes_{A^e} M)$.

"Tate–Hochschild cohomology, homology"

Formalism to explain its meaning:

Theorem (Buchweitz '86): Let A := Gorenstein + Noetherian:

$$\left\{\begin{matrix}\text{Acyclic complexes}\\ \text{of projective } A\text{-mods}\end{matrix}\right\}\Big/\text{homotopy} \xrightarrow{\sim} \overbrace{\left\{\begin{matrix}\text{Maximal Cohen-Macaulay}\\ \text{modules } / A\end{matrix}\right\}}^{\text{"MCM"}}\Big/\text{projectives} \xrightarrow{\sim} \overbrace{D^b(A)\Big/\underbrace{\mathrm{Perf}(A)}_{}}^{\text{"singularity category"}}$$

MCM modules = fg modules M s.t. $\mathrm{Ext}^{>0}_A(M, A) = 0$.

Perfect complexes = complexes of fg proj A-mods

$$\tilde{P} \longmapsto \mathrm{coke}(\tilde{P}_1 \to \tilde{P}_0)$$

$$M \longmapsto [M]$$

Defn $D^b_{sg}(A) := D^b(A)/\mathrm{Perf}(A)$ "singularity category"

Cor: $\widehat{HH}^{\cdot}(A;M) \cong \mathrm{Ext}^{\cdot}_{D^b_{sg}(A^e)}(A, M)$, if $\tilde{P}$ as in ($\star$)

/Defn Tate-Hochschild — sing bimodule cat ($\widehat{HH}_{\cdot} \cong \mathrm{Tor}$)

RHS makes sense $\forall$ associative algebras A! (5/)

Examples: Truncated polynomial algebra:

$\{x^n=0\} \subset \mathbb{A}^1$ fat point

$A = k[x]/(x^n)$: have 2-periodic bimodule resolution

Not a group alg.

$$\cdots \to A\otimes A \xrightarrow[\cdot(x^{n-1}\otimes 1 + x^{n-2}\otimes x + \cdots + 1\otimes x^{n-1})]{} A\otimes A \xrightarrow{\cdot(x\otimes 1 - 1\otimes x)} A\otimes A \to A$$

$= \dfrac{x^n\otimes 1 - 1\otimes x^n}{\underset{u}{x\otimes 1} - \underset{v}{1\otimes x}}$, $A^e \cong k[u,v]/u^n, v^n$

$\Rightarrow \hat{HH}^i(A;M)$ depends only on parity of i

$$\hat{HH}^i(A;A) \cong \begin{cases} A/x^{n-1}A \cong k[x]/(x^{n-1}), & i \text{ even} \\ {}^{x}A \cong k[x]/(x^{n-1}), & i \text{ odd} \end{cases}$$

$H^\cdot(\underset{\text{odd}}{A} \xrightarrow{\cdot x^{n-1}} \underset{\text{even}}{A})$

Observe: actually $\hat{HH}^i$ are all isomorphic.

Note: By definition, $\hat{HH}^\cdot$ is always an associative algebra.
Thm (Z. Wang '15): in fact always Gerstenhaber!

II. Hypersurfaces.

Let $A := \underbrace{k[x_1, \dots, x_n]}_{\substack{\| \\ R,\ \text{polynomial ring.}}} / (f)$.

MOTIVATION: k = field.

Eisenbud (1980): Every fg A-module has a projective resolution which is eventually 2-periodic (MCM modules have 2-periodic resolutions from the start).

$\rightsquigarrow$ connection MCM $\longleftrightarrow$ matrix factorisations:

$$\cdots \to A^{m'} \xrightarrow{d} A^{m} \xrightarrow{d'} A^{m'} \xrightarrow{d} A^{m} \twoheadrightarrow M \longleftrightarrow \text{matrices} \begin{cases} d \in \mathrm{Mat}_R(m, m') \\ d' \in \mathrm{Mat}_R(m', m) \end{cases}$$

$$\boxed{MF^{\mathbb{Z}/2}(f) := \{\text{such } (d, d')\}}$$

Factorisation of fI. $\begin{cases} d \circ d' = f I_m \\ d' \circ d = f I_{m'} \end{cases}$ $\quad R^m \underset{\longleftarrow}{\longrightarrow} R^{m'}$

Buenos Aires Cyclic Homology (BACH) group (1992): gave explicit 2-periodic bimodule resolution of A.

$$\Rightarrow \underbrace{HC^{\cdot}(A;A)}_{\text{Hochschild cochain complex}} \simeq \Big(\underbrace{K_*\big(A;\tfrac{\partial f}{\partial x_1},\dots,\tfrac{\partial f}{\partial x_n}\big)}_{\text{Koszul complex}}[t],\ t\underbrace{\partial_K}_{\text{Koszul differential}}\Big) \quad (7/)$$

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$|t| = 2$

$$\therefore\ \widehat{HH}(A;A) \simeq HH^{\cdot}(A;A)[t^{-1}]$$ Localisation!

> ⚠ Caution: By Künneth formula, for complete intersections (non-hypersurface) $HH^i(A;A)$, $HH_i(A,A)$ grow with i $\Rightarrow$ $\widehat{HH}$ is NOT a localisation

Case $\{f=0\}$ has only isolated singularities:

$$\widehat{HH}^i(A;A) \simeq \begin{cases} A\big/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right) \text{ "Tjurina ring"} \\ = \operatorname{coker}\left(J \xrightarrow{\cdot f} J\right),\ J := R\big/\left(\frac{\partial f}{\partial x_i}\right) \text{ "Milnor ring"}, & i \text{ even} \\ \ker\left(J \xrightarrow{\cdot f} J\right) \text{ nonzero}, & i \text{ odd.} \end{cases}$$

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Observe: Since $\widehat{HH}(A;A) \cong \underbrace{HH(A;A)}_{\text{Gerstenhaber algebra}}[\underset{\text{central}}{t^{-1}}]$

$\Rightarrow \widehat{HH}(A;A)$ is also a Gerstenhaber algebra.

[Thm (Z. Wang): $\widehat{HH}(A;A)$ is actually Gerstenhaber for arbitrary A associative.]

Theorem (CS–Wang): $\widehat{HH}(A;A)$ is Batalin–Vilkovisky:

$\exists\, \Delta : \widehat{HH} \to \widehat{HH}^{\cdot-1}$, $\Delta^2 = 0$, $\{x,y\} = \Delta(x \cup y) \pm \Delta(x) \cup y \pm x \cup \Delta(y)$.

Moreover, $\left(\widehat{HH}^{\cdot}(A;A), \widehat{HH}_{\cdot}(A;A)\right)$ form a Tamarkin–Tsygan calculus, with duality $\widehat{HH}^{\cdot} \xrightarrow{\sim} \widehat{HH}_{n-\cdot}$.

Van den Bergh, Ginzburg: Ordinary $HH^{\cdot}, HH_{\cdot}$ has BV, duality when A is n-Calabi–Yau. (NOT for hypersurfaces!)

Question: does $\widehat{HH}^{\cdot}, \widehat{HH}_{\cdot}$ always form a T-T calculus?

Explicit description of algebraic structures:

Theorems with Wang.

Isolated singularities: $J := R/\left(\frac{\partial f}{\partial x_i}\right)$, here finite-dim$/k$

Jacobi ring

$\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0$ = sing locus.

$$\widehat{HH}(A;A) \cong H^{\cdot}\left(J[\alpha, t, t^{-1}],\ d = f\frac{\partial}{\partial \alpha}\right)$$

$|\alpha| = -1$, $|t| = 2$, $\alpha^2 = 0$

$\| \quad J[t]\alpha_{odd} \xrightarrow{\cdot f} J[t]_{even}$

$\widehat{HH}^0 \cong$ Tjurina J/fJ

Cup product: $\widehat{HH}^{even} \times \widehat{HH} \to \widehat{HH}$ is usual multiplication by Tjurina ring.

For $g, h \in \ker(J \xrightarrow{f} J)$, write:

zero in J $\to$ $\left\{ \underline{fg = \sum g_i \frac{\partial f}{\partial x_i}} \right.$, $\quad fh = \sum h_i \frac{\partial f}{\partial x_i}$ $\quad \left(\frac{\partial^{(2)}}{\partial x_i \partial x_j} = \begin{cases} \frac{1}{2}\frac{\partial^2}{\partial x_i^2}, & i = j \\ \frac{\partial^2}{\partial x_i \partial x_j}, & i \neq j \end{cases} \right)$

Then $\underbrace{g\alpha}_{\in \widehat{HH}^{-1}} \cup h\alpha = t^{-1} \sum_{i \leq j} g_i h_j \frac{\partial^{(2)} f}{\partial x_i \partial x_j}$, extend $k[t, t^{-1}]$-linearly.

$\boxed{\text{Has Batalin-Vilkovisky structure!}}$ $\quad \Delta|_{\widehat{HH}^{even}} = 0$; for g, g_i as above,

$$\Delta(\underbrace{g\alpha t^m}_{\deg 2m-1}) = -\sum \frac{\partial g_i}{\partial x_i} t^{m-1} - (m-1) g t^{m-1}.$$

DOES NOT come from HH: A is NOT Calabi-Yau!!

Says: D_{sg} behaves "like" n-CY category...

General case:

$$HH^{\cdot} \simeq \left(T_{poly}(\mathbb{A}^{n|1}), [ft, -]_{S-N}\right)$$

$$\parallel$$

$$R \otimes \Lambda^{\cdot}\langle \partial_1, \dots, \partial_n \rangle \otimes R[\alpha, t]$$ polyvectors on $\mathbb{A}^{0|1}$

Here, $[-,-]$ is Schouten-Nijenhuis bracket, interpreting $t = \frac{\partial}{\partial \alpha}$. $(\Rightarrow [t, \alpha] = 1)$.

$$\widehat{HH} \cong HH[t^{-1}].$$

$$\widehat{HH}_{\cdot} \simeq (\Omega C\mathbb{A}^{n|1})[t^{-1}], \mathcal{L}_{ft})$$

$\Leftarrow$ Lie derivative

($\Rightarrow$ invariance under Knörrer periodicity $f \mapsto f + yz \in R[y,z]$)

$$\widehat{HH}^{\cdot} \xrightarrow[\sim]{\iota_\omega = \text{"duality"}} \widehat{HH}_{n-\cdot}, \quad \omega = dx_1 \wedge \cdots \wedge dx_n$$ volume form

B (Connes differential) $= d_{DR}$ (on cohomology)

$\Delta = \iota_\omega^{-1} \circ B \circ \iota_\omega$. (always for calculus + duality)

Also we give the A_∞ structure on $HH^{\cdot}$, $\widehat{HH}^{\cdot}$:

III. Deformation theory

Theorem (Keller '18): If $\underbrace{D^b(A) \text{ is homologically smooth}}_{\text{e.g., } A = \text{f.g. commutative algebra/ perfect field}}$

and A noetherian, then:

$$\hat{HH}^\cdot(A;A) \cong HH^\cdot(D_{sg}(A))$$

dg enhancement

Hochschild cohomology of dg category

Thus, Tate-Hochschild cohomology controls deformation theory of $D_{sg}(A)$ (of singularities of A).

Hypersurfaces with isolated singularities:

$\therefore HH^2(D_{sg}(A)) \cong \underbrace{J/fJ}_{\text{Tjurina}} \cong \mathrm{Harr}^2(A)$ Harrison cohomology

This controls commutative deformations of A.

Question: is this still true for complete intersections?

Have always a canonical map ~~$HH(D_{sg})$~~ (Keller)

$\beta^{\cdot} : \underbrace{HH^{\cdot}(A;A)}_{\text{deform } A} \longrightarrow \underbrace{\widehat{HH}^{\cdot}(A;A)}_{\text{deform } D_{sg}}$ (in hypersurface case, just inverting t)

<u>Theorem</u> (S–Wang): Deformations in $\ker(\beta^2)$ of A induce trivial infinitesimal deformations of $D_{sg}(A)$.

Thus, Keller's isomorphism $\widehat{HH} \xrightarrow{\sim} HH^{\cdot}(D_{sg}(A))$ is consistent with β: should compose to <u>canonical</u> $HH^{\cdot}(A,A) = HH^{\cdot}(D(A)) \to HH^{\cdot}(D_{sg}(A))$.

IV. $\mathbb{Z}$-graded vs $\mathbb{Z}/2$-graded categories.

Dyckerhoff studied the $\mathbb{Z}/2$-dg category (hypersurface, isolated sings)

$$D_{sg}^{\mathbb{Z}/2}(A) \cong MF^{\mathbb{Z}/2}(A) \quad \text{(matrix factorisations)}$$

<u>Thm (Dyckerhoff '11)</u>: $HH^0(D_{sg}^{\mathbb{Z}/2}(A)) \cong J$, $HH^1(D_{sg}^{\mathbb{Z}/2}(A)) = 0$.

($J = R/(\partial f)$)

(13/)

Have: $F : \underline{\mathbb{Z}\text{-dg vs/cat}} \rightleftarrows \underline{\mathbb{Z}/2\text{-dg vs/cat}} : G$

F = "forgetful functor"

G = "unrolling functor" get periodic $\mathbb{Z}$-graded

Then $D_{sg}(A) \simeq G(D_{sg}^{\mathbb{Z}/2}(A))$. for A hypersurface

Note: $HH^{\cdot}(D_{sg}(A)) \not\cong \underbrace{G(HH^{\cdot}(D_{sg}^{\mathbb{Z}/2}(A)))}_{\text{concentrated in even degrees}}$

This guess is too naïve.

Example: Suppose $\underset{\mathbb{Z}/2\text{-dga}}{B} \cong F(\underset{\mathbb{Z}\text{-dga}}{C})$. (E.g., C always exists if $B_{odd} \cdot B_{odd} = 0$.)

Then $G(B) \cong G(F(C)) \cong \underline{C[t, t^{-1}]}$

$HH^{\cdot}(G(B)) \overset{\text{Künneth}}{\cong} HH^{\cdot}(C) \otimes HH^{\cdot}(k[t, t^{-1}])$ polyvect fds on $\mathbb{A}^1 \setminus \{0\}$

$$\Rightarrow \boxed{HH^{\cdot}(G(B)) \cong HH^{\cdot}(B)[\alpha, t, t^{-1}]} \quad \left(\alpha = \text{"}\frac{\partial}{\partial t}\text{"}, |\alpha| = -1\right)$$

corrected naïve guess. odd and even.

Since $|t|=2$, this makes sense though B is $\mathbb{Z}/2$-graded.

⚠ NOT true for arbitrary $\mathbb{Z}/2$-graded B!

Back to $D_{sg}(A)$, $A = R/(f)$, $\{f=0\}$ has isolated sings:

f (Quasi) homogeneous: $f=0$ in J $(f = m\sum x_i \frac{\partial f}{\partial x_i},\ m = \deg f)$

$$HH^\cdot(D_{sg}(A)) \cong HH^\cdot(D_{sg}^{\mathbb{Z}/2}(A))[\alpha, t, t^{-1}] \cong J[\alpha, t, t^{-1}]$$

f general:

$$HH^\cdot(D_{sg}(A)) \cong \Big(\ \vdash\!\!\dashv\ , \ \underbrace{d = ft\frac{\partial}{\partial \alpha} = [ft, -]}_{\text{perturbation.}} \Big).$$

This motivates:

Theorem (S-Segal-Wang): For $\mathcal{C}$ = any $\mathbb{Z}/2$-dg category/algebra,

$HH^\cdot(G(\mathcal{C}))$ = perturbation of $HH^\cdot(\mathcal{C})[\alpha, t, t^{-1}]$. $t = \frac{\partial}{\partial \alpha}$, $|\alpha| = -1$

The periodic perturbation is trivial if $\mathcal{C} = F(\partial\theta)$, or $d_{\mathcal{C}} = 0$ (no diff'l) and char $k \neq 2$

NOTE: For f (quasi) homogeneous, it is not clear that $MF^{\mathbb{Z}/2}(f) = F(\text{some } \mathbb{Z}\text{-graded category})$:

$MF^{\mathbb{Z}/2}(f) \simeq D_{sg}^{\mathbb{Z}/2}(A)$

$$\Big(\underset{\text{odd}}{R^{m'}} \underset{d'}{\overset{d}{\rightleftarrows}} \underset{\text{even}}{R^{m}}, \quad d \circ d' = f I_m,\ d' \circ d = f I_{m'} \Big)$$

need not lift to $\mathbb{Z}$-graded $d^0, d^1, d^2, \dots$

So the Theorem does not immediately explain why

$$HH^\bullet(D_{sg}) \cong HH^\bullet(D_{sg}^{\mathbb{Z}/2}) \otimes HH^\bullet(k[t, t^{-1}]).$$

(corrected naïve guess holds.)

I think, f quasihomogeneous $\Rightarrow$ $\mathrm{End}(P) \cong F(C)$,

P = compact generator of $MF^{\mathbb{Z}/2}(f)$ (Dyckerhoff found it),

C = some $\mathbb{Z}$-graded algebra.

$\Rightarrow \exists$ $\mathbb{Z}$-dg cat $\mathcal{D}$, $MF^{\mathbb{Z}/2}(f) \simeq F(\mathcal{D})$. ($\mathcal{D} = D(C\text{-mod})$.)