Families of algebraic structures

Joint works with Loïc Foissy, Xing Gao and Yuanyuan Zhang

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2. Family dendriform algebras
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4. Marcelo Aguiar's results (2020)
5. Main results

Family Rota-Baxter algebras

Ebrahimi-Fard–Gracia-Bondía–Patras/Li Guo, 2007

- Rota-Baxter algebra : (A, R) with

$$R(a)R(b) = R\big(R(a)b + aR(b) + \lambda ab\big).$$

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- Rota-Baxter algebra : (A, R) with
$$R(a)R(b) = R\big(R(a)b + aR(b) + \lambda ab\big).$$
- Family Rota-Baxter algebra : $\big(A, (R_\omega)_{\omega\in\Omega}\big)$ with Ω semigroup and
$$R_\alpha(a)R_\beta(b) = R_{\alpha\beta}\big(R_\alpha(a)b + aR_\beta(b) + \lambda ab\big).$$

- **First instance** (EGP), coming from the momentum scheme in renormalization (with $\Omega = \mathbb{N}$).

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- **Simplest example**, coming from minimal subtraction scheme (with $\Omega = \mathbb{Z}$) : Algebra of Laurent series $A = \mathbf{k}[z^{-1}, z]]$.

  Rota-Baxter family algebra of weight -1, with $\Omega = (\mathbb{Z}, +)$. Here $R_\omega =$ projection onto the subspace $A_{<\omega}$ generated by $\{z^k, k < \omega\}$ parallel to the supplementary subspace $A_{\geq\omega}$ generated by $\{z^k, k \geq \omega\}$.

- **Another interesting example in weight zero** : $\Omega = (\mathbb{R}, +)$, and let A be the $\mathbb{R}$-algebra of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. For any $\alpha \in \mathbb{R}$, define $R_\alpha : A \to A$ by

$$R_\alpha(f)(x) = e^{-\alpha a(x)} \int_0^x e^{\alpha a(t)} f(t)\, dt,$$

where a is a fixed nonzero element of A. Then $(R_\alpha)_{\alpha \in \Omega}$ is a Rota-Baxter family of weight zero.

Family dendriform algebras

X. Gao - Y. Y. Zhang

- Ω semigroup,
- $(D, \prec_\omega, \succ_\omega)_{\omega\in\Omega}$ such that for $x, y, z \in D$ and $\alpha, \beta \in \Omega$,

$$\begin{aligned}(x \prec_\alpha y) \prec_\beta z &= x \prec_{\alpha\beta} (y \prec_\beta z + y \succ_\alpha z),\\ (x \succ_\alpha y) \prec_\beta z &= x \succ_\alpha (y \prec_\beta z),\\ (x \prec_\beta y + x \succ_\alpha y) \succ_{\alpha\beta} z &= x \succ_\alpha (y \succ_\beta z).\end{aligned}$$

- The free Ω-family dendriform algebra generated by a set X can be described in terms of planar binary trees with internal nodes decorated by X and edges typed by Ω (X. Gao - DM - Y. Y. Zhang).

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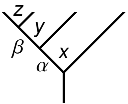
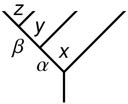
- The free Ω-family Rota-Baxter algebra of weight λ generated by a set X can be described in terms of planar rooted trees with internal node angles decorated by X and edges typed by Ω (X. Gao - DM - Y. Y. Zhang).



An X-decorated Ω-typed PBT



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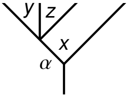
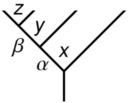


An angularly X-decorated Ω-typed Schröder tree



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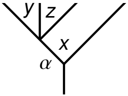

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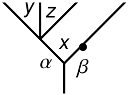

An angularly X-decorated Ω-typed planar rooted tree

- Any Ω-family Rota-Baxter of weight zero (resp. one) is an Ω-family dendriform (resp. tridendriform) algebra (family version a well-known result of M. Aguiar, resp. K. Ebrahimi-Fard).

- The natural embedding of planar binary trees into planar rooted trees is the embedding of the free Ω-family dendriform algebra into its enveloping Rota-Baxter algebra of weight zero.

- The natural embedding of Schröder trees into planar rooted trees is the embedding of the free Ω-family dendriform algebra into its enveloping Rota-Baxter algebra of weight one.

Family pre-Lie algebras
DM - Y. Y. Zhang

- Let Ω be a **commutative** semigroup.
- Left pre-Lie family algebra : $\big(A, (\rhd_\omega)_{\omega\in\Omega}\big)$ such that

$$x \rhd_\alpha (y \rhd_\beta z) - (x \rhd_\alpha y) \rhd_{\alpha\beta} z = y \rhd_\beta (x \rhd_\alpha z) - (y \rhd_\beta x) \rhd_{\beta\alpha} z, \quad (1)$$

where $x, y, z \in A$ and $\alpha, \beta \in \Omega$.

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where $x, y, z \in A$ and $\alpha, \beta \in \Omega$.

- If A is an Ω-family dendriform algebra with Ω commutative, it is an Ω-family pre-Lie algebra with

$$x \rhd_\omega y := x \succ_\omega y - y \prec_\omega x, \text{ for } \omega \in \Omega.$$

- Description of the free Ω-family pre-Lie algebra generated by X in terms of X-decorated Ω-typed non-planar rooted trees.

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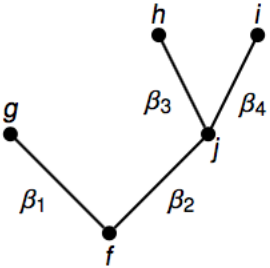

- These examples call for a general approach.
- **What is a family $\mathcal{P}$-algebra for an operad $\mathcal{P}$?**

Marcelo Aguiar's approach (2020)

- **Principle :** A is an Ω-family $\mathcal{P}$-algebra if and only if $A \otimes \mathbf{k}\Omega$ is a graded $\mathcal{P}$-algebra.

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- The family version of an operation of arity n is parametrized by Ω^n, where Ω is the semigroup at hand :

$$\alpha(a_1 \otimes \omega_1, \ldots, a_n \otimes \omega_n) = \alpha_{\omega_1,\ldots,\omega_n}(a_1, \ldots, a_n) \otimes \omega_1 \cdots \omega_n.$$

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- In particular, the natural family version of a binary operation necessitates two parameters in Ω.
- The semigroup Ω must be commutative unless the operad is **non-sigma**, e.g. Assoc, Dup or Dend.

- Example : family associative algebras.

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- The family associative algebra is commutative if moreover

$$x \cdot_{\alpha,\beta} y = y \cdot_{\beta,\alpha} x.$$

This immediately yields the commutativity of the semigroup Ω.

Our approach

(L. Foissy - DM - Y. Y. Zhang)

- **Same Principle :** A is an Ω-family $\mathcal{P}$-algebra if and only if $A \otimes \mathbf{k}\Omega$ is a graded $\mathcal{P}$-algebra.

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- **But** Ω need not be a semigroup : it is just a set a priori.
- the starting (linear) operad $\mathcal{P}$ together with its presentation

$$\mathcal{P} = \mathcal{M}_E/\mathcal{R} = \mathbf{k}.\mathbf{M}_E/\mathcal{R}$$

defines a Ⓟ -algebra structure on Ω, where Ⓟ is a set operad.

- The set operad $\textcircled{\mathcal{P}}$ depends on $\mathcal{P}$ and its presentation :

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- If $\mathcal{P}$ is quadratic and if the Koszul dual $\mathcal{P}^!$ of $\mathcal{P}$ is a set operad, we have

$$\textcircled{\mathcal{P}} = \mathcal{P}^!.$$

Upshot

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- The **color-mixing operad** $\widetilde{\widetilde{\mathcal{P}}}$ is a subquotient of the uniform Ω-colored operad $\mathcal{P}^{\Omega}$.
- If Ω is a $\textcircled{\mathcal{P}}$ -algebra, an Ω-family $\mathcal{P}$-algebra is an Ω-graded algebra over $\widetilde{\widetilde{\mathcal{P}}}$, for which the underlying Ω-graded object is **uniform**.

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- In the color-mixing operad, the color of the output is obtained by combining the n input colors by means of an operation of arity n in $\textcircled{\mathcal{P}}$.

Thank you for your attention !