Imperial College London

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Coherence and Regularity of Modelled Distributions

Higher structures emerging from renormalisation - ESI

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joint work with
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Motivation: Want a systematic understanding of how to glue together deterministic ODE/PDE arguments at larger time (and space) scales with local solution theory from Gubinelli's Branched Rough Paths/Hairer's Theory of Regularity Structures to get a priori bounds on solutions with the aim of getting global in time (and space) well-posedness.

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Recent work: Studying equations with strong damping terms and proving strong "coming down from infinity" type bounds:

The Φ^4 equation:

$$(\partial_t - \Delta)\varphi = -\varphi^3 + \xi ,$$

where formally $\varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and ξ is a rough random space-time process (like space-time white noise).

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Previous work on a priori bounds/global well-posedness:
For for Φ^4_3: [Mourrat, Weber], [Gubinelli, Hofmanová], [Albeverio, Kusuoka], [Moinat, Weber]
[C., Moinat, Weber] extends analysis to the entire subcritical/super-renormalizable regime.

Rough Differential Equations with strong damping:

$$\partial_t Y(t) = -|Y(t)|^{m-1} Y(t)dt + \sum_{\mu=1}^{d} \sigma_\mu(Y(t))dX^\mu(t)$$

where $m > 1$, $Y : \mathbb{R}_+ \to \mathbb{R}^k$ and $X = (X^\mu)_{\mu=1}^d : \mathbb{R}_+ \to \mathbb{R}^d$ is a rough (α-Hölder, for $\alpha > 0$ but arbitrarily small) driving noise, and $\sigma_\mu : \mathbb{R}^k \to \mathbb{R}^k$ are relatively nice vector fields.

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Previous work on this class of equations by [Riedel, Scheutzow]. Joint work in progress using approach from SPDE with Bonnefoi, Moinat, and Weber. Example of coming down from infinite bound: if σ_μ has a certain class of derivatives bounded and if Y solves the above equation on $[0,1]$ then

$$\sup_{t \in (0,1]} |Y(t)| \lesssim \max\{t^{-\frac{1}{m-1}}, \max_{h \in \mathring{\mathcal{T}}} [\mathbf{X} : h]^{\frac{1}{m\alpha|h|}}\}$$

uniformly over the initial data $Y(0)$.

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- Solution: Can construct new spaces of function/distributions that admit good local approximations (non-classical Taylor Expansions) in terms of these iterated integrals - and the equation can be reformulated on this space in a way that makes it locally well-posed there.

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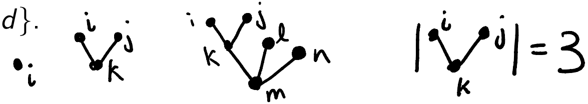
- Problem: Unlike classical spaces, there is no general relation between regularity of derivatives of different "order" in these new spaces, and since all these derivatives enter our estimate we cannot close our ODE argument.
- Solution - Algebra/Combinatorics: Can show that a solution to a Rough Differential Equation has additional relations between these generalised derivatives (*coherence* - [Bruned, Chevyrev, Preiß, Friz], [Bruned, Chevyrev, C., Hairer])

Quick overview of branched Rough Paths

- Let $N = \lfloor \frac{1}{\alpha} \rfloor$, we define $\mathring{\mathcal{T}}$ to consist of all rooted trees with no more than N nodes, each of which is decorated by the set $\{1, \ldots, d\}$.

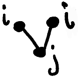


- Driving noise along with Iterated integrals encoded by map $\mathbf{X}_{s,t} : \mathcal{T} \to \mathbb{R}$, $\mathcal{T} \ni h \mapsto \langle \mathbf{X}_{s,t}, h \rangle$.

$\langle \mathbf{X}_{s,t},$  $\rangle = \int_s^t dX^i(r) = X^i(t) - X^i(s)$

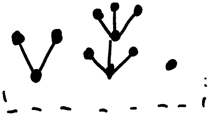
$\langle \mathbf{X}_{s,t},$  $\rangle = \int_s^t \int_s^{r_1} dX^i(r_2) dX^j(r_1)$

$\langle \mathbf{X}_{s,t},$  $\rangle = \int_s^t \int_s^{r_1} \int_s^{r_2} dX^i(r_3) dX^j(r_2) dX^k(r_1)$

$\langle \mathbf{X}_{s,t},$  $\rangle = \int_s^t \Big(\int_s^{r_1} dX^i(r_2) \Big)^2 dX^j(r_1)$

- We define $\mathcal{F}$ to be the set of forests of trees in $\mathring{\mathcal{T}}$. We also set $\mathcal{T} = \{\mathbb{1}\} \sqcup \mathring{\mathcal{T}}$.

forest f

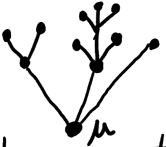


commutative product of trees

$f = h_1 h_2 h_3$

Empty forest $\mathbb{1}$

$f \longmapsto [f]_\mu =$



Can turn forest into a new tree

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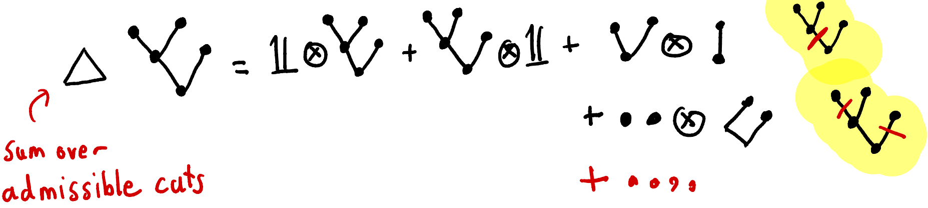
- $\Delta : \mathrm{Span}(\mathcal{F}) \to \mathrm{Span}(\mathcal{F}) \otimes \mathrm{Span}(\mathcal{F})$ is the Grassman-Larson co-product



$$\Delta(h_1 h_2 \cdots h_n) = (\Delta h_1)(\Delta h_2)\cdots(\Delta h_n)$$

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- We call $\mathbf{X}$ a branched rough path if for all $h \in \mathring{\mathcal{T}}$,

$$[\mathbf{X}, h] \stackrel{\text{def}}{=} \sup_{0<|s-t|<1} \left| \frac{\langle \mathbf{X}_{s,t}, h \rangle}{|s-t|^{\alpha|h|}} \right| < \infty$$

$$\text{and } \langle \mathbf{X}_{s,u} \otimes \mathbf{X}_{u,t}, \Delta h \rangle = \langle \mathbf{X}_{s,t}, h \rangle$$

Our solution is a Controlled Rough Path: $\mathbf{Y} : [0,1] \to \mathrm{Span}(\mathcal{T})$, that is a family of local expansions

$$\mathbf{Y}_s = \sum_{h \in \mathcal{T}} \langle h, \mathbf{Y}_s \rangle h = Y_s \mathbb{1} + \sum_{h \in \mathring{\mathcal{T}}} \langle h, \mathbf{Y}_s \rangle h \,.$$

"generalised h-derivative of Y"

Encodes Taylor expansion of Y

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with the following regularity property,

Taylor expansion of $\langle h, Y_s \rangle$ evaluated at t

$$R^h_{s,t} = \langle h, \mathbf{Y}_t \rangle - \langle \mathbf{X}_{s,t} \otimes h, \Delta \mathbf{Y}_s \rangle, \text{ want } |R^h_{s,t}| \lesssim |s - t|^{(N - |h|)\alpha}$$

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Space of controlled rough paths is a vector space where we can solve the equation as a fixed point problem, but in this space there are no clear relations between $R^h_{s,t}$ for different h.

Can show that solutions to equation solve the non-linear coherence constraint $\langle h, \mathbf{Y}_t \rangle = \Upsilon[h](Y_s)$ for some functions $\Upsilon[h](\cdot)$ built inductively in h - if $h = [h_1 \cdots h_n]_\mu$ then

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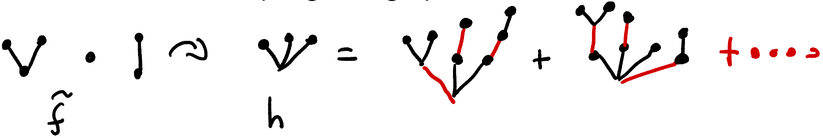

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Combining this with the adjoint relation $\langle \tilde{f} \otimes h, \Delta \bar{h} \rangle \approx \langle \tilde{f} \curvearrowright h, \bar{h} \rangle$ allows us to show that

$$|R^h_{s,t}| \lesssim |\nabla \Upsilon[h](Y_s)| \cdot |R^{\mathbb{1}}_{s,t}| + \text{ higher order terms}$$

which allows us to close our ODE argument!

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