Renormalization of Quasisymmetric Functions

Li GUO
(joint work with Jean-Yves Thibon, Houyi Yu and Bin Zhang)

Rutgers University-Newark

Abstract

- Canonical bases of quasisymmetric functions, in particular the monomial quasisymmetric functions, are nested sum formal power series generated by compositions, that is, by vectors of positive integers.
- Motivated by a suggestion of Rota that Rota-Baxter algebras "represent the ultimate and most natural generalization of the algebra of symmetric functions", we would like to extend this generation of quasisymmetric functions to weak compositions (vectors of nonnegative integers), called weak composition quasisymmetric functions. But this leads to divergence of formal power series.
- To deal with the divergence, a naive regularization realizes the weak quasisymmetric functions as formal power series with semigroup exponents (L. G., J.-Y. Thibon and H. Yu).
- Then a more faithful renormalization of weak composition quasisymmetric functions is taken, following the Connes-Kreimer approach to renormalization applying the algebraic Birkhoff factorization (L. G., H. Yu and B. Zhang).

Rota-Baxter algebras

- Fix λ in a base ring $\mathbf{k}$. A Rota-Baxter operator of weight λ on a $\mathbf{k}$-algebra R is a linear map $P : R \to R$ such that

$$P(x)P(y) = P(xP(y)) + P(P(x)y) + \lambda P(xy), \ \forall\, x, y \in R.$$

- **Examples.** Integration: $R = \mathbf{Cont}(\mathbb{R})$ (ring of continuous functions on $\mathbb{R}$).

$$P : R \to R, P[f](x) := \int_0^x f(t)dt$$

defines a Rota-Baxter operator of weight 0:

$$F(x) := P[f](x) = \int_0^x f(t)\,dt, \quad G(x) := P[g](x) = \int_0^x g(t)\,dt.$$

Then the integration by parts formula states

$$\int_0^x F(t)G'(t)dt = F(x)G(x) - \int_0^x F'(t)G(t)dt$$

$$P[P[f]g](x) = P[f](x)P[g](x) - P[fP[g]](x),$$

$$P[f]P[g] = P[fP[g]] + P[P[f]g].$$

- Partial sum: Let A be a commutative algebra and $\mathfrak{A} := A^{\mathbb{P}}$ be the algebra of sequences in A, with componentwise operations. The partial sum operator

$$P : \mathfrak{A} \to \mathfrak{A}, (a_1, a_2, a_3, \cdots) \mapsto (0, a_1, a_1 + a_2, \cdots)$$

is a Rota-Baxter operator of weight 1:

$$P[f](x)\, P[g](x) = P[P[f]g](x) + P[fP[g]](x) + P[fg](x).$$

- Laurent series: Let $R = \mathbb{C}[z^{-1}, z]]$ be the ring of Laurent series $\sum_{n=-k}^{\infty} a_n z^n,\ k \geq 0$. Define the pole part projection

$$P(\sum_{n=-k}^{\infty} a_n z^n) = \sum_{n=-k}^{-1} a_n z^n.$$

Then P is a Rota-Baxter operator of weight -1.

Rota's standard Rota-Baxter algebra

- The first construction of free commutative Rota–Baxter algebras was given by Rota, called the standard Rota–Baxter algebra.
- Let X be a given set. Let $t_n^{(x)}, n \geq 1, x \in X$, be distinct symbols.
- Denote
$$\overline{X} = \bigcup_{x \in X} \left\{ t_n^{(x)} \mid n \geq 1 \right\}$$
and let $\mathfrak{A} := \mathfrak{A}(X) := \mathbf{k}[\overline{X}]^{\mathbb{P}}$ denote the algebra of sequences with entries in the polynomial algebra $\mathbf{k}[\overline{X}]$.
- Define
$$P_X^r : \mathfrak{A}(X) \to \mathfrak{A}(X), \quad (a_1, a_2, a_3, \cdots) \mapsto (0, a_1, a_1+a_2, a_1+a_2+a_3, \cdots)$$
to be the partial sum Rota–Baxter operator of weight 1.
- The standard Rota–Baxter algebra on X is defined to be the Rota–Baxter subalgebra $\mathfrak{S}(X)$ of $\mathfrak{A}(X)$ generated by the sequences $t^{(x)} := (t_1^{(x)}, \cdots, t_n^{(x)}, \cdots),\ x \in X$.
- Theorem (Rota, 1969) $(\mathfrak{S}(X), P_X^r)$ is the free commutative Rota–Baxter algebra on X.

Spitzer's Identity

- Spitzer's Identity. Let (R, P) be a unitary commutative Rota-Baxter $\mathbb{Q}$-algebra of weight 1. Then for $a \in R$, we have

$$\exp\left(P(\log(1+\lambda a t))\right) = \sum_{n=0}^{\infty} t^n \underbrace{P(P(P(\cdots(P(a)a)a)a))}_{n\text{-iterations}}$$

in the ring of power series $R[[t]]$ (still a Rota-Baxter algebra).

- With the notation $P_a(c) := P(ac)$, this becomes

$$\exp\left(-\sum_{k=1}^{\infty} \frac{(-t)^k P(a^k)}{k}\right) = \sum_{n=0}^{\infty} t^n P_a^n(1).$$

- Take $X = \{x\}, x_n := t_n^{(x)}$, $R = \mathbf{k}[x_n, n \geq 1]^{\mathbb{P}}$, P the partial sum operator and $a := (x_1, \cdots, x_n, \cdots)$.

Rota-Baxter algebras and Symmetric functions

- Then
$$P_a^n(1) = (0, e_n(x_1), e_n(x_1, x_2), e_n(x_1, x_2, x_3), \cdots)$$
where $e_n(x_1, \cdots, x_m) = \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq m} x_{i_1} x_{i_2} \cdots x_{i_n}$ is the elementary symmetric function of degree n in the variables $x_1, \cdots, x_m$ with the convention that $e_0(x_1, \cdots, x_m) = 1$ and $e_n(x_1, \cdots, x_m) = 0$ if $m < n$.
- Also by definition,
$$P(a^k) = (0, p_k(x_1), p_k(x_1, x_2), p_k(x_1, x_2, x_3), \cdots),$$
where $p_k(x_1, \cdots, x_m) = x_1^k + x_2^k + \cdots + x_m^k$ is the power sum symmetric function of degree k in the variables $x_1, \cdots, x_m$.
- So Spitzer's Identity becomes Waring's formula:
$$\exp\left(-\sum_{k=1}^{\infty} (-1)^k t^k p_k(x_1, x_2, \cdots, x_m)/k\right)$$
$$= \sum_{n=0}^{\infty} e_n(x_1, x_2, \cdots, x_m) t^n \text{ for all } \ m \geq 1.$$

Rota's Conjecture/Question

- With this discovery, Rota conjectured in 1995:

  a very close relationship exists between the Baxter identity and the algebra of symmetric functions.

- and concluded

  The theory of symmetric functions of vector arguments (or Gessel functions) fits nicely with Baxter operators; in fact, identities for such functions easily translate into identities for Baxter operators. $\cdots$ In short: Baxter algebras represent the ultimate and most natural generalization of the algebra of symmetric functions.

- Rota Program: Study generalizations of symmetric functions in the context of Rota-Baxter algebras.
- As it turns out, Rota-Baxter algebras are closely relates to quasi-symmetric functions.

Symmetric functions and generalizations

- Sym: Symmetric functions
- QSym: Quasi-symmetric functions (Gessel, Stanley, 1984)
- NSym: Noncommutative symmetric functions (I. Gelfand, Thibon, ..., 1995)
- SSym: Symmetric functions of permutations (Malvenuto, Reutenauer, 1995)
-

$$\begin{tikzcd}
\mathrm{SSym} \arrow[d, two heads] & \mathrm{NSym} \arrow[l, tail] \arrow[d, two heads] & \\
\mathrm{QSym} & \mathrm{Sym} \arrow[l, tail] \arrow[r, dashed, leftrightarrow] & \text{Spitzer's identity}
\end{tikzcd}$$

- Combinatorial Hopf algebras.

Free commutative Rota-Baxter algebras

- After Rota's construction, a second construction of free commutative Rota-Baxter algebras was given by Cartier in terms of what was later called stuffles (joint shuffle product, etc).
- A third construction was given by L. G. and W. Keigher in terms of mixable shuffle product (overlapping shuffle product or motivic shuffle product) which turned out to be recursively defined by the quasi-shuffle product.
- Let A be a commutative algebra. On the underlying space of the tensor algebra $T(A) := \bigoplus_{n\geq 0} A^{\otimes n}$, define the mixable shuffle product (recursively the quasi-shuffle product). Let $Ш^+(A) = QS(A)$ denote the resulting algebra.
- **Theorem** The tensor product algebra $Ш(A) = A \otimes Ш^+(A)$, with the shift operator $P(\mathfrak{a}) := 1 \otimes \mathfrak{a}$, is the free commutative Rota-Baxter algebra on A.
- Let $A = \mathbf{k}\,1 \oplus A^+$. The restriction to $Ш(A)^0 := \oplus_{k\geq 0}(A^{\otimes k} \otimes A^+)$ is the free commutative nonunitary Rota-Baxter algebra on A.

Weak compositions

- When $A = \mathbf{k}[x]$, we have $A^{\otimes k} = \mathbf{k}\{x^{a_1} \otimes \cdots \otimes x^{a_k} \mid a_i \geq 0, 1 \leq i \leq k\}$.
- Hence a linear basis of $Ш^+(A) = T(A) = \oplus_{k\geq 0} A^{\otimes k}$ is

$$\{x^\alpha := x^{a_1} \otimes \cdots \otimes x^{a_k} \mid \alpha = (a_1, \cdots, a_k) \in WC\},$$

parameterized by the set of weak compositions

$$WC := \{\alpha := (a_1, \cdots, a_k) \mid a_i \geq 0, 1 \leq k, k \geq 0\}.$$

- A linear basis of $Ш^+(x\mathbf{k}[x])$ is

$$C := \{x^\alpha := x^{a_1} \otimes \cdots \otimes x^{a_k} \mid \alpha = (a_1, \cdots, a_k) \in C\},$$

parameterized by the set of compositions

$$\{\alpha := (a_1, \cdots, a_k) \mid a_i \geq 0, 1 \leq k, k \geq 0\}.$$

Previous progress on the Rota Program

- The quasi-shuffle algebra on $A := x\mathbb{Q}[x]$ is identified with the algebra $QS(A)$ of quasi-symmetric functions, spanned by **monomial quasi-symmetric functions**

$$M_{(a_1,\cdots,a_k)} := \sum_{1\leq i_1<\cdots<i_k} x_{i_1}^{a_1}\cdots x_{i_k}^{a_k} \in \mathbb{Q}[x_1,\cdots,x_n,\cdots],$$

  for compositions $\alpha := (a_1,\cdots,a_k), a_i \geq 1$.
- At the same time, $QS(x\mathbb{Q}[x])$ is the main part of the free nonunitary Rota-Baxter algebra $Ш(x\mathbb{Q}[x])^0$. Thus to pursue the Rota Program, one should identify the whole commutative Rota-Baxter algebra $Ш(\mathbb{Q}[x])$ with a suitable generalization of quasi-symmetric functions.
- We achieved this in two steps, first for nonunitary Rota-Baxter algebras, then for unitary Rota-Baxter algebras.

Rota-Baxter algebra and symmetric functions

- $$\begin{tikzcd}
 & \text{free nonunital RBA} \arrow[r, hook] & \text{free unital RBA} \\
QS(x\mathbf{k}[x]) \arrow[r, hook] \arrow[d, leftrightarrow] & QS(\mathbf{k}[x])^0 \arrow[r, hook] \arrow[u, hook] \arrow[d, leftrightarrow] & QS(\mathbf{k}[x]) \arrow[u, hook] \arrow[d, leftrightarrow] \\
\mathrm{QSym} \arrow[r, hook] \arrow[d, leftrightarrow] & \color{blue}{\mathrm{LWQSym}} \arrow[r, hook] \arrow[d, leftrightarrow] & \color{blue}{\mathrm{WCQSym}} \arrow[d, leftrightarrow] \\
\text{Compositions} \arrow[r, hook] & \text{Left weak compositions} \arrow[r, hook] & \text{Weak compositions} \arrow[u, bend right=60, "\text{renormalization}"']
\end{tikzcd}$$

The nonunitary case

- A weak composition $\alpha := (a_1, \cdots, a_k) \in \mathbb{Z}^k_{\geq 0}$ is called a left weak composition if $a_k > 0$.
- For a left weak comp composition α, define a monomial quasi-symmetric function

$$M_\alpha := \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \in \mathbb{Q}[[x_1, \cdots, x_n, \cdots]].$$

- Let LWCQSym be the subalgebra of $\mathbb{Q}[[x_1, \cdots, x_n, \cdots]]$ spanned by these M_α.
- Theorem (L. G., H. Yu, J. Zhao, 2017) $\mathbb{Q}[x]LWCQSym$ is the free commutative nonunitary Rota-Baxter algebra on x.

The unitary case by semigroup exponents

- In order to apply this approach to free commutative unitary Rota-Baxter algebras, we need to consider weak compositions, not just left weak compositions.
- For a weak composition $\alpha := (a_1, \cdots, a_k), a_i \geq 0$, the expression M_α might not make sense.
- Example: $\alpha = (0)$ gives $M_\alpha = \sum_{n\geq 1} x_n^0 = \sum_{n\geq 1} 1$. This is not defined.
- To fix this problem, we "modify" the rule $x^0 = 1$ by considering formal power series and quasi-symmetric functions with semigroup exponents.

Power series with semigroup exponents

- In a formal power series, a monomial $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$ can be regarded as the locus of the map from $X := \{x_n \mid n \geq 1\}$ to $\mathbb{N}$ sending x_{i_j} to α_j, $1 \leq j \leq k$, and everything else in X to zero.
- Our generalization of the formal power series algebra is simply to replace $\mathbb{N}$ by a suitable additive monoid with a zero element.
- Let B be a commutative additive monoid with zero such that $B \backslash \{0\}$ is a subsemigroup. Let X be a set. The set of ***B*-valued maps** is defined to be $B^X := \{f : X \to B \mid \mathcal{S}(f) \text{ is finite }\}$, where $\mathcal{S}(f) := \{x \in X \mid f(x) \neq 0\}$ denotes the support of f.
- The addition on B equips B^X with an additive monoid by

$$(f+g)(x) := f(x) + g(x) \quad \text{for all } f, g \in B^X \text{ and } x \in X.$$

- As with formal power series, we identify $f \in B^X$ with its locus $\{(x, f(x)) \mid x \in \mathcal{S}(f)\}$ expressed in the form of a formal product

$$X^f := \prod_{x \in X} x^{f(x)} = \prod_{x \in \mathcal{S}(f)} x^{f(x)},$$

called a ***B*-exponent monomial**, with the convention $x^0 = 1$.

- By abuse of notation, the addition on B^X becomes

$$X^f X^g = X^{f+g} \quad \text{for all } f, g \in B^X.$$

- We then form the semigroup algebra $\mathbf{k}[X]_B := \mathbf{k}B^X$ consisting of linear combinations of B^X, called the algebra of ***B*-exponent polynomials**.
- Similarly, we can define the free $\mathbf{k}$-module $\mathbf{k}[[X]]_B$ consisting of possibly infinite linear combinations of B^X, called ***B*-exponent formal power series**.
- If B is **additively finite** in the sense that for any $a \in B$ there are finite number of pairs $(b, c) \in B^2$ such that $b + c = a$, then the multiplication above extends by bilinearity to a multiplication on $\mathbf{k}[[X]]_B$, making it into a $\mathbf{k}$-algebra, called the algebra of formal power series with semigroup exponents.

Back to weak compositions

- For example, taking B to be the additive monoid $\mathbb{N}$ of nonnegative integers, then B^X is simply the free monoid generated by X and $\mathbf{k}[X]_B$ is the free commutative algebra $\mathbf{k}[X]$.
- Now taking $B := \tilde{\mathbb{N}} := \mathbb{N} \cup \{\varepsilon\}$, with $0+\varepsilon=\varepsilon, n+\varepsilon=n, n \in \tilde{\mathbb{N}} \backslash \{0\}$, the expressions

$$M_\alpha := \sum_{1 \leq i_1 < \cdots < i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \in \mathbb{Q}[[x_1, \cdots, x_n, \cdots]]_{\tilde{\mathbb{N}}}$$

  are well-defined for $\alpha \in WC(\cong C(\tilde{\mathbb{N}})$ via $0 \leftrightarrow \varepsilon)$. The resulting space WCQSym is a Hopf algebra which has QSym as both a sub and quotient Hopf algebra.
- Theorem (G.-Yu-Thibon, 2019) $\mathbb{Q}[x]$ *WCQSym* is isomorphic to the free commutative unitary Rota-Baxter algebra $Ш(x)$.

The unitary case by renormalization

- The treatment of WCQSym by ε is a naive regularization/renormalization of weak composition quasisymmetric functions. Also the resulting expressions are not formal power series.
- We next give a renormalization by applying the algebraic Birkhoff factorization in the Connes-Kreimer approach.
- Algebraic Birkhoff Factorization. Let H be a connected filtered Hopf algebra, (R, P) an itempotent commutative Rota-Baxter algebra of weight -1 and $\phi : H \to R$ an algebra homomorphism. There is a unique factorization
$$\phi = \phi_-^{-1} \star \phi_+$$
where
$$\begin{aligned} \phi_- &: H \to \mathbf{k} + P(R) \ \text{(counter term)} \\ \phi_+ &: H \to \mathbf{k} + (\mathrm{id} - P)(R) \ \text{(renormalization)} \end{aligned}$$
are algebra homomorphisms.

Directional regularizations of weak qsym

- Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k)$ be a weak composition and $\beta = (\beta_1, \beta_2, \cdots, \beta_k)$ a composition. The matrix

$$\left[{\alpha \atop \beta} \right] := \left[{\alpha_1, \alpha_2, \cdots, \alpha_k \atop \beta_1, \beta_2, \cdots, \beta_k} \right].$$

  is called a weak bicomposition.
- Let H_{wb} denote the space spanned by the weak bicompostions. Equipped with the quasi-shuffle product and the deconcatenation coproduct, H_{wb} is a connected filtered Hopf algebra.
- Let LWQSym be the algebra of left weak quasisymmetric functions, a a variable and $R := LWQSym[a][z^{-1}, z]]$ the Rota-Baxter algebra of Laurent series with coefficients in $LWQSym[a]$.
- The assignment

$$\phi\left(\left[{\alpha \atop \beta} \right] \right) := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k} e^{(i_1+a)\beta_1 z} e^{(i_2+a)\beta_2 z} \cdots e^{(i_k+a)\beta_k z},$$

  defines an algebra homomorphism

$$\phi : H_{wb} \longrightarrow LWQSym[a][z^{-1}, z]].$$

Renormalized weak quasisymmetric functions

- Applying the Algebraic Birkhoff Factorization, we obtain

$$\phi_+ : H_{wb} \longrightarrow LWQSym[a][[z]] \subset \mathbf{k}[[X]][a][z],$$

where $X = \{x_n\}_{n\geq 1}$, and with $z \mapsto 0$,

$$Z : H_{wb} \longrightarrow LWQSym[a] \subset \mathbf{k}[[X]][a].$$

- The values depend on the composition β in the second row vector. Taking permutation invariants of the zero tail in α, we obtain a value $M(\alpha)$ independent of the choice of β.
- Thus we obtain an algebra homorphism (in fact an isomorphism)

$$M : QS(\mathbf{k}[x])(= Ш^+(\mathbf{k}[x]) = \mathbf{k}WC \to LWQSym[a] \subset \mathbf{k}[[X]][a]$$

such that $M(\alpha)$ coincides with the monomial quasisymmetric function M_α when α is a left weak composition. $M((0)) = -a - 1/2$.

- Thus M gives a one-point renormalization of weak composition quasisymmetric functions, similar to the one-point regularzation $\zeta(1) = a$ of Ihara-Kaneko-Zagiar for MZVs.
- Since the weak quasisymmetric functions $WCQSym \cong \mathbf{k}WC$, the two renormalizations of weak composition quasisymmetric functions are isomorphic, giving a power series realization of $WCQSym$.
- For free, this also gives a power series realization of the free commutative Rota-Baxter algebra $Ш(x) \cong \mathbf{k}[x] \otimes Ш^{+}(k[x])$. ($\sim$ polynomial realizations of combinatorial Hopf algebras Foissy, Novelli, Thibon, Maurice).
- Observation: The (free) Rota-Baxter algebra associated to quasisymmetric functions (domain of ϕ) is identified with (the coefficients of) the Rota-Baxter algebra in renormalization (range of ϕ).

References

- G.-C. Rota, Baxter operators, an introduction, In: "Gian-Carlo Rota on Combinatorics, Introductory Papers and Commentaries", Birkhäuser, Boston, 1995.
- L. Guo, H. Yu and J. Zhao, Rota-Baxter algebras and left weak composition quasisymmetric functions, *Ramanujan Jour* **44** (2017), 567-596, arXiv:1601.06030.
- L. Guo, J.-Y. Thibon and H. Yu, Weak quasi-symmetric functions, Rota-Baxter algebras and Hopf algebras, *Adv Math* **344** (2019), 1-34, arXiv:1702.08011.
- Y. Li, On weak peak quasisymmetric functions, *J. Combin. Theory, Ser. A* **158** (2018), 449-491.
- L. Guo, J.-Y. Thibon and H. Yu, The Hopf algebra of signed permutations, of weak quasi-symmetric functions and of Malvenuto-Reutenauer, *Adv Math*, online, arXiv:1912.12721.
- L. Guo, H. Yu and B. Zhang, Renormalization of quasisymmetric functions, in preparation.

- **Thank You!**