Blobbed topological recursion of the quartic analogue of the Kontsevich model

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität Münster

arXiv:2008.12201 with Johannes Branahl & Alex Hock
building on previous collaborations with
Harald Grosse, Erik Panzer & Jörg Schürmann

Introduction

This project started in 1998 as an attempt to understand quantum field theories on noncommutative geometries.

- Many nice results were obtained on renormalisation, β-function, extension to tensor models.
- We aimed at an analytic construction of these models (of course combined with combinatorics).

During the last two years we understood: the constructive path is misleading!

- It is about miracles in algebra, about amazing solutions of non-linear problems.
- It is another example for topological recursion, thus links to structures in complex algebraic geometry.

We just entered a new world and admire the structures seen so far. But much more remains to explore.

Interlude: Topological recursion [Eynard-Orantin 07]

Universal structure that governs, e.g.

one- and two-matrix models, Kontsevich model, Weil-Petersson volumes, Hurwitz numbers, Gromov-Witten numbers, ...

Starting from a spectral curve consisting of

- a branched covering $x : \Sigma \to \Sigma_0$ of Riemann surfaces,
- meromorphic differentials $\omega_{0,1}$ on Σ and $\omega_{0,2}$ on $\Sigma \times \Sigma$,

recursively construct family $\omega_{g,n}$ of meromorphic differentials on Σ^n, with poles at zeros of dx (ramification points), by

$$\omega_{g,n}(z_1,...,z_n) = \sum_a \operatorname*{Res}_{q\to a} K(z_1,q,\sigma_a(q))dz\Big(\omega_{g-1,n+1}(q,\sigma_a(q),z_2,...,z_n) + \sum_{\substack{g_1+g_2=g \\ I_1\uplus I_2=\{z_2,\dots,z_n\}}}' \omega_{g_1,|I_1|+1}(q,I_1)\omega_{g_2,|I_2|+1}(\sigma_a(q),I_2)\Big)$$

[sum over ramification points a of x; local involution $x(q) = x(\sigma_a(q))$ near a; recursion kernel $K(z_1,z_2,z_3) = \dfrac{\frac{1}{2}\int_{z'=z_3}^{z_2} \omega_{0,2}(z_1,z')}{\omega_{0,1}(z_2)-\omega_{0,1}(z_3)}$]

Free Euclidean fields on noncommutative geometries

Let X_N be the real vector space of self-adjoint $N \times N$-matrices, and $(E_1, \dots, E_N)$ be (increasing) positive real numbers.

Theorem [Bochner 1933, Schur 1911]

For any inner product $\langle\ ,\ \rangle$ on X_N there exists a unique probability measure $d\mu_0$ on the dual space X_N' with

$$\exp\Big(-\frac{1}{2}\langle M, M\rangle\Big) = \int_{X_N'} d\mu_0(\Phi)\, e^{\mathrm{i}\Phi(M)} \quad \forall M{=}(M_{kl}) \in X\ .$$

Choose $\langle M, M'\rangle_E = \frac{1}{N}\sum_{k,l=1}^{N} \frac{M_{kl}M'_{lk}}{E_k + E_l}$ and corresponding $d\mu_{E,0}$

- Defines the free Euclidean scalar field on N-dimensional approximation of a noncommutative geometry.
- $(E_1, \dots, E_N)$ is truncated spectrum of the Laplacian.
- All moments can be described explicitly.

Two deformations

3. The Kontsevich model $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{3}\mathrm{Tr}(\Phi^3)} d\mu_{E,0}(\Phi)}$
   - Computes intersection numbers of tautological characteristic classes on the moduli space $\overline{\mathcal{M}}_{g,n}$ of stable complex curves.
   - It is integrable via a relation (suggested by Witten) to the KdV hierarchy. Its moments satisfy topological recursion.

4. A quartic analogue $d\mu_{E,\lambda}(\Phi) = \dfrac{e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}{\int_{X'_N} e^{-\frac{\lambda N}{4}\mathrm{Tr}(\Phi^4)} d\mu_{E,0}(\Phi)}$
   - Although perturbatively far apart, we find very similar algebraic geometrical structures. Our solutions are exact in λ.

Overview

Aim: Say something about moments $\int d\mu_{E,\lambda}(\Phi)\ \Phi_{k_1 l_1} \cdots \Phi_{k_n l_n}$

- As in all matrix models, best approach is $1/N$-expansion.
- After 10 years of work, we found in [Grosse-Hock-W 19] the $1/N$-leading contribution to the 2-point function as exact function of $\lambda, E_1, \ldots, E_N$.
- This involved the solution of a non-linear problem (identified in [Grosse-W 09], main step is [Panzer-W 18]).

Exact solutions of non-linear problems are only possibly in presence of a profound algebraic structure

- By analogy with the Kontsevich model we expected that this structure consists in a topological recursion which governs the entirerty of moments of $d\mu_{E,\lambda}$.
- Recent discovery [Alex Hock]: The structure *is* present, but at an unexpected place!

The objects for blobbed topological recursion

Recall that $d\mu_{E,\lambda}$ depends on given family $E_1, \dots, E_N$. Introduce

$$\sum_{g=0}^{\infty} N^{2-2g-n} \Omega^{(g)}_{a_1,\dots,a_n} := \frac{\partial^{n-1}\Big(N \sum_{k=1}^{N} \int d\mu_{E,\lambda}(\Phi)\Phi_{a_1k}\Phi_{ka_1}\Big)}{\partial E_{a_1} \cdots \partial E_{a_{n-1}}} + \frac{\delta_{n,2}}{(E_{a_1}-E_{a_2})^2}$$

- As substitute for unavailable direct E-derivatives we derive Dyson-Schwinger equations for the Ω.
- They extend naturally (not uniquely) to (very complicated) equations for meromorphic functions in several complex variables.
- Their solutions are strikingly simple and structured: The Ω extend to meromorphic forms $\omega_{g,n}$ which exactly follow the rules of blobbed topological recursion [Borot-Shadrin 15].

Equations of motion

Fourier transform $\mathcal{Z}(M) := \int_{X'_N} d\mu_{E,\lambda}(\Phi)\, e^{\mathrm{i}\Phi(M)}$ satisfies

1. $$-N(E_p - E_q)\sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}} = \sum_{k=1}^{N}\Big(M_{kp}\frac{\partial \mathcal{Z}(M)}{\partial M_{kq}} - M_{qk}\frac{\partial \mathcal{Z}(M)}{\partial M_{pk}}\Big)$$
2. $$\frac{1}{N}\frac{\partial \mathcal{Z}(M)}{\partial E_p} = \sum_{k=1}^{N} \frac{\partial^2 \mathcal{Z}(M)}{\partial M_{pk}\partial M_{kp}} + \mathcal{Z}(M)\int d\mu_{E,\lambda}(\Phi)\frac{1}{N}\sum_{k=1}^{N}\Phi_{pk}\Phi_{kp}$$

They allow to express $\sum_{k=1}^{N}\frac{\mathcal{Z}(M)}{\partial M_{pk}\partial M_{kq}}$ in Dyson-Schwinger equations by fewer derivatives, i.e. of same or lower order.

Eq. 1 (essentially due [Disertori-Gurau-Magnen-Rivasseau 06]) can be used for $p \neq q$, whereas $p = q$ requires 2.

Meromorphic continuation

These Dyson-Schwinger equations complexify to equations for meromorphic functions in several complex variables in which we can admit multiplicities $(E_1,\dots,E_N)=(\underbrace{e_1,\dots,e_1}_{r_1},\dots,\underbrace{e_d,\dots,e_d}_{r_d})$

Example: The two-point function

For $p\neq q$, set $\sum_{g=0}^{\infty} N^{-2g} G^{(g)}_{|pq|} = N\int d\mu_{E,\lambda}(\Phi)\;\Phi_{pq}\Phi_{qp}$.

Then $G^{(g)}_{|pq|} = G^{(g)}(e_p,e_q)$ with initial equation [Grosse-W 09]

$$\boxed{\begin{aligned}&(\xi+\eta)G^{(0)}(\zeta,\eta)\\ &=1-\lambda G^{(0)}(\zeta,\eta)\Omega_1^{(0)}(\zeta)+\frac{1}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(e_k,\eta)-G^{(0)}(\zeta,\eta)}{e_k-\zeta}\,,\\ &\Omega_1^{(0)}(\zeta)=\frac{1}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)\end{aligned}}$$

Main achievement: We can guess $\Omega_1^{(0)}$!

$$\Big(\eta+\zeta+\frac{\lambda}{N}\sum_{k=1}^{d} r_k G^{(0)}(\zeta,e_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{e_k-\zeta}\Big)G^{(0)}(\zeta,\eta)=1+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\frac{G^{(0)}(e_k,\eta)}{e_k-\zeta}$$

Ansatz [Schürmann-W 19, building on Grosse-Hock-W 19]

Assume there is a branched cover $R:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$ with

1. R has degree $d+1$ and maps (for $\lambda>0$) some domain $\mathcal{U}$ bijectively to neighbourhood of $e_1,\dots,e_d$
2. $\zeta=R(z)$, $\eta=R(w)$, $e_k=R(\varepsilon_k)$, $G^{(0)}(\zeta,\eta)=\mathcal{G}^{(0)}(z,w)$
3. $$\boxed{R(z)+\frac{\lambda}{N}\sum_{k=1}^{d} r_k\mathcal{G}^{(0)}(z,\varepsilon_k)+\frac{\lambda}{N}\sum_{k=1}^{d}\frac{r_k}{R(\varepsilon_k)-R(z)}=-R(-z)}$$

Gives $(R(w)-R(-z))\mathcal{G}^{(0)}(z,w)=1+\dfrac{\lambda}{N}\displaystyle\sum_{k=1}^{d} r_k\frac{\mathcal{G}^{(0)}(\varepsilon_k,w)}{R(\varepsilon_k)-R(z)}$

Rationality

With techniques for Lagrange interpolation polynomials [Schechter 59] it is *easy* to establish:

Theorem [Schürmann-W 19]

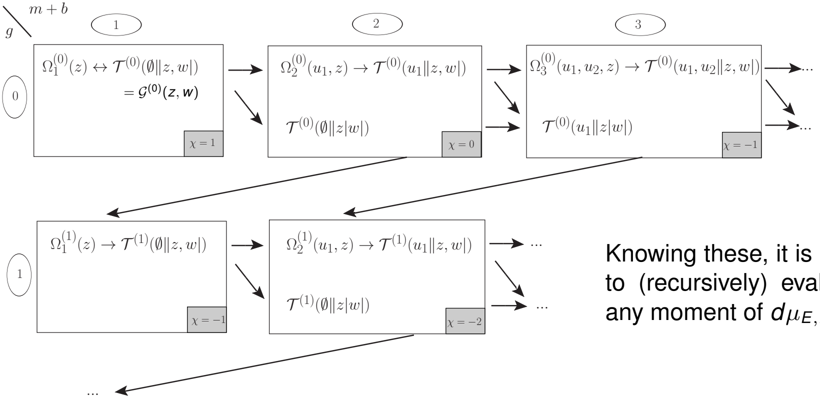
- $R(z) = z - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{\varrho_k}{z+\varepsilon_k}$ where $e_k = R(\varepsilon_k)$, $r_k = R'(\varepsilon_k)\varrho_k$
- $$\mathcal{G}^{(0)}(z,w) = \frac{1 - \frac{\lambda}{N}\sum_{k=1}^{d} \frac{r_k \prod_{j=1}^{d} \frac{R(w)-R(-\widehat{\varepsilon}_k^{\,j})}{R(w)-R(\varepsilon_j)}}{(R(z)-R(\varepsilon_k))(R(\varepsilon_k)-R(-w))}}{R(w)-R(-z)}$$
  where $u \in \{z, \hat{z}^1, \dots, \hat{z}^d\}$ are all solutions of $R(u) = R(z)$.
- $\mathcal{G}^{(0)}(z,w) = \mathcal{G}^{(0)}(w,z)$
- The ansatz 3 is identically satisfied!

Thus, planar 2-point function solved by the composition of a rational function $\mathcal{G}^{(0)}$ with inverse of another rational function R.

Solution procedure

Three types of functions involved:

- $\Omega^{(g)}_m(u_1, ..., u_m)$ objects of BTR, most difficult to compute
- $\mathcal{T}^{(g)}(u_1, ..., u_m\|z, w|)$ auxiliary functions, easy to compute
- $\mathcal{T}^{(g)}(u_1, \dots, u_m\|z|w|)$ auxiliary functions, easy to compute



Knowing these, it is easy to (recursively) evaluate any moment of $d\mu_{E,\lambda}$

$\Omega_2^{(0)}(u,z)$

$$\Omega_2^{(0)}(u,z)R'(z)\mathfrak{G}_0(z) - \frac{\lambda}{N^2}\sum_{n,k=1}^{d}\frac{r_k r_n \mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)}{(R(\varepsilon_k)-R(z))(R(\varepsilon_n)-R(-z))}$$
$$= -\frac{\partial}{\partial R(u)}\big(\mathcal{G}^{(0)}(u,z)+\mathcal{G}^{(0)}(u,-z)\big)$$

where $\mathfrak{G}_0(z) = \mathrm{Res}_{w\to -z}\,\mathcal{G}^{(0)}(z,w)$.

- Seems to need $\mathcal{T}^{(0)}(u\|\varepsilon_k,\varepsilon_n|)$ which itself needs $\Omega_2^{(0)}$.
- But poles separate by partial fraction decomposition

$$\mathcal{G}^{(0)}(z,u) = \frac{\mathfrak{G}_0(z)}{u+z} + \frac{\lambda^2}{N^2}\sum_{k,l,m,n=1}^{d}\frac{C_{k,l}^{m,n}}{(z+\widehat{\varepsilon}_l{}^n)(z-\widehat{\varepsilon}_k{}^m)(u-\widehat{\varepsilon}_l{}^n)}$$

Proposition

$$\Omega_2^{(0)}(u,z) = \frac{1}{R'(u)R'(z)}\Big(\frac{1}{(u-z)^2}+\frac{1}{(u+z)^2}\Big)$$

One recognises the Bergman kernel of topological recursion!

Proposition: $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z,w|)$

The solution is easier than the equation itself:

$$
\begin{aligned}
&\mathcal{T}^{(g)}(u_1,...,u_m\|z,w|)\\
&=\lambda\mathcal{G}^{(0)}(z,w)\operatorname*{Res}_{t\to z,-\hat{w}^j}\frac{R'(t)\,dt}{(R(z)-R(t))(R(w)-R(-t))\mathcal{G}^{(0)}(t,w)}\\
&\times\Bigg[\sum_{\substack{I_1\uplus I_2=\{u_1,...,u_m\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)}}\Omega^{(g_1)}_{|I_1|+1}(I_1,t)\mathcal{T}^{(g_2)}(I_2\|t,w|)\\
&+\sum_{i=1}^{m}\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..\check{u}_i..u_m\|u_i,w|)}{R(u_i)-R(t)}\\
&+\frac{\mathcal{T}^{(g-1)}(u_1,...,u_m\|t|w|)-\mathcal{T}^{(g-1)}(u_1,...,u_m\|w|w|)}{R(w)-R(t)}\\
&+\mathcal{T}^{(g-1)}(u_1,...,u_m,t\|t,w|)\Bigg]
\end{aligned}
$$

Proposition: $\mathcal{T}^{(g)}(u_1,\dots,u_m\|z|w|)$

$$
\begin{aligned}
&\mathcal{T}^{(g)}(u_1,...,u_m\|z|w|)\\
&=\frac{\lambda\prod_{j=1}^d \frac{R(z)-R(\alpha_j)}{R(z)-R(\varepsilon_j)}}{(R(z)-R(-z))}\operatorname*{Res}_{t\to z,\alpha_j}\frac{R'(t)\,dt}{(R(z)-R(t))\prod_{j=1}^d\frac{R(t)-R(\alpha_j)}{R(t)-R(\varepsilon_j)}}\\
&\times\Bigg[\sum_{\substack{I_1\uplus I_2=\{u_1,...,u_m\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)}}\Omega^{(g_1)}_{|I_1|+1}(I_1,t)\mathcal{T}^{(g_2)}(I_2\|t|w|)+\mathcal{T}^{(g-1)}(u_1,...,u_m,t\|t|w|)\\
&+\sum_{i=1}^m\frac{\partial}{\partial R(u_i)}\frac{\mathcal{T}^{(g)}(u_1,..\check{u}_i..,u_m\backslash u_i\|u_i|w|)}{R(u_i)-R(t)}\\
&+\frac{\mathcal{T}^{(g)}(u_1,...,u_m\|t,w|)-\mathcal{T}^{(g)}(u_1,...,u_m\|w,w|)}{R(w)-R(t)}\Bigg],
\end{aligned}
$$

where $\{0,\pm\alpha_1,..\pm\alpha_d\}$ are all solutions of $R(z)-R(-z)=0$.

$\Omega^{(g)}(u_1,\dots,u_m)$ for $2-2g-m<0$

Proposition ($g=0$) / Conjecture ($g\geq 1$)

$$R'(z)\Omega^{(g)}_{m+1}(u_1,...,u_m,z) = \operatorname*{Res}_{q\to 0,-u_l,\beta_i} \frac{dq}{(q-z)\mathfrak{G}_0(q)}\Bigg[\sum_{\substack{I_1\uplus I_2=\{u_1,\dots,u_m\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \Omega^{(g_1)}_{|I_1|+1}(I_1,q)\frac{\lambda}{N}\sum_{n=1}^{d}\frac{r_n\mathcal{T}^{(g_2)}(I_2\|q,\varepsilon_n|)}{R(\varepsilon_n)-R(-q)}$$

$$+\sum_{i=1}^{m}\frac{\partial}{\partial R(u_i)}\mathcal{T}^{(g)}(u_1,..\check{u}_i..,u_m\|u_i,q|)$$

$$+\frac{\lambda}{N}\sum_{n=1}^{d}\frac{r_n\mathcal{T}^{(g-1)}(u_1,...,u_m,q\|q,\varepsilon_n|)}{R(\varepsilon_n)-R(-q)}$$

$$+\frac{\lambda}{N}\sum_{n=1}^{d}\frac{r_n\mathcal{T}^{(g-1)}(u_1,...,u_m\|q|\varepsilon_n|)}{(R(\varepsilon_n)-R(q))(R(\varepsilon_n)-R(-q))}-\mathcal{T}^{(g-1)}(u_1,...,u_m\|q|q|)\Bigg]$$

$\beta_1,\dots\beta_{2d}$ – ramification points, $R'(\beta_i)=0$

Differential forms $\omega_{g,m}(z_1,\dots,z_m)=\lambda^{2-2g-m}\Omega^{(g)}_m(z_1,\dots,z_m)\prod_{k=1}^m dR(z_i)$

There is not much to improve for the $\mathcal{T}^{(g)}$. In contrast, the $\Omega^{(g)}_m$ simplify tremendously (by a factor >100) compared with the formula. Their structure is only visible after these simplifications.

$$\omega_{0,3}(u_1,u_2,z)=-\sum_{i=1}^{2d}\frac{\big(\frac{1}{(u_1-\beta_i)^2}+\frac{1}{(u_1+\beta_i)^2}\big)\big(\frac{1}{(u_2-\beta_i)^2}+\frac{1}{(u_2+\beta_i)^2}\big)du_1\,du_2\,dz}{R'(-\beta_i)R''(\beta_i)(z-\beta_i)^2}$$
$$+\Big[d_{R(u_1)}\Big(\frac{\omega_{0,2}(u_2,-u_1)}{(dR)(-u_1)}\frac{dz}{R'(u_1)(z+u_1)^2}\Big)+u_1\leftrightarrow u_2\Big]$$

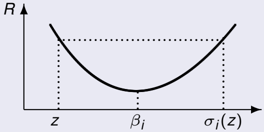
$$\omega_{1,1}(z)=\sum_{i=1}^{2d}\frac{dz}{R'(-\beta_i)R''(\beta_i)}\Big\{-\frac{1}{8(z-\beta_i)^4}+\frac{R'''(\beta_i)}{24R''(\beta_i)(z-\beta_i)^3}$$
$$+\frac{\frac{R''''(\beta_i)}{48R''(\beta_i)}-\frac{(R'''(\beta_i))^2}{48(R''(\beta_i))^2}+\frac{R''(-\beta_i)R'''(\beta_i)}{48R'(-\beta_i)R''(\beta_i)}+\frac{(R''(-\beta_i))^2}{48(R'(-\beta_i))^2}-\frac{1}{8\beta_i^2}}{(z-\beta_i)^2}\Big\}$$
$$-\frac{dz}{8(R'(0))^2z^3}+\frac{R''(0)dz}{16(R'(0))^3z^2}$$

Abstract loop equations [Borot-Eynard-Orantin 13]

Proposition $(g, m) \in \{(0,2), (0,3), (0,4), (1,1)\}$ / Conjecture

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the ramified cover identified in the solution of $\mathcal{G}^{(0)}(z, w)$.

Let $\beta_1, \ldots, \beta_{2d}$ be the ramification points of R and σ_i be the corresponding local Galois involution in the vicinity of β_i.



Define $\omega_{0,1}(z) = -R(-z)R'(z)dz$ and for $2 - 2g - m \leq 0$ the $\omega_{g,m}$ as before. Then:

1. linear loop equation:
$$\omega_{g,m}(u_1, \ldots, u_{m-1}, z) + \omega_{g,m}(u_1, \ldots, u_{m-1}, \sigma_i(z)) = \mathcal{O}(z-\beta_i)dz$$
2. quadratic loop equation:
$$\omega_{g-1,m+1}(u_1, \ldots, u_{m-1}, z, \sigma_i(z)) + \sum_{\substack{I_1 \uplus I_2 = \{u_1, \ldots, u_{m-1}\} \\ g_1+g_2=g}} \omega_{g_1,|I_1|+1}(I_1, z)\omega_{g_2,|I_2|+1}(I_2, \sigma_i(z)) = \mathcal{O}((z-\beta_i)^2)(dz)^2$$

Blobbed topological recursion [Borot-Shadrin 15]

Theorem

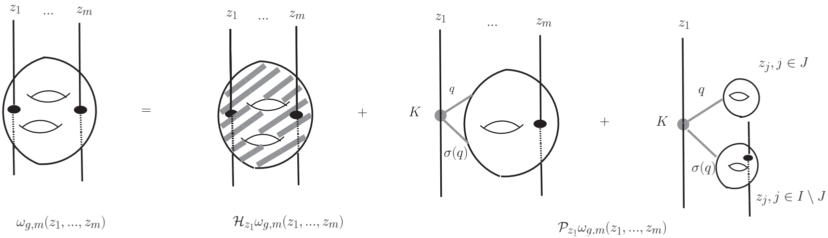
Let $\{\omega_{g,m}\}_{g\geq 0, m>0}$ be a family of meromorphic differential forms which satisfy the abstract loop equations. Then their parts $\mathcal{P}\omega_{g,m}$ containing the poles at ramification points are given by

$$\begin{aligned}
&\mathcal{P}_z\omega_{g,m}(u_1,\dots,u_{m-1},z) \\
&= \sum_{i=1}^{2d} \operatorname*{Res}_{q\to\beta_i} \frac{\frac{1}{2}\int_{q'=\sigma(q)}^{q'=q} B(z,q')}{\omega_{0,1}(q)-\omega_{0,1}(\sigma_i(q))} \Big(\omega_{g-1,m+1}(u_1,\dots,u_{m-1},q,\sigma_i(q)) \\
&\qquad + \sum_{\substack{I_1\uplus I_2=\{u_1,\dots,u_{m-1}\}\\ g_1+g_2=g\\ (I_1,g_1)\neq(\emptyset,0)\neq(I_2,g_2)}} \omega_{g_1,|I_1|+1}(I_1,q)\,\omega_{g_2,|I_2|+1}(I_2,\sigma_i(q))\Big)
\end{aligned}$$

where $B(u,z)=\frac{du\,dz}{(u-z)^2}$ is the Bergman kernel (for $x:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$).

$\mathcal{H}_z\omega_{g,m}(\dots,z) := \omega_{g,m}(\dots,z)-\mathcal{P}_z\omega_{g,m}(\dots,z)$ is made of blobs.

A picture



$\omega_{g,m}$ = meromorphic forms on space of compactified complex lines through the marked points on a genus-g Riemann surface.

- The universal formula of topological recursion produces the parts $\mathcal{P}\omega_{g,m}$ from the entire $\omega_{g',m'}$ of smaller degree.
- The part $\mathcal{H}\omega_{g,m}$ is an additional input at every recursion step. We have a residue formula for them.

The quartic analogue of the Kontsevich model distinguishes a unique such form $\omega_{g,m}$ for every (g, m). What is its significance?

Intersection numbers and integrability

Fact [Borot-Shadrin 15]

- Forms $\omega_{g,m}$ which satisfy BTR encode intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,m}$ of stable complex curves.
- These are several copies of the same intersections of ψ, κ-classes as in the Kontsevich model, coupled via blobs.

We do not know whether this coupling function encoded in the quartic Kontsevich model is interesting or not.

Integrability

- Understanding better our recursion should give access to the partition function itself, a function of λ and (E_i).
- Is it a τ-function for a Hirota equation, i.e. is it integrable? [not known in general BTR]

Outlook: $\lambda\Phi^4$ on noncommutative Moyal space

2. Moyal in $2D$ and $\theta \to \infty$ described by a Lambert curve $x(z) = R(z) = z + \lambda \log(1+z)$, $y(z) = -R(-z)$ [Panzer-W 18]

- It has a single ramification point of infinite order.
- According to the [Bouchard-Mariño 07] conjecture, *another* Lambert curve generates simple Hurwitz numbers.
- Is this more than a formal analogy?
  See [Borot-Eynard-Mulase-Safnuk 09]

4. Moyal in $4D$ and $\theta \to \infty$ according to [Grosse-Hock-W 20] described by

$$R(z) = z \cdot {}_2F_1\left(\begin{matrix}\alpha_\lambda, 1-\alpha_\lambda \\ 2\end{matrix}\middle| -z\right), \quad \alpha_\lambda = \begin{cases} \frac{\arcsin(\lambda\pi)}{\pi} & \text{for } |\lambda| \leq \frac{1}{\pi} \\ \frac{1}{2} + \mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi} & \text{for } \lambda \geq \frac{1}{\pi} \end{cases}$$

- Ramification points accumulate to a curve. Can one turn the TR-formula into an integral representation?
- Can one generalise this to finite θ?