Regularity structures and paracontrolled calculus

Joint works with M. Hoshino

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1. Pointwise description devices

2. Fourier-type description devices

3. From paracontrolled systems to models and modelled distributions

The multiplication problem in singular PDEs

Singular PDEs = **multiplication problem**, e.g.

$$(\partial_t - \Delta)u = u\,\zeta, \quad \text{in 2-dimensional torus,}$$
$$(\partial_t - \partial_x^2)u = \xi + (\partial_x u)^2, \quad \text{in 1-dimensional torus,}$$

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for some reference objects Z_i *using extra arguments*, you can make sense of the ill-defined term $F(u, \nabla u, \zeta)$ for **functions/distributions** u **that look like the** Z_i's.

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Leads to **regularity structures, models and modelled distributions**, and **paracontrolled calculus and paracontrolled systems**.

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Our **aim** in a singular PDE setting

microscopic description $\Longleftrightarrow$ **macroscopic** description

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Consistency of repeated re-expansion around different points and requirement that the $g(\tau)$ form a sufficiently rich family to describe an algebra of functions, directly lead to the definition of a concrete regularity structure $\mathscr{T}$ and a model (g, Π) on it.

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  for all $\tau \in T^+$,
- There is a map $\Pi : T \to \mathcal{S}'(\mathbb{T}^d)$, such that
  $$\Pi_x \tau = (\Pi \otimes g_x^{-1})\Delta\tau$$
  has $C^{|\tau|}$-regularity at x (only).

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$$\mathcal{D}^\gamma(T, g) := \Big\{ \mathbf{f} := \big(f^\tau(x)\big)_{\tau \in \mathcal{B},\, x \in \mathbb{T}^d} \,;\, \Big| \big\langle \mathbf{f}(z) - \widehat{g_{zy}}\big(\mathbf{f}(y)\big), \tau \big\rangle \Big| \lesssim |z-y|^{\gamma - |\tau|}, \ \forall \tau \in T, \forall y, z \in \mathbb{T}^d \Big\} : \ \textbf{modelled distributions}$$

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such that

$$\Big|\big\langle \mathbf{Rf} - \sum_\tau f^\tau(x)\Pi_x\tau, \varphi_x^\lambda\big\rangle\Big| \lesssim \lambda^{|\tau|};$$

this map $\mathbf{R}$ is unique if $\gamma > 0$. It is called the **reconstruction map**.

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$$\Delta \sigma = \sum_{\mu \leq \sigma} \mu \otimes (\sigma / \mu) \in T \otimes T^+, \quad \Delta^+ \tau = \sum_{\nu \leq^+ \tau} \nu \otimes (\tau /^+ \nu) \in T^+ \otimes T^+.$$

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2. Fourier-type description devices: micro to macro

By Littlewood-Paley, a distribution $a = \sum a_i$, with a_i smooth and $\mathrm{supp}(\widehat{a_i}) \subset \{\text{annulus of size} \simeq 2^i\}$. Write

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\begin{aligned}
ab &= \sum_{i \ll j} a_i b_j + \sum_{i \sim j} a_i b_j + \sum_{j \ll i} a_i b_j \\
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Theorem (B.-Hoshino 2018) – *Fix a regularity structure $\mathscr{T}$ and a model* $\mathsf{M} = (\mathsf{g}, \Pi)$ *on $\mathscr{T}$. One can construct 'reference functions/distributions'* $\{[\tau]^{\mathsf{g}} \in C^{|\tau|}(\mathbb{T}^d)\}_{\tau \in \mathcal{B}^+}$ *and* $\{[\sigma]^{\mathsf{M}} \in C^{|\sigma|}(\mathbb{T}^d)\}_{\sigma \in \mathcal{B}}$ *such that*

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We talk of **para-remainders** $[\tau]^{\mathsf{g}}, [\sigma]^{\mathsf{M}}$; *they depend continuously on the model M.*

2. Fourier-type description devices: micro to macro

Theorem (B.-Hoshino 2018) – *Fix a regularity exponent* $\gamma \in \mathbb{R}$, *and a model* $\mathsf{M} = (\mathrm{g}, \Pi)$ *on the regularity structure* $\mathscr{T}$.

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Theorem (B.-Hoshino 2018) – *Fix a regularity exponent* $\gamma \in \mathbb{R}$, *and a model* $\mathsf{M} = (\mathsf{g}, \Pi)$ *on the regularity structure* $\mathscr{T}$. *One can associate to any modelled distribution*

$$\mathbf{f}(\cdot) = \sum_{\sigma \in \mathcal{B}; |\sigma| < \gamma} f^\sigma(\cdot)\sigma \in \mathcal{D}^\gamma(T, g),$$

a distribution $[\mathbf{f}]^{\mathsf{M}} \in \mathcal{C}^\gamma(\mathbb{T}^{\mathbf{d}})$ *such that one defines a reconstruction* $\mathbf{Rf}$ *of* $\mathbf{f}$ *setting*

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Each coefficient f^{σ} *also has a representation*

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$$\mathbf{f} \mapsto \Big([\mathbf{f}]^{\mathsf{M}}, \big([f^\sigma]^{\mathsf{g}}\big)_{\sigma \in \mathcal{B}}\Big)$$

from $\mathcal{D}^\gamma(T, g)$ *to* $\mathcal{C}^\gamma(\mathbb{T}^d) \times \prod_{\tau \in \mathcal{B}} \mathcal{C}^{\gamma - |\tau|}(\mathbb{T}^d)$*, is continuous.*

3. From paracontrolled systems to models and modelled distributions

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Proposition – *If g is given, then for any family $\big([\sigma] \in C^{|\sigma|}(\mathbb{T}^d)\big)_{\sigma \in \mathcal{B}_\bullet, |\sigma| < 0}$ there exists a unique model (g, Π) on $\mathscr{T}$ such that*

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One assumes fairly weak assumptions on $\mathscr{T}$, satisfied by all reasonable regularity structures, like the regularity structures used for the study of singular PDEs. Assume in particular $\mathcal{B}^+$ freely generated by $\mathcal{B}^+_\bullet$ and monomials.

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- **Main assumption on (T^+, Δ^+)**

  (1 – Generating set) *There exists a finite subset $\mathcal{G}^+_\bullet$ of $\mathcal{B}^+_\bullet$ such that*

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Theorem (B. Hoshino 2019) – *For any reasonable regularity structure $\mathscr{T}$, one has a bi-Lipschitz parametrization of the space of models by*

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(A result similar to Corollary proved by Singh and Teichmann 2018.)

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(Generalizes greatly a result by Tapia and Zambotti (2018) on the parametrization of the set of branched rough paths – they used completely different methods.)

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One can create dynamics on the space of models solving (controlled) ODEs or (stochastic) PDEs in the parametrization space.

3. From paracontrolled systems to models: applications

Extension theorem for rough paths (Lyons & Victoir 2007) – *Given any* $\mathbb{R}^\ell$ *-valued Hölder control* h *on a bounded time interval, one can lift* h *into a rough path.*

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via the paracontrolled representation

$$f^\sigma = \sum_{\sigma<\mu;|\mu|<\gamma} \mathsf{P}_{f^\mu}[\mu/\sigma]^{\mathrm{g}} + [f^\sigma]^{\mathrm{g}}, \qquad (1)$$

for $\mathbf{f}(\cdot) = \sum_{\sigma\in\mathcal{B}} f^\sigma(\cdot)\sigma \in \mathcal{D}^\gamma(T,g)$ – *recall* $\mathcal{B}_\bullet = \mathcal{B}\backslash(\text{non-constant monomials})$.

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(Proving this statement happens to be equivalent to an extension problem for the map g.)

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The use of paracontrolled systems like (1) is the starting point of the paracontrolled approach to singular PDEs.

Thank you for your attention!