Preliminaries

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A geometrical interpretation of unitary Master Ward Identity Ward Identity

Based on an ongoing project with Eli Hawkins

Main idea: model classical observables in field theory as functionals on the space of field configurations $\mathcal{E}$ and perform quantization by deforming the product of such functionals.

In this talk I will focus on algebraic structures and neglect functional-analytic aspects.

Background M
- → globally hyperbolic manifold (e.g. $\mathbb{M}_4 = (\mathbb{R}^4, (^{1}{}_{-1}{}_{-1}{}_{-1}))$) ⇝ physical situation in QFT
- → finite set of points equipped with a partial order relation (causal structure) ⇝ causal sets (finite-dimensional analog)

$\mathcal{E}$ configuration space, here: $\mathcal{E} := \Gamma(E \to M)$, smooth sections of a vector bundle over M

$\mathcal{F} \subset \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{C})$ space of functionals ⇝ classical interaction terms, observables
↳ Bastiani calculus

$\mathcal{F}$
- → regular unctionals → $F^{(n)}(\varphi)$ smooth for all $\varphi \in \mathcal{E}$, $n \in \mathbb{N}$ (∞-dim toy model)
- → all smooth functions on $\mathcal{E}$, if M is just a finite poset (finite dim. model)

$\mathcal{F}_{loc}$ – local functionals (∞-dim. physical model) ⇝ e.g. $F(\varphi) = \int \varphi^4(x) f(x) d^4x$

Classical dynamics:

- In the infinite dimensional case: $L : \mathcal{D} \longrightarrow \mathcal{F}_{loc}$, $\mathcal{D} = \mathcal{C}_c^\infty(M, \mathbb{R})$

Definition 1.1. A *generalized Lagrangian* on a fixed spacetime M is a map $L : \mathcal{D} \to \mathcal{F}_{\text{loc}}$ such that

i) $L(f+g+h) = L(f+g) - L(g) + L(g+h)$ for $f, g, h \in \mathcal{D}$ with $\operatorname{supp} f \cap \operatorname{supp} h = \varnothing$ (**Additivity**).

ii) $\operatorname{supp}(L(f)) \subseteq \operatorname{supp}(f)$ (**Support**).

iii) Let $\mathcal{G}$ be the isometry group of the spacetime M (for Minkowski spacetime we set $\mathcal{G}$ to be the proper orthochronous Poincaré group $\mathcal{P}_+^\uparrow$.). We require that $L(f)(g^*\varphi) = L(g_*f)(\varphi)$ for every $g \in \mathcal{G}$ (**Covariance**).

Define: $\boxed{\delta L(\underset{\in \mathcal{D}}{\psi})[\underset{\in \mathcal{E}}{\varphi}] \doteq L(f)[\varphi + \psi] - L(f)[\varphi]}$, where $f \equiv 1$ on $\operatorname{supp} \psi$

- In the finite-dimensional analog $L \in \mathcal{F}$

Define: $\boxed{\delta L(\psi)[\varphi] \doteq L(\varphi + \psi) - L(\varphi)}$

Equations of motion:

$dL \in \Omega^1(\mathcal{E})$, 1-form defined by: $\langle dL(\varphi), \psi \rangle = \frac{d}{dt} \delta L(\varphi + \psi t) \big|_{t=0}$

$\psi \in T_\varphi \mathcal{E}$ ($\cong \mathcal{E}_c$ in the infinite dim. case)
↳ conf. w/ comp. supp.

$\boxed{dL(\varphi) = 0 \text{ equation of motion}}$

↪ solution space ≡ zero locus of dL

$\rotatebox{90}{$\equiv$}$

$\mathcal{E}_L$

Classical dynamics

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$\rightsquigarrow$ derived critical locus of dL (Koszul complex using multivector fields)

Def: $\mathcal{E}^* := \Gamma(E^* \to M)$, $\mathcal{E}^! := \mathcal{E}^* \otimes \mathrm{Dens}(M)$ (in the fin. dim. analog: $\mathcal{E}^! \equiv \mathcal{E}^*$)

$\hookrightarrow$ Let $\bar{\mathcal{E}} \equiv \mathcal{E} \times \mathcal{E}^! \subset T^*\mathcal{E}$

Def: Vector fields on $\mathcal{E}$ are smooth sections $\Gamma(T\mathcal{E})$.

Consider $\mathcal{V} \subset \Gamma(T\mathcal{E})$ to be:

$\mathcal{V}$ $\to$ Regular v. fields (have smooth derivatives) in ∞ dim

$\mathcal{V}$ $\to$ All smooth v. fields (finite dim.)

$\mathcal{V}_{loc}$ - local v. fields $\rightsquigarrow$ derivations of $\mathcal{F}_{loc}$

Def: Let $\bar{\mathcal{E}}_{odd} \subset T^*[-1]\mathcal{E}$ odd cotangent bundle

$\bar{\mathcal{E}}_{odd} = \mathcal{E} \oplus \mathcal{E}^![-1]$

$\mathcal{O}(\bar{\mathcal{E}}_{odd}) \equiv$ (regular) functions on $\bar{\mathcal{E}}_{odd}$

$\hookrightarrow$ multivector fields $\equiv \Lambda\mathcal{V}$

Def: Koszul operator $\delta_L : \mathcal{V} \to \mathcal{F}$ insertion of dL:

$$(\delta_L X)(\varphi) \doteq \langle dL(\varphi), X(\varphi)\rangle$$

$\hookrightarrow$ formally: $\int_M \frac{\delta L}{\delta\varphi(x)} X(\varphi)(x)$

Note: $\delta_L X(\varphi) = 0$ if $\varphi \in \mathcal{E}_L$ (zero locus)

Koszul complex: $(\Lambda\mathcal{V}, \delta_L) \equiv \mathcal{K}$

$$\dots \to \underset{2}{\Lambda^2\mathcal{V}} \to \underset{1}{\mathcal{V}} \to \underset{0}{\mathcal{F}} \to 0$$

$H_0(\mathcal{K}) = \mathcal{F}/\delta_L(\mathcal{V})$ "on-shell" functionals (derived critical locus)

<u>Note:</u> $H_1(\mathcal{K})$ describes non-trivial local symmetries

S-matrices and quantum dynamics

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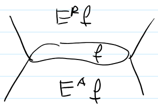
Start with quadratic L, so that $dL(\varphi) = P\varphi$ → P is a diff. op. in the ∞-dim case; → P is a matrix in the finite dim. analog

We need the following additional data:

$E^{R/A}$ – retarded / advanced Green functions for P ($E^{R/A}: \mathcal{E}_c \to \mathcal{E}$ for ∞-dim)

$P \circ E^{R/A} = \mathrm{id} = E^{R/A} \circ P|_{\mathcal{E}_c}$ + support properties

(for the finite dim. analog, see: https://arxiv.org/abs/1908.01973)



Def.: $E^D \doteq \frac{1}{2}(E^R + E^A)$, $E = E^R - E^A$

↳ Dirac propagator ↳ commutator function

Def.: Star-product (quantum product) on $\mathcal{F}[[\hbar]]$

$$(F \star G)(\varphi) = e^{i\hbar \underbrace{\langle \frac{1}{2}E, \frac{\delta^2}{\delta\varphi_1\delta\varphi_2}\rangle}_{D}} F(\varphi_1)\, G(\varphi_2)\Big|_{\varphi_1 = \varphi_2 = \varphi}$$

Time-ordered product:

$$(F \cdot_T G)(\varphi) = e^{i\hbar \underbrace{\langle E^D, \frac{\delta^2}{\delta\varphi_1\delta\varphi_2}\rangle}_{D^D}} F(\varphi_1)\, G(\varphi_2)\Big|_{\varphi_1 = \varphi_2 = \varphi}$$

Can also be written as:

$$(F \cdot_T G) = J(J^{-1}F \cdot J^{-1}G), \quad \text{where} \quad J = e^{\frac{i\hbar}{2}\underbrace{\langle E^D, \frac{\delta^2}{\delta\varphi^2}\rangle}_{\mathcal{D}^D}}$$

This allows to model the S-matrices from Klaus' talk as:

$$\boxed{\mathcal{S}(F) \doteq e_T^{iF/\hbar}, \ F \in \mathcal{F}} \qquad \boxed{\mathcal{S}(F_1)\mathcal{S}(F_2) \doteq (e_T^{iF_1/\hbar}) \star (e_T^{iF_2/\hbar})}$$

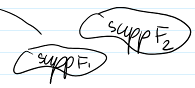
You can now verify:

S1 Identity preserving: $\mathcal{S}(0) = 1$.

S2 Locality: $\mathcal{S}$ satisfies the Hammerstein property, i.e. $F_1 \prec F_2$ implies that

$$\mathcal{S}(F_1 + F + F_2) = \mathcal{S}(F_1 + F)\mathcal{S}(F)^{-1}\mathcal{S}(F + F_2),$$

where $F_1, F, F_2 \in \mathcal{F}_{\mathrm{loc}}$.



S3 Schwinger-Dyson equation:

$$\mathcal{S}(F)\mathcal{S}(\delta L(\psi)) = \mathcal{S}(F^\psi + \delta L(\psi)) = \mathcal{S}(\delta L(\psi))\mathcal{S}(F),$$

↳ infinitesimal (linear order in ψ):

$$(e_T^{iF/\hbar}) \star \int \frac{\delta L}{\delta\varphi(x)}\psi(x) = (e_T^{iF}) \cdot \int \frac{\delta L}{\delta\varphi(x)}\psi(x) + i\hbar \int \frac{\delta}{\delta\varphi(x)}(e_T^{iF}) \underbrace{E(x,y)P_y}_{=0}\psi(x)$$

$$(e_T^{iF/\hbar}) \cdot_T \int \frac{\delta L}{\delta\varphi(x)}\psi(x) = (e_T^{iF}) \cdot \int \frac{\delta L}{\delta\varphi(x)}\psi(x) + i\hbar \int \frac{\delta}{\delta\varphi(x)}(e_T^{iF}) \underbrace{E^D(x,y)P_y}_{\delta(x,y)}\psi(x)$$

Hence:
$$(e_T^{iF}) \star \int \frac{\delta L}{\delta\varphi(x)}\psi(x) = (e_T^{iF}) \cdot_T \underbrace{\left(\int \frac{\delta L}{\delta\varphi(x)}\psi(x) + \int \frac{\delta F}{\delta\varphi(x)}\psi(x)\right)}_{\frac{d}{dt}\big|_{t=0}\left(\delta L(t\psi) + F^{t\psi}\right) = \partial_\psi L + \partial_\psi F}$$

ψ seen as a constant vector field $\int \psi(x)\frac{\delta}{\delta\varphi(x)}$

Generalization: for $X \in \Lambda\mathcal{V}$, we write:

$$X = \int X(x)[\varphi]\underbrace{\frac{\delta}{\delta\varphi(x)}}_{\varphi^\ddagger(x) \text{ odd generators}}$$

Infinitesimal Master Ward Identity:

$\int \frac{\delta}{\ } \ (\ e^{i(F+X)} \ ) \ \frac{\delta L}{\ } \qquad e^{i\overbrace{(F+X)}^{V}}$

→ graded laplacian $\Delta X = \int \frac{\delta^2 X}{\delta\varphi(x)\,\delta\varphi^\ddagger(x)}$

$\varphi^\ddagger(x)$ odd generators

Infinitesimal Master Ward Identity:

$$\int \frac{\delta}{\delta\varphi^\ddagger(x)} \left(e_T^{i(F+X)} \right) \star \frac{\delta L}{\delta\varphi(x)} = e_T^{i(\overbrace{F+X}^{V})} \cdot_T \Big(\underbrace{\partial_X F + \partial_X L}_{=\ \frac{1}{2}\underbrace{\{L+V, L+V\}}_{\text{BV bracket (Schouten bracket)}} - i\hbar \Delta X} - i\hbar \Delta X \Big)$$

graded laplacian $\quad \Delta X = \int \frac{\delta^2 X}{\delta\varphi(x)\,\delta\varphi^\ddagger(x)}$

(up to signs...)

$F \in \mathcal{F}$, $X \in \Lambda\mathcal{V}$

identified with free BV op. $s\left(e_T^{i(F+X)}\right)$

Hence: Infinitesimal MWI tells us that

$$\boxed{s\left(\mathcal{S}(F+X)\right) = \mathcal{S}(F+X) \cdot_T \left(\tfrac{1}{2}\{L+V, L+V\} - i\hbar \Delta X\right)}$$

related to quantum master equation

Functions on the cotangent bundle

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Let $\bar{\mathcal{F}} \subset \mathcal{C}^\infty(\bar{\mathcal{E}})$, where $\bar{\mathcal{E}} = \mathcal{E}\times\mathcal{E}^! \subset T^*\mathcal{E}$

↳ assumed regular

E.g. $X \in \mathcal{V}$ can be identified with an element of $\bar{\mathcal{F}}$:

"$\int X(x)\frac{\delta}{\delta\varphi(x)}$"

$X(\varphi,q) = \int X(x)[\varphi]\, q(x)$ is linear in q

To encode the action of diffeomorphisms on $\mathcal{F}$, we introduce the following products:

$$F \underset{\star}{\,} G = m \circ e^{-i\triangle_\otimes}(F\otimes G)\,,\qquad F \,\bar{\underset{\star}{\,}}\, G = m\circ e^{i\triangle^T_\otimes}(F\otimes G)\,,$$

$$F \star G = m \circ e^{\frac{i}{2}\triangle^{\mathrm{as}}_\otimes}(F\otimes G)\,,$$

where:

$$\triangle_\otimes \doteq \int \frac{\delta}{\delta q(x)}\otimes\frac{\delta}{\delta\varphi(x)}\,. \qquad \triangle^{\mathrm{as}}_\otimes \doteq \triangle^T_\otimes - \triangle_\otimes\,.$$

Note: $\overline{F \underline{\star} G} = G \bar{\underline{\star}} F\,.$

We also have the analogue of the BV Laplacian: $\triangle \doteq \int \frac{\delta^2}{\delta q(x)\delta\varphi(x)}\,.$

For $X, Y \in \bar{\mathcal{F}}$, we have:

$$e^{i\triangle}(X\underline{\star}Y) = (e^{i\triangle}X)\bar{\underline{\star}}(e^{i\triangle}Y)\,.$$
$$e^{\frac{i}{2}\triangle}(X\underline{\star}Y) = (e^{\frac{i}{2}\triangle}X)\star(e^{\frac{i}{2}\triangle}Y)\,.$$
$$e^{-\frac{i}{2}\triangle}(X\bar{\underline{\star}}Y) = (e^{-\frac{i}{2}\triangle}X)\star(e^{-\frac{i}{2}\triangle}Y)\,.$$

Let $X \in \mathcal{V}$, $F \in \mathcal{F}$. The following hold:

$$e^{iX}_{\underline{\star}}\underline{\star}F\underline{\star}e^{-iX}_{\underline{\star}} = e^X_{\mathrm{op}}F \equiv F^X$$
$$e^{iX-\triangle X/2}_\star \star F\star e^{-iX+\triangle X/2}_\star = F^X$$
$$e^{iX}_{\bar{\underline{\star}}}\bar{\underline{\star}}\bar{F}\bar{\underline{\star}}e^{-iX}_{\bar{\underline{\star}}} = \overline{F^X}$$

exponential of X as a differential operator
↳ local diffeomorphism

Note: We want to think of $e_T^{iF/\hbar}$ as something related to path integral:

$$\mathcal{T}(e^{iF/\hbar})(0) = \int \underbrace{e^{(iF/\hbar)(\varphi)}d\varphi}_{\substack{\text{transforms under } g=\exp(X)\\ \text{with a Jacobian factor } \mathrm{Jac}(g)}}$$

Proposition 2.3. *Let $X \in \mathcal{V}$, $F \in \mathcal{F}$. The following holds*

$$e^{iX+\triangle(X)}_{\underline{\star}}\underline{\star}F\underline{\star}e^{-iX}_{\underline{\star}} = e^{iX+\triangle(X)/2}_\star \star F\star e^{-iX+\triangle(X)/2}_\star = \mathrm{Jac}(e^X_{\mathrm{op}})\,F^X\,.$$

Here are some geometrical facts that can be thought of as classical analogs of MWI:

$\mathcal{A}(X) \equiv \log(\mathrm{Jac}(e^X_{\mathrm{op}}))$

$$e^{-iL}_{\underline{\star}}\,e^{iX+\triangle X}_{\underline{\star}}\underline{\star}e^{iF}\underline{\star}e^{iL}_{\underline{\star}} = e^{i(F^X+\delta L(X)-i\mathcal{A}(X))}e^{iX}_{\underline{\star}}\,,$$

$$e^{-iL}\star e^{iX+\triangle X/2}_\star\star e^{iF}\star e^{iL}\star e^{-iX+\triangle X/2}_\star = e^{i(F^X+\delta L(X)-i\mathcal{A}(X))}\,,$$

Define:
$$\langle L',\psi\rangle = \frac{d}{dt}\delta L(t\psi)\big|_{t=0}\,,$$

and $\delta L(\partial_q) := \left\langle L', \frac{\delta}{\delta q}\right\rangle + \frac{1}{2}\left\langle L'', \frac{\delta^2}{\delta q^2}\right\rangle$

$$\langle L'',\psi_1\otimes\psi_2\rangle = \frac{d^2}{dtds}\delta L(t\psi_1+s\psi_2)\big|_{t=s=0}\,,$$

We have the following identities: $A \in \bar{\mathcal{F}}$

$$e^{-iL} \underline{\star} A \underline{\star} e^{iL} = e^{i\delta L(-i\partial_q)} A,$$
$$e^{-iL} \overline{\star} A \overline{\star} e^{iL} = e^{-i\delta L(i\partial_q)} A,$$

this is the finite analogue of of insertion of a 1-form dL

Unitary MWI analogue

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Disclaimer:
here I drop h-bar and ignore UV and IR problems (work in the finite dimensional analogue)

We need to bring J and $\star$ into the game!

First take on the unitary MWI:

$$\boxed{\underbrace{e^{iL}\,\overline{\star}\,J(e_{\overline{\star}}^{iX}\,\overline{\star}\,e_{\overline{\star}}^{iF})\,\overline{\star}\,e^{-iL}}_{= e^{-i\delta L(i\partial_q)}(J(e_{\overline{\star}}^{iX}\,\overline{\star}\,e_{\overline{\star}}^{iF}))} = J\left(e_{\underline{\star}}^{i(F^X+\delta L(X)-i\mathcal{A}(X))}\,\underline{\star}\,e_{\underline{\star}}^{iX}\right)} = \overline{J}\left(e_{\overline{\star}}^{-iX}\,\overline{\star}\,e_{\overline{\star}}^{-i(F^X+\delta L(X)+i\mathcal{A}(X))}\right)$$

anti-time-ordered product

Analogue to the infinitesimal one:

$$\int \frac{\delta}{\delta\varphi^{\ddagger}(x)}\left(e_T^{i(F+X)}\right)\frac{\delta L}{\delta\varphi(x)} = e_T^{i(F+X)} \cdot_T \left(\partial_X F + \partial_X L - i\hbar\,\Delta X\right)$$

To put $\star$ into the game, note:

$F\star G \doteq \mathcal{T}\circ m\circ e^{-iD_A}\,(\mathcal{T}^{-1}F\otimes\mathcal{T}^{-1}G)$, where $D_A = \langle E^A, \frac{\delta}{\delta\varphi}\otimes\frac{\delta}{\delta\varphi}\rangle$

Hence we can work with $(\cdot, \star_A)$, instead of $(\cdot_T, \star)$ (barring divergences)

$$F\,\overline{\star}_A G = m\circ e^{i\Delta_\otimes^T - iD_A}\,(F\otimes G)\,,\qquad F\circledast_A G = m\circ e^{\frac{i}{2}\Delta_\otimes^{\mathrm{as}} - iD_A}\,(F\otimes G)\,,$$

In this setting we have: $e_{\circledast_A}^{iL}\circledast_A X\circledast_A e_{\circledast_A}^{-iL} = e^{i\langle L', i\partial_q\rangle + \Delta/2}X\,.$, $X\in\overline{\mathcal{F}}$

The analogue of unitary MWI is then:

$$\boxed{e_{\circledast_A}^{iL}\circledast_A\left(e_{\underline{\star}}^{iF}\,\underline{\star}\,e_{\underline{\star}}^{iX}\right)\circledast_A e_{\circledast_A}^{-iL} = \overline{\left(e_{\underline{\star}}^{-i(F^X+\delta L(X)+i\mathcal{A}(X))}\,\underline{\star}\,e_{\underline{\star}}^{-iX}\right)}}$$

Work in progress:
- Solve UV and IR problem
- Construct the analogue of the Kuszul complex
- Formulate the C*-algebraic version