Resurgent Asymptotics of Hopf Algebraic Dyson-Schwinger Equations

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Higher Structures Emerging from Renormalisation

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M. Borinsky & GD, 2005.04265; M. Borinsky, GD, M. Meynig, 2020 to appear
O. Costin & GD, 1904.11593, 2003.07451, 2009.01962, ...

[DOE Division of High Energy Physics]
Motivation

- Kreimer-Connes:

\[\text{[perturbative]} \text{ QFT renormalisation } \leftrightarrow \text{ Hopf algebra structure}\]

\Rightarrow \text{ enables perturbative computations to very high order}

- Écalle: resurgent asymptotics

\[\text{[perturbative]} \text{ series } \rightarrow \text{[perturbative + nonperturbative]} \text{ transseries}\]

\Rightarrow \text{ nonperturbative physics encoded in perturbative physics}

**IDEA:** use resurgent trans-series to decode nonperturbative properties of QFT from their perturbative Hopf algebra structure
Trans-series

- an interesting observation by Hardy:


_\textit{No function has yet presented itself in analysis, the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms}_

G. H. Hardy, *Orders of Infinity*, 1910

- deep result: “this is all we need” (J. Écalle, 1980s)

- also as a closed logic system: Dahn and Göring (1980s)
Resurgent Trans-Series

- Écalle: resurgent functions closed under all operations:
  
  (Borel transform) + (analytic continuation) + (Laplace transform)

- basic trans-series expansion in QM & QFT applications:

\[ f(x) \sim \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \sum_{l=1}^{k-1} c_{k,l,p} x^p \left( \exp \left( -\frac{c}{x} \right) \right)^k \left( \ln \left( \pm \frac{1}{x} \right) \right)^l \]

- transmonomial elements: \( x, e^{-\frac{1}{x}}, \ln(x) \), familiar in QFT
- new: analytic continuation encoded in trans-series
- new: trans-series coefficients \( c_{k,l,p} \) are highly correlated
- new: exponentially improved asymptotics
- explored in ODEs, PDEs, difference eqs., QM, matrix models, QFT, string theory, ...
resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities.

J. Écalle

fluctuations about different singularities are quantitatively related
• resurgence is well established in matrix models and QM
• renormalisation makes resurgence in quantum field theory extremely interesting and also difficult
• recent progress for regularised QFTs and lattice QFT
• here: invoke Hopf algebra structure of perturbative QFT
Combinatoric explosion of renormalization tamed by Hopf algebra: 30-loop Padé-Borel resummation

D.J. Broadhurst ¹, D. Kreimer ²
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Exact solutions of Dyson–Schwinger equations for iterated one-loop integrals and propagator-coupling duality

D.J. Broadhurst a,1, D. Kreimer b,2

Nuclear Physics B 600 (2001) 403–422

An Étude in non-linear Dyson–Schwinger Equations*

Dirk Kreimer a,†
Karen Yeats b

Nonlinear ODEs from Dyson-Schwinger Equations

- Broadhurst/Kreimer 1999/2000; Kreimer/Yeats 2006:

for certain QFTs the renormalisation group equations can be reduced to coupled nonlinear ODEs for the anomalous dimension in terms of the renormalised coupling

- resurgence is deeply understood for (nonlinear) ODEs (Écalle, Costin, Kruskal, Ramis, Sauzin, Fauvet, ...)

- so this is a natural place to start

- some paradigmatic cases: Wess-Zumino model (Bellon, Schaposnik, Clavier, 2008, 2016, 2018); 4 dim. Yukawa (Borinsky, GD, 2020); 6 dim. \( \phi^3 \) theory (Bellon & Russo, 2020), (Borinsky, GD, Meynig, 2020)

- also related: Maiezza, Vasquez (2019, 2020)

- future goal: gauge theories
4 dimensional massless Yukawa theory

- renormalised fermion self-energy

\[ \Sigma(q) := \rightarrow \quad = \oint \Sigma(q^2) \]

- Dyson-Schwinger equation

\[ \rightarrow \quad = \rightarrow + \rightarrow + \rightarrow + \cdots \quad - \text{subtractions} \]

- anomalous dimension \( \gamma(\alpha) \) (\( \alpha \equiv \text{renormalised coupling} \)):

\[ \gamma(\alpha) = \frac{d}{d \ln q^2} \ln \left( 1 - \Sigma(q^2) \right) \bigg|_{q^2=\mu^2} \]

- renormalisation group \( \Rightarrow \) non-linear ODE

\[ 2\gamma = -\alpha - \gamma^2 + 2\alpha \frac{d}{d\alpha} \gamma \]
Perturbative Solution  
(rescale: $\gamma(\alpha) := 2 C \left(-\frac{\alpha}{4}\right)$)

\[
\left[C'(x) \left(2 x \frac{d}{dx} - 1\right) - 1\right] C(x) = -x
\]

- perturbative solution: $C(x) = \sum_{n=1}^{\infty} C_n x^n$ (OEIS: A000699)

$C_n = [1, 1, 4, 27, 248, 2830, 38232, 593859, 10401712, 202601898, \ldots]$

- combinatorics: generating function for “connected chord diagrams”

- large order asymptotics

$C_n \sim e^{-1} 2^{n+\frac{1}{2}} \frac{\Gamma \left(n + \frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{5}{2} \frac{2}{n - \frac{1}{2}} - \frac{43}{8} \frac{2^2}{(n - \frac{1}{2}) (n - \frac{3}{2})} - \cdots\right)$
Perturbative Solution (rescale: \( \gamma(\alpha) := 2 C\left(-\frac{\alpha}{4}\right) \))

\[
\left[C(x) \left(2 \frac{d}{dx} x - 1\right) - 1\right] C(x) = -x
\]

- perturbative solution: \( C(x) = \sum_{n=1}^{\infty} C_n x^n \) (OEIS: A000699)

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- combinatorics: generating function for “connected chord diagrams”

- large order asymptotics

\[
C_n \sim e^{-1} \frac{2^{n+\frac{1}{2}}} {\sqrt{2\pi}} \Gamma\left(n + \frac{1}{2}\right) \left(1 - \frac{5}{2} \frac{1}{2} \frac{43}{8} \right)^{-1} \left(2^n \left(2^n - \frac{1}{2}\right) - \frac{3}{2}\right) \ldots
\]

- missing boundary condition parameter?

Écalle: formal series \(\rightarrow\) trans-series: \( C(x) = \sum_{k=0}^{\infty} \sigma^k C^{(k)}(x) \)
• expand \( C(x) = C^{(0)}(x) + \sigma C^{(1)}(c) + \sigma^2 C^{(2)}(x) + \ldots \)

• \( C^{(0)}(x) = \text{previous formal perturbative series solution} \)

• linear inhomogeneous equations for \( C^{(k)}(x) \) for \( k \geq 1 \)

\[
C^{(1)}(x) = \frac{1}{\sqrt{2\pi} C^{(0)}(x)} \frac{\sqrt{x}}{2x} \exp \left[ - \frac{(C^{(0)}(x) + 1)^2}{2x} \right]
\]

\[
\sim \frac{e^{-1/(2x)}}{\sqrt{x}} \frac{e^{-1}}{\sqrt{2\pi}} \left[ 1 - \frac{5}{2} x - \frac{43}{8} x^2 - \frac{579}{16} x^3 - \ldots \right]
\]

• resurgence: \( C^{(1)}(x) \) expressed in terms of \( C^{(0)}(x) \)
Trans-series Solution

M. Borinsky & GD, 2005.04265

- expand $C(x) = C^{(0)}(x) + \sigma C^{(1)}(c) + \sigma^2 C^{(2)}(x) + \ldots$
- $C^{(0)}(x) =$ previous formal perturbative series solution
- **linear** inhomogeneous equations for $C^{(k)}(x)$ for $k \geq 1$

$$C^{(1)}(x) \ = \ \frac{1}{\sqrt{2\pi}} \frac{\sqrt{x}}{C^{(0)}(x)} \exp \left[ -\frac{(C^{(0)}(x) + 1)^2}{2x} \right]$$

$$\sim \ e^{-1/(2x)} \ e^{-1} \left[ 1 - \frac{5}{2} x - \frac{43}{8} x^2 - \frac{579}{16} x^3 - \ldots \right]$$

- **resurgence:** $C^{(1)}(x)$ expressed in terms of $C^{(0)}(x)$
- characteristic signature of resurgent structure:

$$C^{(0)}_n \sim e^{-1} \frac{2^{n+\frac{1}{2}} \Gamma \left( n + \frac{1}{2} \right)}{\sqrt{2\pi}} \left( 1 - \frac{5}{2} \frac{2}{n - \frac{1}{2}} - 2^2 \frac{43}{8} \frac{2}{(n - \frac{1}{2}) (n - \frac{3}{2})} - \ldots \right)$$

- combinatorics of $C^{(1)}_n$: Mahmoud & Yeats, 2020
Resurgent structure

- large order asymptotics of $C^{(1)}_n$ coefficients

$$C^{(1)}_n \sim -2e^{-2} \frac{2^{n+\frac{3}{2}} \Gamma(n + \frac{3}{2})}{2\pi} \left(1 - \frac{5}{2(n + \frac{1}{2})} - \frac{11}{2^2(n + \frac{1}{2})(n - \frac{1}{2})} - \ldots\right)$$

- next nonperturbative solution ($\xi(x) \equiv \frac{1}{\sqrt{x}} e^{-1/(2x)}$):

$$C^{(2)}(x) \sim \xi(x)^2 \frac{e^{-2}}{2\pi} \left[\frac{1}{x} - 5 - \frac{11}{2}x - \frac{97}{2}x^2 - \ldots\right]$$
Resurgent structure

- large order asymptotics of $C_n^{(1)}$ coefficients

\[
C_n^{(1)} \sim -2e^{-2} \frac{2^{n+\frac{3}{2}} \Gamma \left(n + \frac{3}{2}\right)}{2\pi} \left(1 - \frac{5}{2 \left(n + \frac{1}{2}\right)} - 2^2 \left(n + \frac{1}{2}\right) \left(n - \frac{1}{2}\right) - \ldots\right)
\]

- next nonperturbative solution ($\xi(x) \equiv \frac{1}{\sqrt{x}} e^{-1/(2x)}$):

\[
C^{(2)}(x) \sim \xi(x)^2 \frac{e^{-2}}{2\pi} \left[\frac{1}{x} - 5 - \frac{11}{2} x - \frac{97}{2} x^2 - \ldots\right]
\]

- continues to all orders ⇒ all-orders summation

\[
C(x) = \left[\exp \left(\sigma \xi(x) f(x, y) \frac{\partial}{\partial y}\right) \cdot y\right]_{y=C^{(0)}(x)}
\]

generating function: $f(x, y) \equiv \frac{1}{\sqrt{2\pi y}} x \exp \left[-\frac{1}{2x} y(y + 2)\right]

- also follows from Borinsky’s alien derivative on the ring of formal power series
Resurgence in the 4 dimensional massless Yukawa Model

- trans-series: the (asymptotic) perturbative solution to the nonlinear ODE for the anomalous dimension can be extended to a trans-series which resums all nonperturbative orders

- non-perturbative terms $C^{(k)}(x)$ ($k \geq 1$) $\leftrightarrow$ singularities of the Borel transform of the perturbative series

- resurgence: all non-perturbative terms are expressed explicitly in terms of the original formal series $C^{(0)}(x)$

fluctuations about different singularities are quantitatively related
Resurgence in the 6 dim. Scalar $\phi^3$ Theory

• physically more interesting model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{3!} \phi^3$$

, \quad \alpha := \frac{g^2}{(4\pi)^3}

• asymptotically free; $d = 6$ critical dimension; Lipatov instanton; renormalons; $\rightarrow$ non-perturbative physics
Resurgence in the 6 dim. Scalar $\phi^3$ Theory  

- physically more interesting model

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{3!} \phi^3, \quad \alpha := \frac{g^2}{(4\pi)^3}$$

- asymptotically free; $d = 6$ critical dimension; Lipatov instanton; renormalons; $\rightarrow$ non-perturbative physics

- Broadhurst/Kreimer: 3rd order ODE (with quartic nonlinearity) for anomalous dimension

$$\left[ C \left( 2x \frac{d}{dx} - 1 \right) - 1 \right] \left[ C \left( 2x \frac{d}{dx} - 1 \right) - 2 \right] \left[ C \left( 2x \frac{d}{dx} - 1 \right) - 3 \right] C = x$$

- perturbative solution: $C(x) = \sum_{n=1}^{\infty} C_n x^n$: (OEIS: A051862)

$|C_n| : \{1, 11, 376, 20241, 1427156, 121639250, 12007003824, \ldots \}$

- no known combinatorial interpretation of $C_n$
Trans-series Analysis

• Broadhurst/Kreimer: \( C_n \sim (-1)^n \Gamma (n + 2) \)

• with more data

\[
C_n \sim (-1)^n \Gamma \left( n + \frac{23}{12} \right) \left( 1 - \frac{97}{144} - \frac{53917}{124416} \left( n - \frac{1}{2} \right) \right) + \ldots
\]

• now there are 3 “missing” b.c. parameters!
• Broadhurst/Kreimer: \( C_n \sim (-1)^n \Gamma (n + 2) \)

• with more data

\[
C_n \sim (-1)^n \Gamma \left( n + \frac{23}{12} \right) \left( 1 - \frac{97}{144} \frac{1}{2 (n + \frac{11}{12})} - \frac{53917}{124416} \frac{1}{2^2 (n + \frac{11}{12}) (n - \frac{1}{12})} - \ldots \right) + \ldots
\]

• now there are 3 “missing” b.c. parameters !

• transseries ansatz for terms “beyond all orders”

\[
C(x) \sim x^c e^{-b/x} \rightarrow \text{three solutions}
\]

\[
\begin{align*}
  b &= 1 \quad & c &= -\frac{23}{12} \\
  b &= 2 \quad & c &= +\frac{1}{6} \\
  b &= 3 \quad & c &= -\frac{11}{4}
\end{align*}
\]

⇒ three \underline{resonant} Borel singularities at \( t = -1, -2, -3 \)
Trans-series Analysis

- full three-term trans-series

\[ C(x) \sim C_{\text{pert}}(x) + S_1[1] \sum_{k=1}^{\infty} \sigma_k^{[1]} \left( \frac{e^{-\frac{1}{x}}}{x^{23/12}} \right)^k \sum_{n=0}^{\infty} C^{(k)}_{[1],n} x^n \]

\[ + S_2[2] \sum_{k=1}^{\infty} \sigma_k^{[2]} \left( \frac{e^{-\frac{2}{x}}}{x^{-1/6}} \right)^k \sum_{n=0}^{\infty} C^{(k)}_{[2],n} x^n \]

\[ + S_3[3] \sum_{k=1}^{\infty} \sigma_k^{[3]} \left( \frac{e^{-\frac{3}{x}}}{x^{11/4}} \right)^k \sum_{n=0}^{\infty} C^{(k)}_{[3],n} x^n \]

- compute fluctuation coefficients from ODE: e.g. \( C^{(k=1)}_{[1],n} \)

\[ C^{(k=1)}_{[1],n} = \left\{ 1, \frac{97}{144}, \frac{53917}{124416}, \cdots \right\} \]

- resurgence relation:

\[ C^{\text{pert}}_n \sim (-1)^n \Gamma \left( n + \frac{23}{12} \right) \left( 1 - \frac{97}{144} \frac{1}{2 \left( n + \frac{11}{12} \right)} - 2^2 \frac{53917}{124416} \frac{1}{(n - \frac{1}{12}) \left( n - \frac{1}{12} \right)} - \cdots \right) \]
Borel Analysis

- location and nature of singularities, and associated Stokes constants $S_j$, can be efficiently extracted numerically
- perturbative series: Borel singularities on negative axis

\[ \log[B(t)] \]

- implies subleading exponentially small corrections
Borel Analysis

- decoding the full non-perturbative information (e.g. Stokes constants) requires new Borel analysis: Borel-Padé & conformal/uniformizing maps [Costin, GD: 2009.01962]

- 2-instanton fluctuations: Borel singularities on both negative and positive axis
Borel Analysis

- uniformization map in Borel plane enables (optimal) high precision extraction of Stokes constants:

\[
B(t)(1+t)^{35/12}
\]

- conformal map [blue]; uniformizing map [gold]
Borel Analysis

- uniformized Borel analysis → large order growth
- fluctuations about $t = -2$ have interference terms

\[
C^{(k=1)}_{[2],n} \sim (-1)^n \Gamma \left( n + \frac{35}{12} \right) \left[ c_1 + \frac{c_2}{(n + \frac{23}{12})} + \ldots \right] \\
+ \Gamma \left( n + \frac{25}{12} \right) \left[ d_1 + \frac{d_2}{(n + \frac{13}{12})} + \ldots \right]
\]
Resurgence in the 6 dimensional Scalar $\phi^3$ Theory

- richer non-perturbative structure than Yukawa model
- 3rd order ODE with 4th order non-linearity
- 3 different non-perturbative structures, with different fluctuation powers
- resonance: Borel singularity locations are integer multiples of leading one
- large order/low order resurgence relations
- non-perturbative terms expressed in terms of formal perturbative series
Origin of Non-perturbative Physics in 6 dim scalar $\phi^3$ QFT?

- Lipatov instanton $\Rightarrow$ one Borel singularity, repeated
- Hopf algebra iterative structure $\Rightarrow$ 3 independent (but resonant) Borel branch points, repeated
- "renormalon" bubble-chain diagrams $\Rightarrow$ rescaled Lipatov Borel singularity (?)
- dominant effect? other effects?
- diagrammatic interpretation?
Conclusions

perturbative Hopf algebra renormalisation

resurgent ↓ analysis

non-perturbative completion

• does there exist a “natural” Hopf algebraic non-perturbative (trans-series) structure ?

• functional relation & Borinsky’s “alien derivation” ?

• multi-component fields ? (Gracey, 2015; Giombi et al ...)

• relation with instantons and renormalons ?

• other renormalisation schemes ?

• 2d $\sigma$ models, Chern-Simons, SUSY, QED, QCD, ... ?