Renormalization, Categories and Self-similarity

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Higher Structures Emerging from Renormalisation, Erwin Schrödinger Institute, Vienna, October 12-16

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Motivation and Overview

Hopf Algebras and fixed point equations

Gauge symmetries

DSE

Transcendentality, analytic strcuture of amplitudes

Interesting Developments

- Graph operads and Graph complexes: work of Ralph Kaufmann and coll., Marko Berghoff,..., initiated by Maxim Kontsevich, Thomas Willwacher,...

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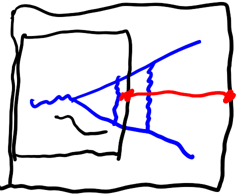
- Rough Paths: work of Kurusch Ebrahimi-Fard, Ynain Bruned, Dominique Manchon, Hans Zanna Munthe-Kaas,...

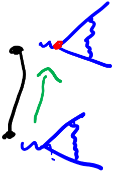
Some references

i) Graph complexes and Feynman rules
Marko Berghoff, Dirk Kreimer, e-Print: 2008.09540 [hep-th].
ii) Self-consistency of off-shell Slavnov-Taylor identities in QCD
J.A. Gracey, H. Kissler, D. Kreimer,
Phys.Rev.D 100 (2019) 8, 085001 • e-Print: 1906.07996 [hep-th].
iii) Outer Space as a Combinatorial Backbone for Cutkosky Rules and Coactions,
Dirk Kreimer,
https://indico.desy.de/indico/event/24049/session/4/contribution/13/material/slides/0.pdf

Hopf Algebras I

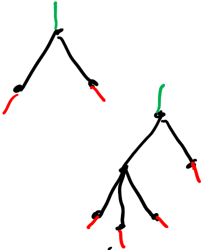


graphs →



rooted trees

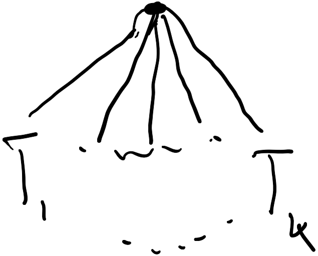
operad structure



$$\Delta B_+(X) = B_+(X) \otimes \mathbb{I} + (\mathrm{id} \otimes B_+)\Delta X$$

fixed point eq → DSE

Hopf Algebras II

$B_+(T_1 \dots T_k) =$ 

$$X(\alpha) = \mathbb{I} + \alpha B_+(X(\alpha))$$

linear DSE $\longleftrightarrow$ CFT$_1$ ...

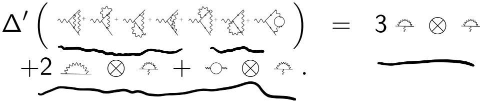
else ... $B_+(F(X(\alpha)))$

QFT$_1$ assembly maps ...

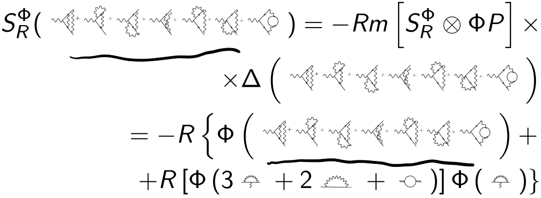
Hopf Algebras III

An Example

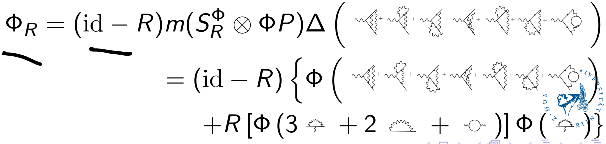
- The co-product



- The counterterm



- The renormalized result



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sub-Hopf algebras

- summing order by order

$$c_k^r = \sum_{|\Gamma|=k,\mathrm{res}(\Gamma)=r} \frac{1}{|Aut(\Gamma)|}\Gamma \Rightarrow \Delta(c_k^r) = \sum_j \mathrm{Pol}_j(c_m^s) \otimes c_{k-j}^r. \quad (7)$$

- Hochschild closedness

$$X^r = 1 \pm \sum_j c_j^r \alpha^j = 1 \pm \sum_j \alpha^j B_+^{r;j}(X^r Q^j(\alpha)), \quad (8)$$

$Q^j = \frac{X^v}{\sqrt{\prod_{\text{edges e at v}} X^e}}$. Evaluates to invariant charge.

- $bB_+^{r;j} = 0$.

$$\Delta B_+^{r;j}(X) = B_+^{r;j}(X) \otimes 1 + (id \otimes B_+^{r;j})\Delta(X). \quad (9)$$

Implies locality of counterterms upon application of Feynman rules $\Phi B_+^{r;j}(X) = \int d\mu_{r;j}\Phi(X)$:

$$\bar{R}(\Gamma) = m(S_\Phi^R \otimes \Phi P))\Delta B_+^{r;j} = \int d\mu_{r;j}\Phi^R(X). \quad (10)$$

Symmetry

- Ward and Slavnov–Taylor ids

$$i_k := c_k^{\bar\psi\psi} + c_k^{\bar\psi A\!\!\!/\psi} \tag{11}$$

span Hopf (co-)ideal I:

$$\Delta(I) \subseteq H \otimes I + I \otimes H. \tag{12}$$

$$\Delta(i_2) = i_2 \otimes 1 + 1 \otimes i_2 + (c_1^{\frac{1}{4}F^2} + c_1^{\bar\psi A\!\!\!/\psi} + i_1) \otimes i_1 + i_1 \otimes c_1^{\bar\psi A\!\!\!/\psi}.$$

- Feynman rules vanish on I $\Leftrightarrow$ Feynman rules respect quantized symmetry:
  $\Phi^R : H/I \to V$.
- Ideals for Slavnov–Taylor ids generated by equality of renormalized charges, also for the master equation in Batalin-Vilkovisky (see Walter van Suijlekom's work)
- Similar ideals for the core Hopf algebra are respected by the BCFW recursion, and fit naturally with the structure of perturbative quantum gravity

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DSE I

DSE II

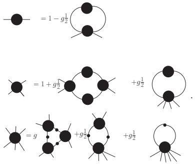
We are led to the following system:

$$X^2(g) = \mathbb{I} - gB_+^{(1)}(X^4/X^2(g)), \tag{28}$$

$$X^4(g) = \mathbb{I} + g\frac{1}{2}B_+^{(1,1)}([X^4]^2/[X^2(g)]^2) + g\frac{1}{2}B_+^{(2)}(X^6(g)/X^2(g)), \tag{29}$$

$$\begin{aligned} X^6(g) = \; & gB_+^{(1,1,1)}([X^4(g)]^3/[X^2(g)]^3) \\ & + g\frac{1}{2}B_+^{(2,1)}([X^6(g)X^4(g)]/[X^2(g)]^2) \\ & + g\frac{1}{2}B_+^{(3)}(X^8(g)/X^2(g)), \end{aligned} \tag{30}$$

and so on, which is best understood graphically (we omit to give contributions obtained by swapping or permuting external edges):



Note that we have only a finite number of one-loop primitives contributing to each fix-point equation, but we have an infinite set of equations to consider. Also, we emphasize that we maintain the B_+ operators to be closed one-cocycles in the Hochschild cohomology of the core Hopf algebra, and claim that the same definition (18) achieves precisely that.

The series X^4 and X^2 which are fixpoints of the above system are the same series as the one obtained in the Hochschild cohomology of the renormalization Hopf algebra above. This is a rather remarkable fact. We have done something very typical for the functional integral actually: we have traded a loop expansion for a leg expansion.

It is instructive to see in an example how this comes about. From $B_+(1,1)$ we get the same graphs as before, but

(31) 

has now three maximal forests (in the core Hopf algebra the number of maximal forests equals the number of non-self-intersecting closed paths we can draw on the graph). So this contribution gets an extra factor $1/3$. The missing $2/3$ is precisely provided from the same graph generated by insertions of

(32) 

into $B_+^{(2)}$, where the number of relevant bijections is two.

DSE III

The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\left(\begin{pmatrix} 1 \\ -Li_1(z) \\ -Li_2(z) \end{pmatrix} \begin{pmatrix} 0 \\ 2\pi i \\ 2\pi i \ln(z) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ (2\pi i)^2 \end{pmatrix}\right) = (C_1, C_2, C_3) \qquad (24)$$

$$\mathrm{Var}(\Im Li_2(z) - \ln|z| \Im Li_1(z)) = 0 \qquad (25)$$

Hodge sructure from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality ⇔ differential equation

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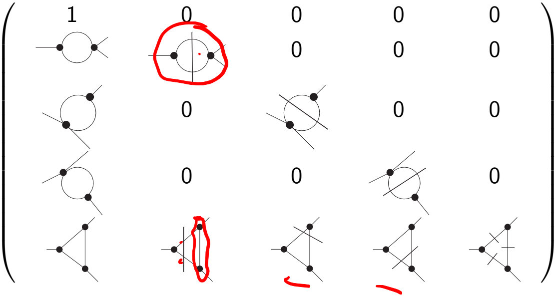
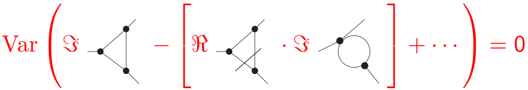
The Feynman graph as a Hodge structure

Hopf algebra structure as above



$= (C_1, C_2, C_3, C_4, C_5)$



Hodge structure: cut-reconstructability: from Hopf algebra structure:
branch cut ambiguities columnwise
Griffith transversality ⇔ differential equation?

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Outer Space and coactions I

Outer Space and coactions II

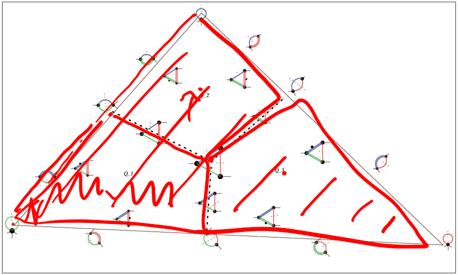
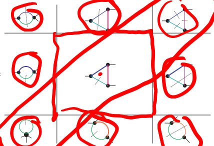


Figure 1: A two dimensional cell $C(t)$ in outer space for the triangle graph t on three different masses (indicated by colored edges). On shell edges are thin and marked by a hashed line, off-shell edges are double lines, a dot orders the two-edge spanning trees with the dotted edge the longer one. For this simple graph the spine gives a simplicial decomposition of $C(t)$ into three 2-cubes Q_1, Q_2, Q_3.

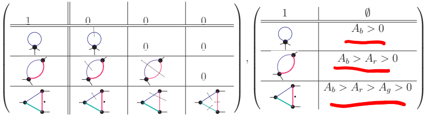
Let us study one cube say for the spanning tree on blue and red edges, so the cube containing  .

It provides nine cells: a two-dimensional square, four one-dimensional edges, and four zero-dimensional corners:

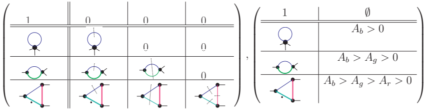
$Q^{(1)} =$ 

The three graphs in the anti-diagonal from the lowest left corner (the origin of the cube) to the upper right corner appear in both Hodge matrices $M^{(1)}$ and $M^{(2)}$ associated to this cube.

Outer Space and coactions III

$M^{(5)} =$ 

,

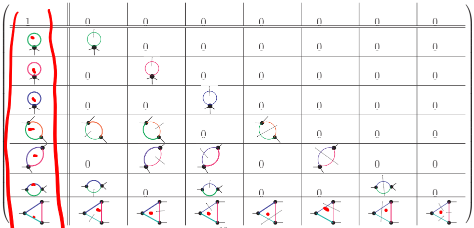
$M^{(6)} =$ 

,

We define

$$\Delta^M = \sum_{j=1}^{6} \Delta^{M^{(j)}},$$

where we find M as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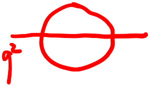
q²