The spectral action principle on Lorentz scattering spaces

Higher structures emerging from renormalisation

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[Connes, Bourbaki Juin 1996] “Mentionnons enfin que nous avons négligé dans cet exposé la nuance importante entre les signatures riemanniennes et lorentziennes”.

What is the spectral action principle ?

- (M,g) compact Riemannian spin manifold, $\dim(M) = n$ even,
- $\not{D}$ first order s.t. $\not{D}^2 = \Delta$ generalized Laplacian acting on spinors.

Example ($\mathbb{R}^4$ or $\mathbb{T}^4$)

Identifying $\mathbb{R}^4 = \mathbb{H}^4$ (quaternions), the Dirac operator reads $D = \begin{pmatrix} 0 & -\frac{\partial}{\partial q} \\ \frac{\partial}{\partial \bar{q}} & 0 \end{pmatrix}$.

$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + \sigma_1 \frac{\partial}{\partial x_1} + \sigma_2 \frac{\partial}{\partial x_2} + \sigma_3 \frac{\partial}{\partial x_3}$ *where* $(\sigma_i)_i$ *Pauli matrices.*

What is the spectral action principle ?

(M, g) Riemannian compact $\Longrightarrow$ Δ discrete spectrum

$$\sigma(\Delta) = \{0 \leqslant \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \dots\}$$

Recall Riemann zeta $\sum_{n=1}^{\infty} n^{-s}$, then spectral zeta :

$$\zeta_\Delta(s) = \sum_{\lambda \in \sigma(\Delta) \setminus \{0\}} \lambda^{-s}. \tag{1}$$

Theorem (Minakshisundaram-Pleijel, Seeley)

The function $\zeta_\Delta(s) = Tr_{L^2(M)}(\Delta^{-s})$ is holomorphic on $Re(s) > \frac{n}{2}$, meromorphic continuation to $s \in \mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \dots, 1\}$.

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Local version with **densities***. Diagonal value $\Delta^{-s}(x, x)$ of* **Schwartz kernel** *of Δ^{-s} holomorphic on $Re(s) > \frac{n}{2}$, meromorphic continuation to $s \in \mathbb{C}$ with poles at $\{\frac{n}{2}, \frac{n}{2} - 1, \dots, 1\}$,* **smooth** *in $x \in M$.*

What is the spectral action principle ?

$\kappa \in C^\infty(M)$ scalar curvature : 2nd order defect from flatness.
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When $\dim(M) = n \geqslant 4$ *:*

$$Res_{s=\frac{n}{2}-1} Tr_{L^2}\left(\Delta^{-s}\right) = \frac{\int_M \kappa}{(12)2^n \pi^{\frac{n}{2}} \Gamma(\frac{n}{2}-1)}. \tag{2}$$

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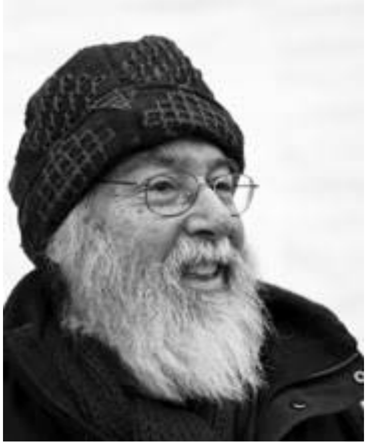
One can *hear Einstein–Hilbert action*. Application : in the standard model of Connes, allows to incorporate the gravitational action, the action of the standard model plus some coupling of the two.

Some problems.

- What about **non compact** space time ?
- What about **Lorentzian** space times ? $(dt^2 - \sum_{i=1}^{d}(dx^i)^2)$
- What about understanding clearly what is needed in the compact Riemannian case ?
- In what sense the Laplace operator which is **unbounded** has real spectrum ? Could all this be solved in the Lorentz case in the 70–80's ?



A boring problem.

$\Delta : L^2(M) \mapsto L^2(M)$ is unbounded therefore consider $\Delta : C_c^\infty(M) \mapsto L^2(M)$, formally self–adjoint and possible ways to close the graph in $L^2(M) \times L^2(M)$.

In practice, show that $P : C_c^\infty(M) \subset L^2(M) \mapsto L^2(M)$ $(P+i)u = f \in C_c^\infty(M)$ always has a solution

Theorem (Gaffney, Roelcke, Chernoff, Strichartz)

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Statement functional analytic in nature but proof uses **PDE methods** : finite propagation speed of solutions of wave equation. Consequence : makes sense to write $Tr(f(\Delta)) = \sum_{\lambda\in\sigma(\Delta)} f(\lambda)$!

A quote from Terence Tao

Thus, to actually verify essential self-adjointness of a differential operator, one typically has to first solve a PDE (such as the wave, Schrödinger, heat, or Helmholtz equation) by some **non-spectral method***... But there is no getting out of that first step, which requires some input (typically of an ODE, PDE, or geometric nature) that is external to what abstract spectral theory can provide.*

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Conjecture of Derezinski : find good classes of space–times for which the wave $\Box_g$ or Klein–Gordon operator $\Box_g + m^2$ are essentially self–adjoint in $L^2(M)$.

Theorem (Deresinski–Siemssen 2017, Vasy 2017, Nakamura–Taira 2019)

If (M, g) close to ultrastatic (DS), Lorentz scattering with **non trapping metric** *(V) or asymptotically Minkowski (NT), then $P = \Box_g + m^2$ essentially self–adjoint in $L^2(M)$.*

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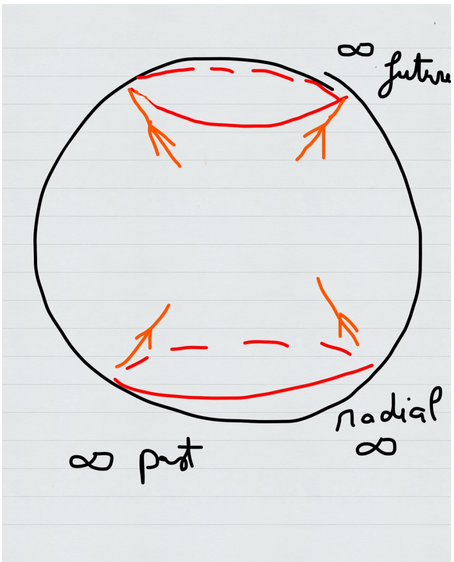

∞ future
radial ∞
∞ past

Simplest example : Minkowski space $\mathbb{R}^{1+d}$.

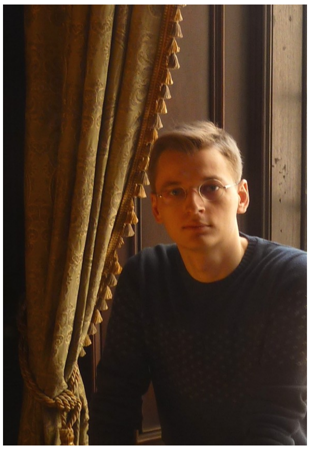
- $P = \partial_t^2 - \sum_{i=1}^d \partial_{x^i}^2 = \partial_t^2 + \Delta_{\mathbb{R}^d}$ self–adjoint by Nelson's Theorem since **commutes** with positive self–adjoint elliptic operator $-\partial_t^2 - \sum_{i=1}^d \partial_{x^i}^2$.
- Observe that formally $(P-z)\underbrace{\frac{e^{i|t-s|\sqrt{\Delta+z}}}{\sqrt{\Delta+z}}}_{(P-z)^{-1}} = Id$ when $Im(z) \geqslant 0, Re(z) > 0$.

  Set **weighted space** $\mathcal{H} = \langle t\rangle^p L_t^2 L_x^2$, $p > \frac{1}{2}$ (weight on time t) s. t. :

$$\begin{aligned} \|(P-z)^{-1}u\|^2_{L_t^{2,p}L_x^2(\mathbb{R}^d)} &= \int_{\mathbb{R}} \langle t\rangle^{-2p} \| \int_{\mathbb{R}} \frac{e^{i|t-s|\sqrt{\Delta+z}}}{\sqrt{\Delta+z}} u(s,.)ds\|^2_{L_x^2(\mathbb{R}^d)} dt \\ &\leqslant C\|u\|^2_{L_t^{2,-p}L_x^2(\mathbb{R}^d)} \end{aligned}$$

- $(P-z)^{-1} : L_t^{2,-p}L_x^2 \mapsto L_t^{2,p}L_x^2$ invertible for $Im(z) \geqslant 0, Re(z) \geqslant m^2 > 0$.

Michal Wrochna, support from Christian Brouder



Main Theorem.

Theorem (D–Wrochna 2020)

Let (M, g) be a globally hyperbolic, non trapping Lorentz scattering space of even dimension n. Set $P = \Box_g + m^2$. Then

1. $(P + i\varepsilon)^{-s} : C_c^\infty(M) \mapsto \mathcal{D}'(M)$ *is a holomorphic family of operators s.t.* $(P + i\varepsilon)^{-s_1}(P + i\varepsilon)^{-s_2} = (P + i\varepsilon)^{-s_1-s_2}$, **global object defined by functional calculus**.

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3. κ *scalar curvature*

$$\boxed{\lim_{m\to 0^+} \lim_{\varepsilon\to 0^+} Res_{s=\frac{n}{2}-1}(P + i\varepsilon)^{-s}(x, x) = \frac{(-1)^{\frac{n}{2}}\pi^{\frac{n}{2}}\kappa(x)}{6i\Gamma(\frac{n}{2} - 1)}.} \quad (4)$$

The wave front set.

A subset of the cotangent $T^\bullet(M \times M)$ that measures the position and codirection of singularities. Convention x position, ξ momentum.

Example (Feynman propagator on $\mathbb{R}^{1+3}$)

Δ_F represents VEV of time ordered products of quantum fields :

$$\langle\Omega| T\left(\phi(x_1)\phi(x_2)\right)|\Omega\rangle = \Delta_F(x_1, x_2) = C\left((x_1^0 - x_2^0)^2 - \sum_{i=1}^{3}(x_1^i - x_2^i)^2 + i0\right)^{-1}.$$

$WF(\Delta_F) = \Lambda \subset T^\bullet(M \times M)$ encodes the fact particles (resp antiparticles) propagate in the future (resp past) :

$$\Lambda = \{(x_1, x_2; \xi_1, \xi_2) \ s.t. \ (x_1; \xi_1) \prec (x_2; \xi_2) \ if \ \xi_1^0 > 0\}$$
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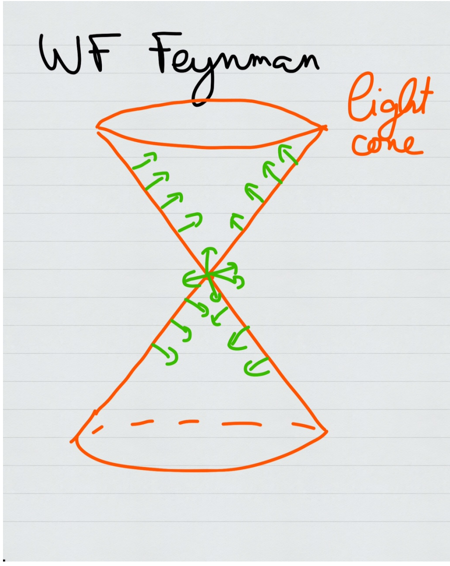
$\Lambda \subset T^\bullet(M \times M) =$ Feynman wave front, $\mathcal{D}'_\Lambda$ space of distributions with WF in Λ with normal topology, $\mathcal{C}^\alpha$ Hölder–Zygmund spaces.


WF Feynman
light cone

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3. **Formal** Hadamard parametrix $H_N(z)$ of $(P-z)^{-1}$ near the diagonal $d_2 \subset M\times M$, $H_N(z,x,y) = \sum_{k=0}^N \underbrace{u_k(x,y)}_{\in C^\infty}\underbrace{\mathbf{F}_k(z,x,y)}_{\in \mathcal{D}'_\Lambda}$, for Q signature $(1,d)$

$$\boxed{\mathbf{F}_k(z,x,y) = \frac{1}{(2\pi)^n}\int_{T_xM} (Q(\xi)-z)^{-s-1} e^{i\left\langle \xi, \exp_x^{-1}(y)\right\rangle} d^n\xi.} \tag{5}$$

Hölder regularity $\mathbf{F}_k \in \mathcal{C}^{<k+1-n}$. **Control** for large $Im(z)$, $H_N(z) = \mathcal{O}_{\mathcal{D}'_\Lambda}(|Im(z)|^{-1})$.

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4. Prove that $(P-z)^{-1} = H_N(z) + E_N(z)$ where error $E_N(z) \in \mathcal{C}^p$ for p large. Contour integration yields

$$(P+i\varepsilon)^{-s} = \sum_{k=0}^N c(k,s)\underbrace{u_k(x,y)}_{\in C^\infty}\underbrace{\mathbf{F}_{s+k-1}(z,x,y)}_{\in \mathcal{D}'_\Lambda} + \underbrace{\text{Error}(x,y)}_{\text{regular no poles}}$$

where $c(k,s)$ combinatorial factors.

Residue calculation by Wick rotation.

Extract residue of diagonal restriction $\mathbf{F}_{s+k-1}(z,x,x)$. Euclidean residue by polar coordinates + volume spheres :

$$\boxed{Res|_{s=\frac{n}{2}} \int_{\mathbb{R}^n} (\|\xi\|^2+1)^{-s} d^n\xi = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.} \tag{6}$$

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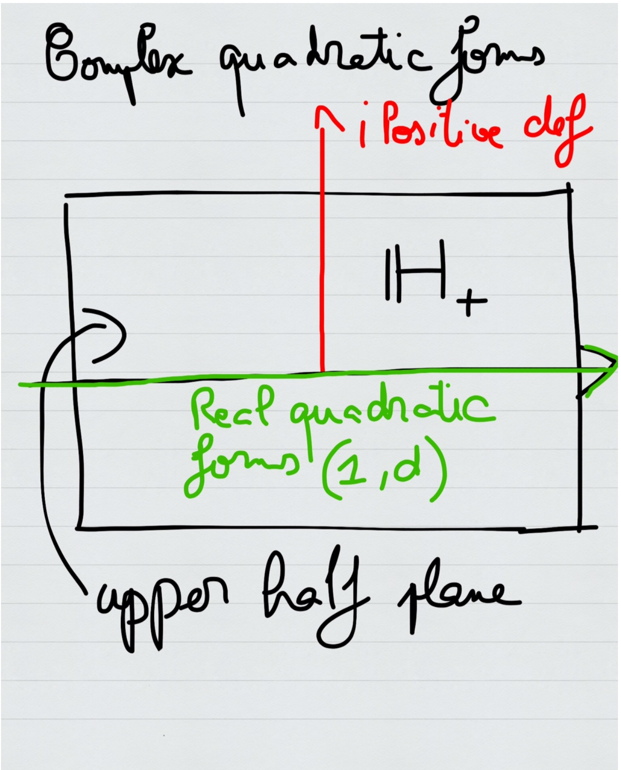
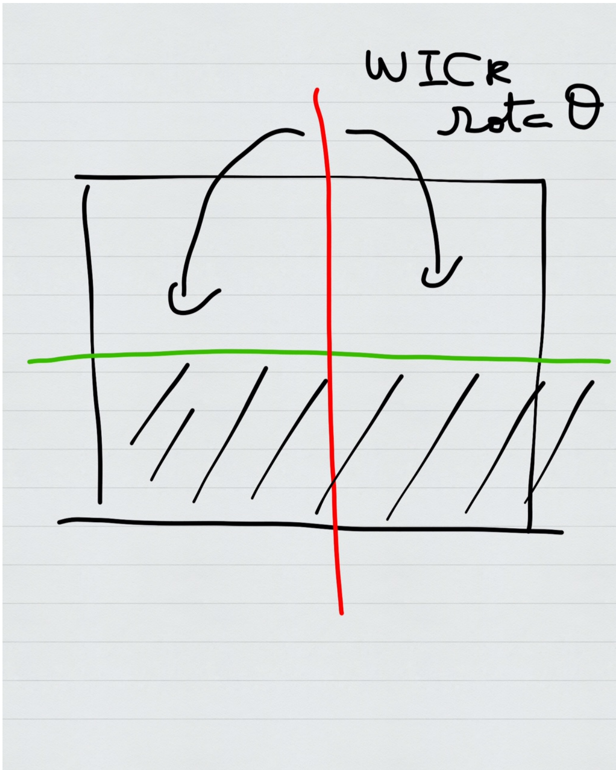
Pass from $\|\xi\|^2$ to $Q(\xi)$ with Q sign $(1,d)$. Wick rotation (Gelfand–Shilov) complexify and view integral as holomorphic of both s and quadratic form Q.

$$\boxed{Res|_{s=\frac{n}{2}} \int_{\mathbb{R}^n} (Q(\xi)+1\pm i0)^{-s} d^n\xi = \pm(-1)^{\frac{n}{2}} i \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.} \quad (7)$$

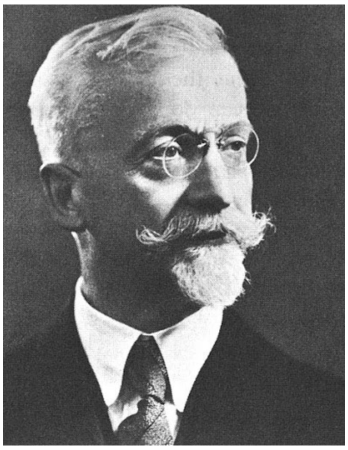
Wick and upper half–space of quadratic forms





Some personal heroes of curvature.

Gelfand–Retakh–Shubin : "Connection (or parallel transport) is the main object in the geometry of manifolds. Indeed, connections allow comparison between geometric quantities associated with different (distant) points of the manifold . . .According to Cartan, connection is a mathematical alias for an observer traveling in space-time and carrying measuring instruments."




Gelfand (90th birthday) :"And never forget E. Cartan, and always learn from Atiyah and Singer"

Where does κ come from ?

Connection Γ_{ij}^k torsion free preserves g. Normal coordinates $(x^i)_{i=1}^n$ solution

$$\boxed{\Gamma_{ij}^k(x)x^i x^j = 0, \forall k.} \tag{8}$$

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In normal coordinates, metric reads

$$g_{ij}(x) = \eta_{ij} + \underbrace{R_{ikjl}x^kx^l}_{\text{curvature}} + \mathcal{O}(\|x\|^3). \tag{9}$$

Hierarchy $(u_k)_{k\in\mathbb{N}}$ solves transport equations.

$$u_0 = \det(|g|)^{-\frac{1}{4}} = 1 - \frac{1}{12}\mathbf{Ric}_{kl}x^kx^l + \mathcal{O}(\|x\|^3)$$

hence $u_1 = -Pu_0(0) = \frac{1}{6}\kappa$.

Thanks for your attention !