A dimorphic description of the Grothendieck-Teichmüller Lie algebra

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Part I: Multizeta values and the Drinfeld associator

For each sequence $(k_1, \ldots, k_r)$ of strictly positive integers, $k_1 \geq 2$, the **multiple zeta value** is defined by the convergent series

$$\zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

These real numbers have been studied since Euler (1775).

They form a $\mathbb{Q}$-algebra, the *multizeta algebra* $\mathcal{Z}$.

Two multiplications of multizeta values

1. Shuffle multiplication

Straightforward integration shows that

$$\zeta(k_1,\ldots,k_r) = (-1)^r \int_0^1 \int_0^{t_1} \cdots \int_0^{t_{n-1}} \frac{dt_n}{t_n-\epsilon_n} \cdots \frac{dt_2}{t_2-\epsilon_2}\frac{dt_1}{t_1-\epsilon_1}$$

where

$$(\epsilon_1,\ldots,\epsilon_n) = (\underbrace{0,\ldots,0}_{k_1-1},1,\underbrace{0,\ldots,0}_{k_2-1},1,\ldots,\underbrace{0,\ldots,0}_{k_r-1},1).$$

The product of two simplices is a union of simplices, giving an expression for the product of two multizeta values as a sum of multizeta values. This is the **shuffle product**.

Example. We have

$$\zeta(2) = \int_0^1 \int_0^{t_1} \frac{dt_2}{1-t_2} \frac{dt_1}{t_1}$$

$$\zeta(2,2) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4} \frac{dt_3}{t_3} \frac{dt_2}{1-t_2} \frac{dt_1}{t_1}$$

$$\zeta(3,1) = \int_0^1 \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \frac{dt_4}{1-t_4} \frac{dt_3}{1-t_3} \frac{dt_2}{t_2} \frac{dt_1}{t_1}$$

and

$$\zeta(2)^2 = 2\,\zeta(2,2) + 4\,\zeta(3,1).$$

Convergent words in x, y

A **convergent word** $w \in \mathbb{Q}\langle x, y\rangle$ is a word $w = xvy$.

The reason for this notation is that it gives a bijection

$$\begin{aligned}\{\text{tuples with } k_1 \geq 2\} &\leftrightarrow \{\text{convergent words}\} \\ (k_1, \dots, k_r) &\leftrightarrow x^{k_1-1}y \cdots x^{k_r-1}y.\end{aligned}$$

Definition. For two monomials $u, v \in \mathbb{Q}\langle x, y\rangle$, the **shuffle product** $sh(u, v)$ is the set or formal sum of permutations of the letters of u and v where the letters of each word remain ordered.

Example. $sh(y, xy) = yxy + 2xyy$.

We use the x, y-notation to define $\zeta(w)$ for any convergent word $w = x^{k_1-1}y \cdot x^{k_r-1}y$ by setting

$$\zeta(x^{k_1-1}y \cdots x^{k_r-1}y) := \zeta(k_1, \ldots, k_r).$$

We then extend the definition of $\zeta(w)$ to all words w by writing $w = y^a u x^b$ with u convergent and setting

$$\zeta(w) := \sum_{r=0}^{a} \sum_{s=0}^{b} (-1)^{r+s} \zeta\big(sh(y^r, y^{a-r} u x^{b-s}, x^s)\big).$$

Theorem. *For all words u, v in x, y, we have*

$$\zeta(u)\zeta(v) = \zeta\big(sh(u, v)\big).$$

In particular, this multiplication law shows that the $\mathbb{Q}$-vector space $\mathcal{Z}$ has a $\mathbb{Q}$-algebra structure.

2. Stuffle multiplication

The product of two series over ordered indices can be expressed as a sum of series over ordered indices. This is the **stuffle product** of multizeta values.

Example. We have

$$\zeta(2)^2 = \Big(\sum_{n>0} \frac{1}{n^2}\Big)\Big(\sum_{m>0} \frac{1}{m^2}\Big)$$

$$= \sum_{n>m>0} \frac{1}{n^2 m^2} + \sum_{m>n>0} \frac{1}{n^2 m^2} + \sum_{n=m>0} \frac{1}{n^4}$$

$$= 2\zeta(2,2) + \zeta(4).$$

The x, y notation for stuffle

We write

$$st\big((k_1, \ldots, k_r), (l_1, \ldots, l_s)\big)$$

for the formal sum of sequences that come from this way of calculating the product of $\zeta(k_1, \ldots, k_r)\zeta(l_1, \ldots, l_s)$. For example

$$st\big((2), (3)\big) = (2, 3) + (3, 2) + (5).$$

We translate this operation on sequences over to convergent words.

Letting $y_i = x^{i-1}y$, we write a convergent word $w = x^{k_1-1}y \cdots x^{k_r-1}y$ as $w = y_{k_1} \cdots y_{k_r}$, and define the *stuffle product* of convergent words $st(u, v)$ as above. For example

$$st(xy, x^2y) = st(y_2, y_3) = y_2y_3 + y_3y_2 + y_5.$$

Thus

$$\zeta(2)\zeta(3) = \zeta(2, 3) + \zeta(3, 2) + \zeta(5).$$

The Drinfeld associator

Definition. The *Drinfeld associator* is the power series given by

$$\Phi_{KZ}(x, y) = 1 + \sum_{w \in \mathbb{Q}\langle x,y \rangle} (-1)^{d_w} \zeta(w) w$$

where d_w is the number of y's in the word w. Let Φ^*_{KZ} denote the product

$$\exp\Big(\sum_{n \geq 2} \frac{(-1)^{n-1}}{n} \zeta(n) y_1^n\Big) \cdot \pi_y(\Phi_{KZ}),$$

where π_y forgets all words ending in x, and rewrites words ending in y in the letters y_i.

Theorem. *For all words w in the letters y_i, set $\zeta^*(w) = (\Phi^*_{KZ} \,|\, w)$. Then*
(i) For all convergent words w, we have $\zeta^*(w) = \zeta(w)$.
(ii) For every pair of words u, v in the y_i, we have

$$\zeta^*(u)\zeta^*(v) = \zeta^*\big(st(u, v)\big).$$

Since non-convergent words in the y_i are exactly those starting with $y = y_1$, assigning a value $\zeta^*(w)$ to these words exactly means giving a *regularized value* to the divergent quantities $\zeta(1, k_2, \ldots, k_r)$.

The shuffle and stuffle relations between multizetas can be encoded directly as properties of the Drinfeld associator.

Shuffle relations: The shuffle relations satisfied by the $\zeta(w)$ are encoded in the property

$$\Delta(\Phi_{KZ}) = \Phi_{KZ} \otimes \Phi_{KZ}, \tag{1}$$

where $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$ is the standard coproduct on the Hopf algebra $\mathbb{Q}\langle\langle x, y\rangle\rangle$.

Stuffle relations: The stuffle relations satisfied by the $\zeta^*(w)$ are encoded in the condition

$$\Delta_*(\Phi_*) = \Phi_* \otimes \Phi_*, \tag{2}$$

where Δ_* be the coproduct on $\mathbb{Q}\langle\langle y_1, y_2, \ldots\rangle\rangle$ defined by

$$\Delta_*(y_i) = \sum_{k+l=i} y_k \otimes y_l.$$

Part II

The Grothendieck-Teichmüller and double shuffle Lie algebras

Definition of the n-strand braid Lie algebra. Let $\mathrm{Lie}\, P_n$ be the Lie algebra given by:

Generators: x_{ij}, $1 \le i, j \le n$ with $x_{ij} = x_{ji}$, $x_{ii} = 0$

Relations: $[x_{ij}, x_{ij} + x_{ik} + x_{jk}] = 0$ for sets $|\{i,j,k\}| = 3$ $\quad \sum_{j \neq i} x_{ij} = 0$

$[x_{ij}, x_{kl}] = 0$ if $\{i,j\} \cap \{k,l\} = \emptyset$.

In particular, $\mathrm{Lie}\, P_4$ is free on x_{12}, x_{23} and $\mathrm{Lie}\, P_5$ can be generated by $x_{12}, x_{23}, x_{34}, x_{45}, x_{51}$ or by $x_{12}, x_{23}, x_{14}, x_{24}, x_{34}$.

Theorem. [Drinfel'd] *The Drinfeld associator also satisfies the* associator relations

$$(I) \qquad \Phi_{KZ}(x,y)\Phi_{KZ}(y,x) = 1$$

$$(II) \quad e^{\pi i x}\Phi_{KZ}(y,x)e^{\pi i y}\Phi_{KZ}(z,y)e^{i\pi z}\Phi_{KZ}(x,z) = 1 \quad \text{with } x+y+z=0,$$

(III) The 5-cycle relation

$$\Phi_{KZ}(x_{12},x_{23})\Phi_{KZ}(x_{34},x_{45})\Phi_{KZ}(x_{51},x_{12})\Phi_{KZ}(x_{23},x_{34})\Phi_{KZ}(x_{45},x_{51}) = 1,$$

where the x_{ij} generate $\mathrm{Lie}\, P_5$.

Lie algebras associated to Φ_{KZ}

Definition. For any ring R, let $DS(R)$ denote the set of *double shuffle* power series in $R\langle\langle x, y\rangle\rangle$ that are group-like, have no linear term and satisfy the double shuffle relations.

Let $DS_0(R)$ denote the subset (pro-unipotent group) of those having no quadratic terms. Then $\Phi_{KZ} \in DS(\mathcal{Z})$, and if $\overline{\Phi}_{KZ}$ denotes the power series Φ_{KZ} with coefficients reduced modulo $\zeta(2)$ then

$$\overline{\Phi}_{KZ} \in DS_0(\mathcal{Z}/\langle\zeta(2)\rangle).$$

Definition. For any ring R, an *associator* is a group-like power series $\Phi(x, y) \in R\langle\langle x, y\rangle\rangle$ with no linear term which satisfies the associator relations (I), (II), (III) with $i\pi$ replaced by $\mu/2$ for some $\mu \in R$.

Let $GRT_0(R)$ denote the set (pro-unipotent group) of associators with no quadratic term. The power series Φ_{KZ} is an associator and for the reduced power series we have

$$\overline{\Phi_{KZ}} \in GRT_0(\mathcal{Z}/\langle\zeta(2)\rangle.$$

Definition. Let $\mathfrak{ds} = \mathfrak{ds}(\mathbb{Q})$ denote the Lie algebra consisting of polynomials $f(x, y) \in \mathbb{Q}\langle x, y\rangle$ such that

(i) f is of degree ≥ 3,

(ii) f is Lie-like for Δ, i.e. $f \in \mathrm{Lie}[x, y]$, and

(iii) f^* is Lie-like for Δ^*, where f^* is obtained from f by the formula

$$f^* = \pi_y(f) + \sum_{n\geq 2} \frac{(-1)^{n-1}}{n} y_1^n.$$

Definition. Let $\mathfrak{grt} = \mathfrak{grt}(\mathbb{Q})$ denote the Lie algebra consisting of polynomials $f(x, y) \in \mathbb{Q}\langle x, y\rangle$ such that

(i) f is of degree ≥ 3,

(ii) $f \in \mathrm{Lie}[x, y]$

(iii) $f(x_{12}, x_{23}) + f(x_{23}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{51}) + f(x_{51}, x_{12}) = 0$, where the x_{ij} generate Lie P_5.

Remark. Clearly $\mathfrak{ds}$ is the Lie version of DS_0. By a theorem of H. Furusho, these properties automatically imply that if $x + y + z = 0$ then

$$f(x, y) + f(y, x) = f(x, y) + f(y, z) + f(z, x) = 0.$$

So this definition really is the graded Lie algebra associated to GRT_0.

The Lie bracket: Every $f \in \mathrm{Lie}[x,y]$ yields a derivation D_f of $\mathrm{Lie}[x,y]$ defined by

$$D_f(x) = 0, \qquad D_f(y) = [y, f(x,y)].$$

We write $\mathrm{Der}^* \mathrm{Lie}[x,y]$ for the derivations of this type. We can put a different Lie bracket on the vector space $\mathrm{Lie}[x,y]$, called the Poisson or Ihara bracket via

$$\{f,g\} = [f,g] + D_f(g) - D_g(f),$$

which corresponds to bracketing derivations:

$$[D_f, D_g] = D_{\{f,g\}}.$$

* Ihara: $\mathfrak{grt}$ is a graded Lie algebra under $\{,\}$ (so GRT_0 is a pro-unipotent group).
* Racinet,Ecalle: $\mathfrak{ds}$ is a graded Lie algebra under $\{,\}$ (so DS_0 is a pro-unipotent group).

Theorem. (Furusho) There is an injection of Lie algebras

$$\mathfrak{grt} \hookrightarrow \mathfrak{ds}$$
$$f(x,y) \mapsto f(x,-y).$$

Part III. A dimorphic definition of $\mathfrak{grt}$

Theorem. *Let* $\mathrm{Lie}_n P_5$ *be the graded part of* $\mathrm{Lie}\, P_5$ *of degree* n*, and let* K_4 *denote the kernel of the morphism that "erases the 4th strand". Then*

(i) $K_4 := \langle x_{14}, x_{24}, x_{34}, x_{45}\rangle \simeq \langle x_{24}, x_{34}, x_{45}\rangle$ *is a free Lie algebra on 3 generators (since* $x_{14} + x_{24} + x_{34} + x_{45} = 0$*).*

(ii) We have the semi-direct product decomposition

$$\mathrm{Lie}\, P_5 \simeq \langle x_{12}, x_{23}\rangle \rtimes K_4.$$

(iii) For fixed $n \geq 1$*, we have the direct sum decomposition*

$$\mathrm{Lie}_n P_5 \simeq \langle x_{12}, x_{23}\rangle_n \oplus (K_4)_n.$$

This theorem implies that if $f \in \mathrm{Lie}_n P_5$, then f can be written uniquely as

$$f = f_0(x_{12}, x_{23}) + F$$

where $F \in K_4$. We call F the *normalization* of f with respect to the 4th strand.

We want to consider the normalization of elements of $\mathrm{Lie}\, P_5$ of the form $f(x_{12}, x_{51})$ where $f(x, y) \in \mathrm{Lie}[x, y]$.

We use the two following facts:

(1) The relation $x_{51} = x_{23} + x_{24} + x_{34}$ in $\mathrm{Lie}\, P_5$

(2) The fact that if $a, b, c \in \mathrm{Lie}\, P_5$ and $[a, c] = [b, c] = 0$, then

$$f(a, b + c) = f(a, b).$$

By (1), we have

$$f(x_{12}, x_{51}) \equiv f(x_{12}, x_{23}) \bmod K_4.$$

Therefore for $f(x_{12}, x_{51})$, we have $f_0(x_{12}, x_{23}) = f(x_{12}, x_{23})$, so the normal form of $f(x_{12}, x_{51})$ in $\mathrm{Lie}\, P_5$ is

$$f(x_{12}, x_{51}) = f(x_{12}, x_{23}) + F$$

with $F \in K_4$.

Recall that an element $f \in \mathfrak{grt}$ is an $f \in \mathrm{Lie}[x, y]$ that satisfies the pentagon relation

$$f(x_{51}, x_{12}) + f(x_{12}, x_{23}) + f(x_{23}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{51}) = 0.$$

Since it also satisfies $f(x, y) + f(y, x) = 0$, we can write this as

$$\begin{aligned} f(x_{12}, x_{51}) &= f(x_{12}, x_{23}) + f(x_{23}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{51}) \\ &= f(x_{12}, x_{23}) + f(x_{23} - c, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{23} + x_{24} + x_{34}) \\ &= f(x_{12}, x_{23}) + f(-x_{24} - x_{34}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{24} + x_{34}) \end{aligned}$$

where $c = x_{23} + x_{24} + x_{34}$ commutes with x_{23} and x_{34}.

This is nothing other than the normalization of $f(x_{12}, x_{51})$ with respect to the 4th strand.

So an equivalent definition of $\mathfrak{grt}$ is the space of $f \in \mathrm{Lie}[x, y]$ such that the normalization F of $f(x_{12}, x_{51})$ is given by

$$F = f(-x_{24} - x_{34}, x_{34}) + f(x_{34}, x_{45}) + f(x_{45}, x_{24} + x_{34}).$$

Set

$$\begin{cases} x := x_{45} \\ y := x_{24} + x_{34} \\ z := x_{34}, \end{cases}$$

and write $K_4 = \langle x, y, z \rangle$. Then this re-definition of $\mathfrak{grt}$ can be expressed as follows: **the space of** $f \in \mathrm{Lie}[x, y]$ **such that the normalization** F **of** $f(x_{12}, x_{51})$ **is of the form**

$$F(x, y, z) = f(-y, z) + f(z, x) + f(x, y).$$

Lemma. *Let* $f \in \mathrm{Lie}[x, y]$ *and let* F *be the normalization of* $f(x_{12}, x_{51}) \in \mathrm{Lie}\, P_5$*. Then*

$$\begin{cases} F(x, y, 0) = f(x, y) \\ F(x, 0, z) = f(z, x) \\ F(0, y, z) = f(-y, z). \end{cases}$$

So the re-definition of $\mathfrak{grt}$ can be expressed as follows: **the space of $f \in \text{Lie}[x,y]$ whose normalization F has no monomials in all three variables**, i.e. satisfies

$$F(x,y,z) = F(x,y,0) + F(x,0,z) + F(0,y,z).$$

The coefficients of F are all linear combinations of the coefficients of f. Therefore we have:

Dimorphic definition of $\mathfrak{grt}$: the space of polynomials $f \in \mathbb{Q}\langle x,y\rangle$ whose coefficients satisfy two families of linear relations:

(i) the shuffle relations (so $f \in \text{Lie}[x,y]$),

(ii) the relations $(F|w) = 0$ for all mixed monomials w in x, y, z.

Part IV. Normal form of braids in $\mathrm{Lie}\, P_5$

The goals of this research project are:

(i) to find an explicit expression for the coefficients of the mixed monomials in F, and

(ii) to relate them to the stuffle relations, so as to recover Furusho's result $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$ and possibly even prove the conjectured isomorphism of these spaces.

Let d, v and u_1 be derivations of $K_4 \simeq \mathbb{Q}\langle x, y, z\rangle$ defined by

$$\begin{cases} d(x) = 0 \\ v(x) = 0 \\ u_1(x) = 0 \end{cases} \quad \begin{cases} d(y) = [y, x] \\ v(y) = [y + x, z] \\ u_1(y) = 0 \end{cases} \quad \begin{cases} d(z) = 0 \\ v(z) = 0 \\ u_1(z) = [y, z], \end{cases}$$

and for any $w \in \mathbb{Q}\langle x, y, z\rangle$, let $u_2(w) = wy$. Let u be the linear operator $u_1 + u_2$ on $\mathbb{Q}\langle x, y, z\rangle$.

Normalization theorem. *Let $f \in \mathrm{Lie}[x, y]$ and write $f = xf^x + yf^y$. Then*

(i) $f^y(d, u) \cdot y = f(x, y)$.

(ii) $f^y(d + v, u) \cdot y = F(x, y, z)$

This helps compute the coefficients $(F|w)$ for mixed monomials w.

Example. Consider the monomial $yx^a zx^b$. Recall that

$$F(x,y,z) = f^y(d+v,u).$$

The words in d,u,v that can produce this monomial are exactly

$$d^b \cdot v \cdot d^a - d^{b-1}h \cdot ud^a + \sum_{i=0}^{a-1} d^i u \cdot d^b h \cdot d^{a-1-i} - \sum_{i=0}^{a} d^i u \cdot d^{b-1}h \cdot d^{a-i}.$$

Therefore since $F(x,y,z) = f^y(d+v,u)$, the coefficient $(F|yx^a zx^b)$ is given by

$$\big(f^y(d+v,u)|d^b \cdot v \cdot d^a\big) + \sum_{i=0}^{a-1}\big(f^y(d+v,u)|d^i u \cdot d^b v \cdot d^{a-1-i}\big)$$

$$-\big(f^y(d+v,u)|d^{b-1}v \cdot ud^a\big) - \sum_{i=0}^{a}\big(f^y(d+v,u)|d^i u \cdot d^{b-1}v \cdot d^{a-i}\big).$$

But for any monomial $w(d,u,v)$, we have

$$\big(f^y(d+v,u)|w(d,u,v)\big) = \big(f^y(d,u)|w(d,u,d)\big),$$

so this is equal to

$$\begin{aligned}
\big(f^y(d,u)|d^{a+b+1}\big) + \sum_{i=0}^{a-y1} \big(f^y(d,u)|d^i u \cdot d^{a+b-i}\big)& \\
- \big(f^y(d,u)|d^b u d^a\big) - \sum_{i=0}^{a}\big(f^y(d,u)|d^i u \cdot d^{a+b-i}\big)& \\
= \big(f^y(d,u)|d^{a+b+1}\big) - \big(f^y(d,u)|d^b u d^a\big) - \big(f^y(d,u)|d^a u d^b\big)& \\
= \big(f^y(x,y)|x^{a+b+1}\big) - \big(f^y(x,y)|x^b y x^a\big) - \big(f^y(x,y)|x^a y x^b\big)& \\
= \big(f(x,y)|y x^{a+b+1}\big) - \big(f(x,y)|y x^b y x^a\big) - \big(f(x,y)|y x^a y x^b\big)&.
\end{aligned}$$

Thus the coefficient of yx^azx^b in $F(x,y,z)$ is zero if and only if f satisfies

$$(f|yx^byx^a) + (f|yx^ayx^b) - (f|yx^{a+b+1}) = 0.$$

Setting $g(x,y) = f(x,-y)$, this means that

$$(g|yx^byx^a) + (g|yx^ayx^b) + (g|yx^{a+b+1}) = 0,$$

which is exactly the stuffle relation $st((a),(b))$ for g.

This example shows how **stuffles can arise naturally from normalizing elements of** Lie P_5**.**

Theorem. *Let $f \in \mathrm{Lie}[x,y]$ and let $F(x,y,z)$ be the normalization of $f(x_{12},x_{51})$ in $\mathrm{Lie}\, P_5$. Then*
(i) the coefficient of $yx^{a_1}\cdots yx^{a_r}zx^b$ in F is $st\big((b),(a_1,\ldots,a_r)\big)$.
(ii) the coefficient of $yx^{a_1}\cdots yx^{a_r}zx^{b_1}\cdots zx^{b_s}$ in F (with $s \le r$) is equal to $st\big((b_s,\ldots,b_1),(a_1,\ldots,a_r)\big)$ up to adding stuffles whose left-hand sequence is of length $< s$.

General rule. Let $A = a_1,\ldots,a_r$ and let $y_A = yx^{a_1}\ldots yx^{a_r}$. Then

$$\begin{aligned}
(F|y_A\cdot zx^b) &= st\big((b),(A)\big)\\
(F|y_A\cdot zx^bzx^c) &= st\big((c,b),(A)\big) - st\big((c),(A,b)\big) - st\big((c+b),(A)\big)\\
(F|y_A\cdot zx^bzx^czx^d) &= st\big((d,c,b),(A)\big) - st\big((d+c,b),(A)\big)\\
&\quad - st\big((d,c+b),(A)\big) + st\big((d+c+b),(A)\big)\\
&\quad - st\big((d,c),(A,b)\big) + st\big((d+c),(A,b)\big)\\
&\quad - st\big((d),(A,b,c)\big).
\end{aligned}$$

Corollary. *If the mixed monomials in $F(x,y,z)$ are all zero, then all the stuffle relations hold for f.*

Thus our method recovers Furusho's result $\mathfrak{grt} \hookrightarrow \mathfrak{ds}$.

In order to prove that $\mathfrak{grt} \simeq \mathfrak{ds}$, we need to prove that the coefficients of all mixed monomials in F are sums of stuffles, or that they can be deduced from the special monomials in the theorem, in which the z's follow the y's. Ongoing...