Fundamental Group of Quantum Field Theory

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September 2, 2020

• Want to introduce a notion of ‘fundamental’ group of QFT.
• Motivations? Fantasy?
• Need to have a ‘definition’ of QFT, a working definition at least.
• Need to construct such a group from such a definition of QFT.
1 Various constructions of fundamental group

Let $X$ be a path connected topological space with some nice properties.

1.1 Definition

Let $p_1(X; x, y)$ be the set of homotopy types of paths from $x$ to $y$, $x, y \in X$.

- $p_1(X; x, x)$ is a group denoted by $\pi_1(X, x)$;
- $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic, and $p_1(X; x, y)$ is the set of isomorphisms;
- $p_1(X; x, y)$ is a torsor of fundamental group.

1.2 Via fiber functors

Let $\text{Cov}(X)$ be the category of covering spaces over $X$.

- Taking the fiber over $x$ for each covering space defines a functor, called a fiber functor, $F_x : \text{Cov}(X) \to \text{Set}$.
- Let $\text{Iso}(F_x, F_y)$ be the set of invertible natural transformations from $F_x$ to $F_y$.
- There is an isomorphism $\text{Iso}(F_x, F_y) \simeq p(X; x, y)$ and $\text{Aut}(F_x) \simeq \pi_1(X; x)$. 
1.3 Via the homotopy category \( \text{hoTop}_* \)

Let \( \text{hoTop}_* \) be the homotopy category of based topological spaces, \( X_* \), \( X'_* \) etc.

- Objects are based topological spaces;
- Morphisms are homotopy types of base point preserving continuous maps:
  \[ [X'_*, X_] : \text{the set of morphisms from } X'_* \text{ to } X_* \]

The based loop space \( \Omega(X_*) \) defines a contravariant functor from \( \text{hoTop}_* \) to the category \( \text{Grp} \) of groups:

\[
\mathcal{F}^{X_*} := [-, \Omega(X_*)] : \text{hoTop}_* \to \text{Grp}
\]

We have \( \mathcal{F}^{X_*}(pt) \simeq \pi_1(X, x) \)

\[
\mathcal{F}^{X_*}(pt) = [pt, \Omega(X_*)] \simeq [S^1, X_*].
\]

In other words \( \mathcal{F}^{X_*} \) is a presheaf of groups over the homotopy category \( \text{hoTop}_* \), that is represented by the based loop space \( \Omega(X_*) \).
1.4 Digression I

Let \(ho\text{-ccdg}C(k)\) be the homotopy category of differential graded cocommutative coalgebras over \(k\), a field of characteristic zero such as \(\mathbb{Q}, \mathbb{R}, \mathbb{C}\).

- Objects are ccdg-Coalgebras over \(k\);
- Morphisms are homotopy types of morphisms of ccdg-Coalgebra.
- \(\text{Hom}_{ho\text{-ccdg}C(k)}(C, C')\): the set of morphisms from \(C\) to \(C'\).
- \(k\) is naturally a ccdg-Coalgebra, denoted by \(k^\vee\), with the degree concentrated to zero.

Let \(C_*\) be a ccdg-Coalgebra with a coaugmentation.

- The Adams cobar construction provides us a ccdg-Hopf algebras \(\Omega(C_*)\).
- We have a contravariant functor
  \[
  \hat{\mathcal{C}}^C : = \text{Hom}_{ho\text{-ccdg}C(k)}(-, \Omega(C_*)) : ho\text{-ccdg}C(k) \to \text{Grp}
  \]
- We call the group \(\hat{\mathcal{C}}^C(k^\vee) = \text{Hom}_{ho\text{-ccdg}C(k)}(k^\vee, \Omega(C_*))\) the fundamental group of \(C_*\).

In other words \(\hat{\mathcal{C}}^C\) is a presheaf of groups over the homotopy category ccdg-Coalgebras that is represented by the cobar construct \(\Omega(C_*))\) of \(C_*\).

We shall use this construction for the would-be fundamental group of QFT.
1.5 Via the rational homotopy category $Qho\text{Top}_*$

The rational homotopy theory, founded by Quillen and Sullivan, replace $ho\text{Top}_*$ with something more manageable $Qho\text{Top}_*$ after forgetting something like torsions.

- For any $X_*$ we have an augmented cdg-Algebra $A_{pl}(X)_*$ over $\mathbb{Q}$ by Sullivan.
- A $k$-rational Sullivan model of $X_*$ is any augmented cdg-Algebra over $k$ quasi-isomorphic to $A_{pl}(X)_* \otimes \mathbb{Q} k$.
- The cdg-Algebra of smooth differential forms on a smooth based and connected manifold is a $\mathbb{R}$-rational Sullivan model.
- A coaugmented ccdg-Coalgebra $C_*$ over $k$ is a $k$-rational Quillen model of $X_*$ if its dual is a $k$-rational Sullivan model of $X_*$. 

If $C_*$ is a $\mathbb{R}$-rational Quillen model of a smooth based and connected manifold $X_*$, the group

$$\hat{\pi}_1(C_*) = \text{Hom}_{bcdgC(k)}(k^\vee, \Omega(C_*))$$

is isomorphic to the pro-unipotent fundamental group $\hat{\pi}_1(X_*)$, constructed by Chen via the iterated line integrals in the dual picture.

Let $A_*$ be a augmented cdg-Algebra of differential forms on $X_*$, the bar construction gives us a cdg-Hopf algebra $B(A_*)$, which define a covariant functor

$$\hat{\mathcal{G}}_{A_*} = \text{Hom}_{bcdgA(\mathbb{R})}(B(A_*), -) : ho\text{cdgA}(\mathbb{R}) \to \text{Grp}$$

such that $\hat{\mathcal{G}}_{A_*}(\mathbb{R}) \simeq \hat{\pi}_1(X_*)$
1.6 Digression II

Return to the presheaf of groups over the homotopy category ccdg-Coalgebras

\[ \hat{\mathcal{H}}^C := \text{Hom}_{\text{ccdg-C}(k)}(-, \Omega(C_*)) : \text{hoccdgC}(k) \to \text{Grp}, \]

that is represented by the cobar construction \( \Omega(C_*) \) of a coaugmented ccdg-Colalgebra \( C_* \).

We are looking for a sort of (dual of dg) Tannakian picture. [with JH Lee]

We consider linear representations of \( \hat{\mathcal{H}}^C \).

- A linear representation is equivalent to a dg-module over the ccdg-Hopf algebra \( \Omega(C_*) \) over \( k \).
- Linear representations of \( \hat{\mathcal{H}}^C \) form a tensor dg-category, which is equivalent to a tensor dg-category \( \text{dgMod}(\Omega(C_*)) \) of dg-modules over \( \Omega(C_*) \).
- We have the forgetful functor \( \varpi : \text{dgMod}(\Omega(C_*)) \to \text{Ch}(k) \), which is a tensor dg-functor, to the underlying tensor dg-category \( \text{Ch}(k) \) of chain complexes over \( k \).
- We have a presheaf of groups over the homotopy category ccdg-Coalgebras

\[ Z_0 \text{Aut}^\otimes(- \otimes \varpi) : \text{hoccdgC}(k) \to \text{Grp} \]

and an isomorphism \( \hat{\mathcal{H}}^C \simeq Z_0 \text{Aut}^\otimes(\varpi) \).

I claim that a (perturbative) QFT with a corresponds to a linear representations of \( \hat{\mathcal{H}}^C \), where the relevant coaugmented ccdg-Colagebra \( C_* \) is that of homotopy equivariant \( \hbar \)-de Rham coalgebra on the space of off-shell quantum observables with an infinitesimal symmetry governed by a unital \( sL_{\infty} \)-algebra.
2 On perturbative QFT

Let \( k \) be a field of characteristic zero. We regard the Planck constant \( \hbar \) as a formal parameter. Let \( X \) be a \( \mathbb{Z} \)-graded vector space over \( k \), whose grading is specified by the ghost number \( gh \).

A perturbative quantum field theory is governed by a morphism

\[
\begin{array}{c}
(\mathcal{X}[\hbar], 1_X, \ell) \longrightarrow (k[\hbar], 1, 0)
\end{array}
\]

of topologically-free unital \( sL_\infty \)-algebras over \( k[\hbar] \):

- \( \mathcal{X}[\hbar] \) is the \( \mathbb{Z} \)-graded topologically-free \( k[\hbar] \)-module of off-shell quantum observables with quantum symmetry encoded by the \( sL_\infty \)-structure \( \ell \) whose unit \( 1_X \in \mathcal{X}^0 \) corresponds to a vacuum.

- \( \ell : \mathcal{X}(\mathcal{X}[\hbar]) \to \mathcal{X}[\hbar] \) is a \( k[\hbar] \)-linear map of \( gb = 1 \)

- \( K : \ell : \mathcal{X}[\hbar] \to \mathcal{X}[\hbar] \) is the quantum differential, making \( (\mathcal{X}[\hbar], 1_X, K) \) a pointed cochain complex, i.e., \( K \circ K = 0 \), over \( k[\hbar] \).

- \( k[\hbar] \) with the multiplicative unit 1 is a unital \( sL_\infty \)-algebra with the zero \( sL_\infty \)-structure 0;

- \( \kappa : \mathcal{X}(\mathcal{X}[\hbar]) \to k[\hbar] \) is a \( k[\hbar] \)-linear map of \( gb = 0 \) is called quantum cumulant functional.

- \( c : \mathcal{X}[\hbar] \to k[\hbar] \) is the quantum expectation, which is a pointed cochain map \( (\mathcal{X}[\hbar], 1_X, K) \xrightarrow{c} (k[\hbar], 1, 0) \); \( c(1_X) = 1 \), \( c \circ K = 0 \).

- \( \kappa_2 : S^2 \mathcal{X}[\hbar] \to k[\hbar] \) is the quantum covariance and etc. etc.

The relation \( c \circ K = 0 \) says that \( K \) is an infinitesimal symmetry of quantum expectation \( c \).
A structure of $sL_{\infty}$-algebra $(\mathcal{X}[\hbar], \ell)$ on $\mathcal{X}[\hbar]$ is equivalent to a structure $(\overline{\mathcal{X}}(\mathcal{X}[\hbar]), \delta_\ell)$ ccdg-Coalgebra on the reduced symmetric coalgebra $\overline{\mathcal{X}}(\mathcal{X}[\hbar])$, such that an $sL_{\infty}$-morphism is equivalent to a ccdg-Coalgebra map.

- From the $sL_{\infty}$-structure $\ell : \overline{\mathcal{X}}(\mathcal{X}[\hbar]) \to \mathcal{X}[\hbar]$, we define $\delta_\ell : \overline{\mathcal{X}}(\mathcal{X}[\hbar]) \to \overline{\mathcal{X}}(\mathcal{X}[\hbar])$ such that, for all $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{X}[\hbar]$,

$$\delta_\ell(x_1 \otimes \cdots \otimes x_n) = \sum_{S_1 \cup S_2} \varepsilon(S_1 \cup S_2)x_{S_1} \otimes \ell(x_{S_2}).$$

Then we have $\delta_\ell \circ \delta_\ell = 0$.

- From the $sL_{\infty}$-morphism $\kappa : \overline{\mathcal{X}}(\mathcal{X}[\hbar]) \to \mathbb{k}[\hbar]$, we define $\overline{\mathcal{X}}(\mathcal{X}[\hbar]) \to \overline{\mathcal{X}}(\mathcal{X}[\hbar])$ such that, for all $n \geq 1$ and $x_1, \ldots, x_n \in \mathcal{X}[\hbar]$,

$$\overline{\mathcal{X}}(\mathcal{X}[\hbar]) \otimes \cdots \otimes \kappa(x_{S_1}) \otimes \cdots \otimes \kappa(x_{S_2}).$$

Then we have $\overline{\mathcal{X}}(\mathcal{X}[\hbar]) \otimes \cdots \otimes \kappa(x_{S_1}) \otimes \cdots \otimes \kappa(x_{S_2})$.

Corollary.

- Define $\ell^h : \overline{\mathcal{X}}(\mathbb{k}[\hbar]) \to \mathbb{k}[\hbar]$ such that $\delta_{\ell^h}(x_1 \otimes \cdots \otimes x_n) = (-\hbar)^{n-1} \delta_\ell(x_1 \otimes \cdots \otimes x_n)$. Then we also have $\delta_{\ell^h} \circ \delta_{\ell^h} = 0$.

- Define $\kappa^h : \overline{\mathcal{X}}(\mathbb{k}[\hbar]) \to \mathbb{k}[\hbar]$ such that $\kappa^h(x_1 \otimes \cdots \otimes x_n) = (-\hbar)^{n-1} \kappa(x_1 \otimes \cdots \otimes x_n)$. Then we also have $\overline{\mathcal{X}}(\mathbb{k}[\hbar]) \otimes \cdots \otimes \kappa(x_{S_1}) \otimes \cdots \otimes \kappa(x_{S_2}) = 0$. 

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The quantum correlation functional $\mu : S(\mathbb{K}[[\hbar]]) \to \mathbb{K}[[\hbar]]$ is defined by

$$\mu := \pi^k \circ \mathfrak{p}(\kappa^k),$$

where $\pi^k(a_1 \otimes \cdots \otimes a_n) = a_1 \cdots a_n$ for all $n \geq 1$ and $a_1, \ldots, a_n \in \mathbb{K}[[\hbar]]$

For example, we have $\mu_1 = \kappa_1 \equiv \mathfrak{c}$ is the quantum expectation, and

$$\mu_2(x_1, x_2) = \kappa_1(x_1) \cdot \kappa_1(x_2) - \hbar \kappa_2(x_1, x_2),$$
$$\mu_3(x_1, x_2, x_3) = \kappa_1(x_1) \cdot \kappa_1(x_2) \cdot \kappa_1(x_3) - \hbar \kappa_2(x_1, x_2) \cdot \kappa_1(x_3) - \hbar \kappa_1(x_1) \cdot \kappa_2(x_2, x_3)$$
$$- \hbar(-1)^{|x_1||x_3|} \kappa_1(x_2) \cdot \kappa_2(x_1, x_3) + \hbar^2 \kappa_3(x_1, x_2, x_3).$$

From the condition that $\kappa$ being an unital $sL_{\infty}$-morphism, we have

- $\mu(1_x) = 1$ and $\mu(1_x \otimes x_1 \otimes \cdots \otimes x_n) = \mu(x_1 \otimes \cdots \otimes x_n)$ for all $n \geq 1$;
- $\mu \circ \delta_{\ell^3} = 0$, which says that $\delta_{\ell^3}$ is an infinitesimal symmetry of the quantum correlation $\mu$. 

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We demand that the quantum correlation functional \( \mu : S(\mathcal{X})[[\hbar]] \to k[[\hbar]] \) factors through the quantum expectation \( c : \mathcal{X}[[\hbar]] \to k[[\hbar]] \) via a cochain map \( \pi : S(\mathcal{X})[[\hbar]] \to \mathcal{X}[[\hbar]] \), called quantum correlator:

\[
\begin{align*}
S(\mathcal{X}[[\hbar]]) & \xrightarrow{\mu} k[[\hbar]], \\
\pi & \xrightarrow{\mu = c \circ \pi, \quad \pi \circ \delta_\ell = K \circ \pi.} \mathcal{X}[[\hbar]]
\end{align*}
\]

Note that \( \pi_1 = 1_{\mathcal{X}[[\hbar]]} \). Combining with the definition \( \mu := \pi^k \circ \beta(\kappa^h) \), we have \( \pi^k \circ \beta(\kappa^h) = c \circ \pi \).

If we fix a quantum correlator \( \pi \), we have

- the \( sL_\infty \)-structure \( \ell \) is determined by the quantum differential \( K \) by the formula \( \pi \circ \delta_\ell = K \circ \pi \). For example, we have \( \ell_1 = K \) and

\[
-\hbar \ell_2 (x_1, x_2) = K \pi_2 (x_1, x_2) - \pi_2 (K x_1, x_2) - \pi_2 (x_1, K x_2),
\]

\[
\hbar \ell_3 (x_1, x_2, x_3) = K \pi_3 (x_1, x_2, x_3) - \pi_3 (K x_1, x_2, x_3) - \pi_3 (x_1, x_2, K x_3) - \hbar \pi_2 (x_2 (x_1, x_2), x_3) - \hbar \pi_2 (x_2 (x_2, x_3), x_3) - \hbar (1)^{x_1 | x_2} \pi_2 (x_2, x_2 (x_1, x_3)).
\]

- the quantum cumulant functional \( \kappa : S(\mathcal{X})[[\hbar]] \to k[[\hbar]] \) is determined by the quantum expectation \( c \) by the formula \( \pi^k \circ \beta(\kappa^h) = c \circ \pi \). For example, we have \( \kappa_1 = c \) and

\[
-\hbar \kappa_2 (x_1, x_2) = c (\pi_2 (x_1, x_2)) - c (x_1) \cdot c (x_2),
\]

\[
\hbar \kappa_3 (x_1, x_2) = c (\pi_3 (x_1, x_2, x_3)) - c (x_1) \cdot c (x_2) \cdot c (x_3) + \hbar \kappa_2 (x_1, x_2) \cdot c (x_3) + \hbar c (x_1) \cdot \kappa_2 (x_2, x_3) + \hbar (1)^{x_1 | x_2} c (x_2) \cdot \kappa_2 (x_1, x_3),
\]

\[
+ \hbar (1)^{x_1 | x_3} c (x_2) \cdot \kappa_2 (x_1, x_3).
\]
Finally, we impose the following $\hbar$-condition to a quantum correlator $\pi : S(\mathcal{X})[[\hbar]] \to \mathcal{X}[[\hbar]]$:

- There is a family $m_0^2, m_0^3, \ldots$ of $k[[\hbar]]$-linear maps
  
  $$m^2_{k+2} : S^k \mathcal{X}[[\hbar]] \otimes \mathcal{X}[[\hbar]] \otimes \mathcal{X}[[\hbar]] \to \mathcal{X}[[\hbar]] \quad k = 0, 1, 2, \ldots$$

  such that

  $$\pi_{n+2}(x_1 \otimes \ldots \otimes x_n \otimes y \otimes z) = \sum_{S_1 \sqcup S_2 = \{x\}} (-\hbar)^{|S_1|} \pi(S_1 \sqcup S_2) \pi_{|S_1|+1}(x_{S_1} \otimes m_{|S_1|+2}^n (x_{S_2} \otimes y \otimes z))$$

- For example we have $m_2^0 = \pi_2$ and

  $$m_2^2 = \pi_2,$$

  $$-\hbar m_2^2 = \pi_3 - m_2^0 \circ (1 \otimes m_2^0)$$

  etc.

Finally we call the resulting tuple $\mathcal{X}_{QFTA} = (\mathcal{X}[[\hbar]], 1, \mathcal{X}, m_2^2, m_3^3, \ldots)$ a structure of (pertubative) QFT algebra on $\mathcal{X}$ over $k$. 
Some examples.

- A QFT algebra is called binary if \( m^\ell_k = 0 \) for all \( k \geq 3 \).
- A binary QFT algebra \( (\mathcal{X}[\hbar], 1_X, \ell, m^\ell_k) \) is a BV-QFT algebra if \( m^\ell_2 \) does not depend on \( \hbar \) and \( \ell_k = 0 \) for all \( k \geq 3 \). For example:

  \[
  \mathcal{K} = -\hbar \Delta_{BV} + (S, -)_{BV}, \quad \ell = (-, -)_{BV}
  \]

  where \( \Delta_{BV} \) is the BV operator and \((-, -)_{BV}\) is the BV-bracket, and, \( S \) is a quantum master action.

\[
-\hbar \Delta + \frac{1}{\hbar} (S, S) = 0.
\]

- \( k_{QFTA} = (k[[\hbar]], 1, \partial, \cdot) \) is a QFT algebra.

We have the natural notions of morphisms of QFT algebras and homotopy of morphisms, so that we can form the (homotopy)category \((\text{bo})\mathcal{QFTA}(k)\) QFT algebras over \( k \).

A quantum expectation \( e : \mathcal{X}[\hbar] \to k[[\hbar]] \) is a morphism of QFT algebra from \( \mathcal{X}_{QFTA} \) to \( k_{QFTA} \). A QFT is such an arrow \( \mathcal{X}_{QFTA} \to k_{QFTA} \).

There is an ample room as well as computational needs to extend the notion of QFT-algebra to that of homotopy QFT-algebra.
Let $C^o(\mathcal{X}[\hbar]) = S^o(\mathcal{X}[\hbar]) \otimes \Lambda^o(\mathcal{X}[\hbar])$ be the tensor product of the symmetric and exterior coalgebras over $k[[\hbar]]$ generated by $\mathcal{X}[\hbar]$.

• bigraded by $(gh, fm)$ where an element in $C_k(\mathcal{X}[\hbar]) := A^k \mathcal{X}[\hbar] \otimes S(\mathcal{X}[\hbar])$ is assigned to $fm = -k$.

• Koszul differential $\partial : C_k(\mathcal{X}[\hbar]) \to C_{k-1}(\mathcal{X}[\hbar])$ of $(gh, fm) = (0, 1)$ such that $(C^o(\mathcal{X}[\hbar]), -\hbar \partial)$ is a ccdg-Coalgebra.

From an $sL_\infty$-structure $(\mathcal{X}[\hbar], \ell)$ of on $\mathcal{X}[\hbar]$, we have a ccdg-Coalgebra $(S^o(\mathcal{X}[\hbar]), \delta_\ell)$ over $k[[\hbar]]$.

Then the differential $\delta_\ell$ on $S(\mathcal{X}[\hbar])$ has a unique extension to a differential $\mathcal{D}_\ell$ on $C(\mathcal{X}[\hbar])$ with $(gh, fm) = (1, 0)$ such that

• $(C(\mathcal{X}[\hbar]), \mathcal{D}_\ell)$ is a dg-comodule over the ccdg-Coalgebra $(S^o(\mathcal{X}[\hbar]), \delta_\ell)$;

• $(C^o(\mathcal{X}[\hbar]), -\hbar \partial + \mathcal{D}_\ell)$ is a ccdg-Coalgebra over $k[[\hbar]]$.

We use the notation

$$C_{hdd}(\mathcal{X}[\hbar], \ell) = (C^o(\mathcal{X}[\hbar]), -\hbar \partial + \mathcal{D}_\ell)$$

call it the homotopy equivariant $\hbar$-de Rham coalgebra cogenerated by the $sL_\infty$-algebra $(\mathcal{X}[\hbar], \ell)$.

Restoring a unit $1_\mathcal{X}$, we are led to completed and coaugmented homotopy equivariant $\hbar$-de Rham Coalgebra $\hat{C}_{hdd}(\mathcal{X}[\hbar], 1_\mathcal{X}, \ell)$, whose coaugmentation is induced from $1_\mathcal{X}$.
A cotwisting coefficient system over $C_{\hbar R}(\mathfrak{X}[\hbar], \ell)$ is a tuple $(\mathcal{V}[\hbar], \omega)$ where

- $\mathcal{V}[\hbar]$ is a cochain complex over $k[[\hbar]]$ and

- $\omega : C(\mathfrak{X}[\hbar]) \otimes \mathcal{V}[\hbar] \to \mathcal{V}[\hbar]$ is a cotwisting matrix of the total degree $gh + fm = 1$, satisfying the following integrability condition $\mathcal{R}(\omega) = 0$, where

$$\mathcal{R}(\omega) := d_{\mathcal{V}[\hbar]} \circ \omega + \omega \circ (-\hbar \partial \otimes \mathcal{I} + D_{\hbar} \otimes \mathcal{I} + \mathcal{I} \otimes d_{\mathcal{V}[\hbar]}) + \omega \circ (\mathcal{I} \otimes \omega) \circ (\Delta \otimes \mathcal{I})$$

See the diagram

$$\begin{array}{c}
C(\mathfrak{X}[\hbar]) \otimes \mathcal{V}[\hbar] \xrightarrow{\nabla_{\omega}} C(\mathfrak{X}[\hbar]) \otimes C(\mathfrak{X}[\hbar]) \otimes \mathcal{V}[\hbar] \xrightarrow{\otimes \omega} C(\mathfrak{X}[\hbar]) \otimes \mathcal{V}[\hbar] \xrightarrow{\omega} \mathcal{V}[\hbar]
\end{array}$$

Equivalently, we have the cotwisted cofree comodule $(C(\mathfrak{X}[\hbar]), \mathcal{V}[\hbar], h^{\nabla})$ over $C_{\hbar R}(\mathfrak{X}[\hbar], \ell)$ with the cotwisted differential

$$h^{\nabla} = -h \partial \otimes \mathcal{I} + D_{\hbar} \otimes \mathcal{I} + \mathcal{I} \otimes d_{\mathcal{V}[\hbar]} + (\mathcal{I} \otimes \omega) \circ (\Delta \otimes \mathcal{I})$$

satisfying $h^{\nabla} \circ h^{\nabla} = 0$.

We say such a cotwising coefficient system over $C_{\hbar R}(\mathfrak{X}[\hbar], \ell)$ is tangential if $\mathcal{V}[\hbar] = \mathfrak{X}[\hbar]$. 

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A homotopy QFT algebra is a tangential cotwisting coefficient system \((X[[\h]], \omega)\) over the completed and coaugmented homotopy equivariant \(\h\)-de Rham coalgebra \(C_{h\text{dR}}(X[[\h]], 1, \ell)\) cogenerated by a \(sL_\infty\)-algebra \((X[[\h]], \ell)\) over \(k[[\h]]\).

Note that the total degree of \(\omega : \hat{C}(X[[\h]]) \otimes X[[\h]] \to X[[\h]]\) is \(gb + fm = 1\). Decomposing \(\hat{\omega}\) into \(\omega_k : C_k(X[[\h]]) \otimes X[[\h]] \to X[[\h]]\) of degree \((gb/fm) = (1-k, k)\), we obtain a family \(\omega_0, \omega_1, \omega_2, \ldots\).

Define \(m_{n+k+1}^{1-k} : S^n X[[\h]] \otimes \Lambda^n X[[\h]] \otimes X[[\h]] \to X[[\h]]\) of \(gb = 1-k\) such that

\[
m_{n+k+1}^{1-k} = \omega_k(x_1 \otimes x_n \otimes x_{n+1} \land \ldots \land x_{n+k} \otimes x_{n+k+1})
\]

We obtain the following set of multi-linear operations on \(X[[\h]]\), indexed by the ghost number and arity:

<table>
<thead>
<tr>
<th>(gb)</th>
<th>(arity)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\ell)</td>
<td>+1</td>
<td>(K)</td>
<td>(\ell_2)</td>
<td>(\ell_3)</td>
<td>(\ell_4)</td>
<td>\ldots</td>
</tr>
<tr>
<td>(\omega_1)</td>
<td>0</td>
<td>(m_2^0)</td>
<td>(m_3^0)</td>
<td>(m_4^0)</td>
<td>\ldots</td>
<td></td>
</tr>
<tr>
<td>(\omega_2)</td>
<td>(-1)</td>
<td>(m_2^{-1})</td>
<td>(m_3^{-1})</td>
<td>(m_4^{-1})</td>
<td>\ldots</td>
<td></td>
</tr>
<tr>
<td>(\omega_3)</td>
<td>(-2)</td>
<td>\ldots</td>
<td>(m_4^{-2})</td>
<td>\ldots</td>
<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>

satisfying the set of relations summarized by the integrability \(\h^n \nabla \omega \circ \h^n \nabla \omega = 0\).

We can form the (homotopy)category \((ho)QFTA_\infty(k)\) of homotopy QFT-algebras. Many nice things happen there...

But we have also opened a door to something unknown.