A Microlocal Approach to Renormalisation in Stochastic PDEs

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joint work with C. Dappiaggi, N. Drago and P. Rinaldi

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Higher Structures Emerging from Renormalisation, ESI

Two communities

This talk is based on a recent arXiv preprint

- *A Microlocal Approach to Renormalisation in Stochastic PDEs*
  by C. Dappiaggi, N. Drago, P. Rinaldi and L.Z.

which is the first step in a project aiming at bringing two distinct research areas closer:

- Singular SPDEs
- Algebraic Quantum Field Theory

In both areas, the notion of renormalisation plays a key role.

The Stochastic Quantisation

In the 80s some theoretical physicists (Giorgio Parisi, Gianni Jona-Lasinio), introduced SPDEs in their models, e.g.

$$\Delta\psi - \psi^3 + \xi = 0, \qquad x \in \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d,$$

where ξ is a white noise, namely a Gaussian random distribution on $\mathbb{R}^d$ characterised by

$$\mathbb{E}\left[\exp\left(\langle \xi, f\rangle\right)\right] = \exp\left(\frac{1}{2}\|f\|^2_{L^2(\mathbb{R}^d)}\right),$$

for all $f \in C_c^\infty(\mathbb{R}^d)$.

(The original equation was parabolic but in this talk we consider the elliptic version for simplicity. The paper considers both in a more general setting.)

The Stochastic Quantisation: $d = 4$

For example for $d = 4$

$$\Delta\psi - \psi^3 + \xi = 0, \quad x \in \mathbb{R}^4/\mathbb{Z}^4 = \mathbb{T}^4,$$

Here ψ is expected to be a random distribution, namely $\psi = \psi(\xi) \in \mathcal{D}'(\mathbb{T}^4)$.

More precisely ψ is expected to belong a.s. to some negative Sobolev space $H^{-\kappa}$, $\kappa > 0$.

Therefore ψ^3 is ill-defined.

This is what makes the above SPDE singular: not only existence and uniqueness are problematic, the very same notion of solution is unclear.

The Stochastic Quantisation: $d = 5$

For example for $d = 5$

$$\Delta\psi - \psi^3 + \xi = 0, \quad x \in \mathbb{R}^5/\mathbb{Z}^5 = \mathbb{T}^5,$$

Here ψ is expected to be a random distribution and to belong to some negative Sobolev space $H^{-1/2}$ and again ψ^3 is ill-defined.

The higher the dimension d, the lower the regularity of ψ.

Singular SPDEs

In the last 8 years there have been many progresses in the study of singular SPDEs, mainly driven by Martin Hairer and Massimiliano Gubinelli.

The result is that one has

- a proper notion of solutions and...
- ...existence and uniqueness of such solutions for a class of equations
- a non-perturbative approach

The outcome is a map $\xi \mapsto \psi = \psi(\xi)$: a random variable.

Renormalisation in Singular SPDEs

The construction of solutions works as follows: one considers a regularisation $\xi_\varepsilon = \rho_\varepsilon * \xi$ of the white noise and the random PDE

$$\Delta\hat{\psi}_\varepsilon - \hat{\psi}_\varepsilon^3 + \left(\frac{C_1}{\varepsilon} + C_2 \log\varepsilon + R\right)\hat{\psi}_\varepsilon + \xi_\varepsilon = 0$$

(here $d = 5$). The constants C_1, C_2 are fixed, while $R \in \mathbb{R}$ can vary. The theory shows that

$$\hat{\psi}_\varepsilon \to \hat{\psi}, \qquad \varepsilon \to 0$$

as a distribution, and $\hat{\psi}$ is the renormalised solution to the equation.

Note that $\hat{\psi} = \hat{\psi}(R)$.

The approach is non-perturbative.

Renormalisation in Singular SPDEs

$$\Delta\hat{\psi}_\varepsilon - \hat{\psi}_\varepsilon^3 + \left(\frac{C_1}{\varepsilon} + C_2 \log\varepsilon + R\right)\hat{\psi}_\varepsilon + \xi_\varepsilon = 0$$

Here we have at the same time

- a finite family of renormalisation parameters
- a uniqueness statement once the renormalisation parameters are fixed.

In particular:

- for every choice of the renormalisation parameters, there is a unique solution.
- for different choices of the renormalisation parameters, the renormalised solutions can possibly differ.

Algebraic QFT

In our recent paper, we use techniques borrowed from AQFT in order to compute the correlation functions which are expected for the solution to our equation

$$\Delta\psi - \lambda\psi^3 + \xi = 0, \qquad x \in \mathbb{R}^d/\mathbb{Z}^d = \mathbb{T}^d.$$

Also with this approach one has to deal with ill-defined distributions and ambiguities due to renormalisation.

Now the approach is perturbative, which is related to the presence of the (small) parameter λ.

Mild formulation

We write the equation in its mild formulation

$$\psi = G * \xi - \lambda\, G * \psi^3$$

where G is the Green function of Δ. Now we consider the functional $C^\infty \ni \varphi \mapsto \Psi = \Psi(\varphi) \in C^\infty$

$$\Psi = \varphi - \lambda\, G * \Psi^3.$$

Note that this is a completely deterministic setting.

Moreover $G * \xi \in \mathscr{D}'$ has been replaced by $\varphi \in C^\infty$.

How can this different problem help to describe the original one?

A power series

$$\Psi = \varphi - \lambda\, G * \Psi^3.$$

Let us write $\Psi = \Psi(\lambda)$ as a formal power series

$$\Psi[\![\lambda]\!] = \sum_{j\geq 0} \lambda^j F_j\,,$$

with $F_j = F_j(\varphi)$ a C^∞-valued, polynomial functional on C^∞.

The equation yields the explicit expressions

$$F_0 = \varphi, \qquad F_1 = -G * \varphi^3, \qquad F_2 = 3G * (\varphi^2\, G * \varphi^3)\,,$$
$$F_j = -\sum_{j_1+j_2+j_3=j-1} G * (F_{j_1} F_{j_2} F_{j_3})\,, \qquad j \geq 3\,.$$

Ill-defined products

Recall:

$$\Psi[\![\lambda]\!] = \sum_{j\geq 0} \lambda^j F_j \,,$$

$$F_0 = \varphi, \qquad F_1 = -G * \varphi^3, \qquad F_2 = 3G * (\varphi^2 \, G * \varphi^3) \,,$$
$$F_j = - \sum_{j_1+j_2+j_3=j-1} G * (F_{j_1} F_{j_2} F_{j_3}) \,, \qquad j \geq 3 \,.$$

Now, the series may not be convergent, and this is the well known conundrum of the perturbative approach.

However this is not the only problem: if we want to go back and replace $\varphi \in C^\infty$ with $G * \xi \in \mathcal{D}'$, then all terms bar F_0 are ill-defined since they contain powers of $G * \xi \in \mathcal{D}'$.

Wick polynomials

Let now $(X_1, \dots, X_k)$ be a $\mathbb{R}^k$-valued random variable such that $\mathbb{E}[|X^n|] < +\infty$ for all $n \in \mathbb{N}^k$.

We have a unique linear map $W : \mathbb{R}[x_1, \dots, x_k] \to \mathbb{R}[x_1, \dots, x_k]$ such that

$$W(1) = 1, \qquad \frac{\mathrm{d}}{\mathrm{d}x_i} \circ W = W \circ \frac{\mathrm{d}}{\mathrm{d}x_i}, \qquad \mathbb{E}(W(X^n)) = 0,$$

for all $i \in \{1, \dots, k\}$, $n \in \mathbb{N}^k \setminus \{0\}$. We call $W(x^n)$ the Wick polynomial of degree $n \in \mathbb{N}^k$.

Deformed products

We can define two natural deformed products $\bullet_1, \bullet_2$ on $\mathbb{R}[x_1, \ldots, x_k]$ (among many others of course) as follows

$$x^n \bullet_1 x^m = W^{-1}(W(x^n) \cdot W(x^m)),$$

$$x^n \bullet_2 x^m = W(W^{-1}(x^n) \cdot W^{-1}(x^m)).$$

For example if $\mathbb{E}[X_i] = 0$, $C_{ij} = \mathbb{E}[X_i X_j]$ for $i, j \in \{1, \ldots, k\}$

$$x_i \bullet_1 x_j = x_i x_j + C_{ij}, \qquad x_i \bullet_2 x_j = x_i x_j - C_{ij}.$$

See [Ebrahimi-Fard, Patras, Tapia, Z., IMRN 2018] for a Hopf-algebraic approach.

Distribution-valued Wick polynomials

Now, if $(X_1, \ldots, X_k)$ is replaced by the random distribution $G * \xi$, with the parameter $i \in \{1, \ldots, k\}$ replaced by $x \in \mathbb{T}^d$, then an analogous construction can be conceived.

Note that $G * \xi$ is Gaussian, centered and with covariance function

$$Q(x, y) = \int G(x, z)\, G(z, y)\, \mathrm{d}z$$

which for $d \geq 4$ is well-defined only for $x \neq y$ due to the singularities of G.

For $d \leq 3$ one can give a meaning to an algebra of polynomial functionals of φ, e.g.

$$\Phi \cdot_Q \Phi(f; \varphi) = \int_{\mathbb{T}^d} \left(\varphi^2(x) + Q(x, x)\right) f(x)\, \mathrm{d}x$$

where $\Phi(f; \varphi) = \int_{\mathbb{T}^d} f(x)\, \varphi(x)\, \mathrm{d}x$ and $f \in \mathscr{D}(\mathbb{T}^d)$.

Distribution-valued Wick polynomials

Recall that $\Phi(f;\varphi) = \int_{\mathbb{T}^d} f(x)\,\varphi(x)\,\mathrm{d}x$,

$$\Phi \cdot_Q \Phi(f;\varphi) = \int_{\mathbb{T}^d} \left(\varphi^2(x) + Q(x,x)\right) f(x)\,\mathrm{d}x.$$

However for $d \geq 4$ this expression is ill-defined since $Q \in \mathscr{D}'(\mathbb{T}^d \times \mathbb{T}^d \setminus \mathrm{Diag})$.

Techniques of microlocal analysis allow to find and characterize all $\widehat{Q} \in \mathscr{D}'(\mathbb{T}^d \times \mathbb{T}^d)$ which extend Q and to give a meaning to the restriction of $\widehat{Q}$ on $\mathrm{Diag} \subset \mathbb{T}^d \times \mathbb{T}^d$.

This is based on the study of the wavefront set and of the scaling degree of the relevant distributions.

We obtain existence of a well defined product

$$\Phi \cdot_Q \Phi(f;\varphi) = \int_{\mathbb{T}^d} \left(\varphi^2(x) + \widehat{P}(x)\right) f(x)\,\mathrm{d}x$$

with $\widehat{P} \in \mathscr{D}'(\mathbb{T}^d)$.

Renormalisation

We have obtained existence of a well defined (renormalised) product

$$\Phi \cdot_Q \Phi(f;\varphi) = \int_{\mathbb{T}^d} \Big(\varphi^2(x) + \widehat{P}(x)\Big) f(x)\,\mathrm{d}x$$

with $\widehat{P} \in \mathcal{D}'(\mathbb{T}^d)$. More generally, we can define an algebra of distribution-valued polynomial functionals of φ.

What about uniqueness?

If $d \geq 4$ then there is a family of possible choices for $\cdot_Q$. These are the ambiguities due to renormalisation and they can be fully characterized.

For example any other possible choice of $\widehat{P'} \in \mathcal{D}'(\mathbb{T}^d)$ must satisfy $\widehat{P'} - \widehat{P} \in \mathbb{R}\mathbf{1}$.

Back to the equation

We go back to our equation

$$\Psi = \varphi - \lambda\, G * \Psi^3, \qquad \Psi[\![\lambda]\!] = \sum_{j\geq 0} \lambda^j F_j\,,$$

with

$$F_0 = \varphi, \qquad F_1 = -G * \varphi^3, \qquad F_2 = 3G * (\varphi^2\, G * \varphi^3)\,,$$
$$F_j = - \sum_{j_1+j_2+j_3=j-1} G * (F_{j_1} F_{j_2} F_{j_3})\,, \qquad j \geq 3\,.$$

Since all this does not properly describe our SPDE, we renormalise by writing...

The renormalised equation

$$\widehat{\Psi} = \Phi - \lambda\, G * (\widehat{\Psi} \cdot_Q \widehat{\Psi} \cdot_Q \widehat{\Psi}), \qquad \widehat{\Psi}[\![\lambda]\!] = \sum_{j \geq 0} \lambda^j \widehat{F}_j\,,$$

with

$$\widehat{F}_0 = \Phi, \qquad \widehat{F}_1 = -G * (\Phi \cdot_Q \Phi \cdot_Q \Phi), \qquad \widehat{F}_2 = 3G * (\Phi \cdot_Q \Phi \cdot_Q G * (\Phi \cdot_Q \Phi \cdot_Q \Phi))\,,$$

$$\widehat{F}_j = -\sum_{j_1+j_2+j_3=j-1} G * (\widehat{F}_{j_1} \cdot_Q \widehat{F}_{j_2} \cdot_Q \widehat{F}_{j_3})\,, \qquad j \geq 3\,.$$

Now all $\widehat{F}_j$'s make sense as distribution-valued polynomial functionals of φ.

Bridging the gap

We have now the following conjecture: if $\hat{\psi}(\lambda)$ is the renormalised solution to

$$\Delta\psi - \lambda\psi^3 + \xi = 0$$

then for all $f \in \mathscr{D}$ (and for $\varphi = 0$)

$$\mathbb{E}[\langle\hat{\psi}(\lambda), f\rangle] = \sum_{j\geq 0} \lambda^j \widehat{F}_j(f;0)$$

or some weaker version of the same formula.

We have an analogous construction which allows to compute correlation functions of $\hat{\psi}$.

We also plan to study the connection between the two different sets of renormalisation constants.