On the spectral problem of a three term difference operator

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Motivation: topological strings

Topological String/Spectral Theory (TS/ST) correspondence

Toric CY 3-fold $M \xrightarrow{\text{Mirror Symmetry}} \rho_M$ (trace class operator)

The spectrum of $\rho_M$ is expected to be related to enumerative invariants of $M$ through the topological string partition functions. Suggested by Aganagic–Dijkgraaf–Klemm–Mariño–Vafa (2006) and materialized by Grassi–Hatsuda–Mariño (2016).

Example: the local $\mathbb{P}^2$

$$\rho^{-1}_{\mathbb{P}^2} = u + v + e^{i \frac{\hbar}{2}} v^{-1} u^{-1}$$

with positive self-adjoint operators $u$ and $v$ in a (separable) Hilbert space satisfying the Heisenberg–Weyl commutation relation

$$uv = e^{i \hbar} vu, \quad \hbar \in \mathbb{R}_{>0}.$$
Fredholm determinant

\[ \det(1 + \kappa \rho_M) = 1 + \sum_{N=1}^{\infty} Z(N, \hbar) \kappa^N \] (convergent series)

where the fermionic spectral traces \( Z(N, \hbar) = e^{F(N, \hbar)} \) provide a non-perturbative definition of the topological string partition functions.

\[ \hbar = \lambda N, \quad N \to \infty, \quad (t’Hooft limit) \]

\[ F(N, \hbar) \simeq \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda) \hbar^{2-2g} \] (asymptotic series)

with the standard topological string genus \( g \) free energies \( \mathcal{F}_g(\lambda) \) in the conifold frame where \( \lambda \) is a flat coordinate for the CY moduli space vanishing at the conifold point.
The spectral problem

Define a positive self-adjoint operator in $L^2(\mathbb{R})$

$$H := e^{2\pi b p} + e^{2\pi b x} + e^{-2\pi b(p+x)}, \quad b := \sqrt{\frac{\hbar}{2\pi}} \in \mathbb{R}_{>0}$$

with normalized Heisenberg’s position and momentum operators

$$\langle x | x = x \langle x |, \quad \langle x | p = \frac{1}{2\pi i} \frac{\partial}{\partial x} \langle x |, \quad [p, x] = (2\pi i)^{-1}$$

Small $\hbar$ limit

$$H = 3 + \hbar \sqrt{3} \left( a^* a + \frac{1}{2} \right) + O(\hbar^{3/2}),$$

$$a := \frac{\sqrt{2\pi}}{\sqrt[4]{3}} \left( x + pe^{\frac{\pi i}{3}} \right), \quad [a, a^*] = 1$$

Power series expansion of eigenvalues $E_n(\hbar) = \sum_{k=0}^{\infty} E_{n,k} \hbar^k$,

$$E_{n,0} = 3, \quad E_{n,1} = 0, \quad E_{n,2} = \sqrt{3} \left( n + \frac{1}{2} \right), \ldots$$
If \( b = e^{i\theta}, \ 0 < \theta < \pi/2 \), then \( H \) is formally normal:

\[
\mathcal{D}(H) \subset \mathcal{D}(H^*), \quad \|Hx\| = \|H^*x\| \quad \forall x \in \mathcal{D}(H).
\]

**Lemma**

Let \( \{a_j\}_{j \in J} \) be a finite set of densely defined operators such that \( A := \sum_{j \in J} a_j \) is densely defined and, for any \( j, k \in J \), the operator \( a_j + a_k \) is formally normal. Then \( A \) is formally normal.

**Proof.** As \( a_j \) is formally normal for any \( j \in J \), it follows that

\[
\mathcal{D}(A) = \bigcap_{j \in J} \mathcal{D}(a_j) \subset \bigcap_{j \in J} \mathcal{D}(a_j^*) = \mathcal{D}\left( \sum_{j \in J} a_j^* \right) \subset \mathcal{D}(A^*).
\]

For any \( j, k \in J \) and \( x \in \mathcal{D}(a_j + a_k) \), one deduces that

\[
\langle a_j x | a_k x \rangle - \langle a_j^* x | a_k^* x \rangle =: M_{j,k}(x) = -M_{k,j}(x).
\]

For any \( x \in \mathcal{D}(A) \), the equality \( \|Ax\| = \|A^*x\| \) follows from

\[
\|Ax\|^2 - \|A^*x\|^2 = \sum_{j,k \in J} M_{j,k}(x) = -\sum_{j,k \in J} M_{k,j}(x) = \|A^*x\|^2 - \|Ax\|^2.
\]
In our case $H = a_1 + a_2 + a_3$ with

$$a_1 = e^{2\pi b p}, \quad a_2 = e^{2\pi b x}, \quad a_3 = e^{-2\pi b (p+x)}$$

and $a_1 + a_2 = U e^{2\pi b x} U^*$ with unitary operator

$$U := \Phi_b(p-x), \quad \Phi_b(x) := \frac{(-q e^{2\pi b x}; q^2)_{\infty}}{(-\bar{q} e^{2\pi b^{-1} x}; \bar{q}^2)_{\infty}}$$

$q := e^{\pi i b^2}, \quad \bar{q} := e^{-\pi i b^{-2}}$, and similarly for two other pairs. Thus, $H$ is at least formally normal and it is expected to admit a unique normal extension.
Principle of F-duality

The common spectral problem for $H$ and $H^*$ is equivalent to constructing an element $\langle x | \Psi \rangle := \Psi(x) \in L^2(\mathbb{R})$ admitting analytic continuation to a domain in $\mathbb{C}$ containing the strip $|\Im z| < \cos \theta$ and satisfying two difference equations

$$\Psi(x - ib) + e^{-\pi ib^2} e^{-2\pi bx} \Psi(x + ib) = (E - e^{2\pi bx})\Psi(x),$$
$$\Psi(x - ib^{-1}) + e^{-\pi ib^{-2}} e^{-2\pi b^{-1}x} \Psi(x + ib^{-1}) = (\bar{E} - e^{2\pi b^{-1}x})\Psi(x)$$

related to each other by the substitutions

$$(b, E) \leftrightarrow (b^{-1}, \bar{E}) \quad \text{(Faddeev’s modular duality=F-duality).}$$

There are two possibilities

$$\Psi(x)|_{x \to +\infty} \sim \psi_k(x) := e^{\pi i (1-3k)x^2 - 2\pi x \cos \theta}, \quad k \in \{0, 1\},$$

with exact solutions $\Psi_k(x) = \psi_k(x) \varphi_k(x)$,

$$\varphi_k(x - \epsilon_k) + e^{-(2x+\epsilon_k)3\pi b} \varphi_k(x + \epsilon_k) = (1 - E e^{-2\pi b x}) \varphi_k(x),$$

$$\epsilon_k := (1 - 2k)ib,$$

+ the F-dual equations $(b, E) \mapsto (b^{-1}, \bar{E})$ and the boundary conditions

$$\lim_{x \to +\infty} \varphi_k(x) = 1.$$
The factorisation Ansatz

The F-dual substitutions

$$\varphi_k(x) = \chi_k(e^{2\pi b x})\bar{\chi}_k(e^{2\pi b^{-1} x}), \quad k \in \{0, 1\},$$

give rise to power series solutions \((q := e^{\pi b^2})\)

$$\chi_k(z) = \phi_{q^{2k-1},E}(1/z), \quad \phi_{q,E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q,E)}{(q^{-2};q^{-2})_n} z^n,$$

$$\bar{\chi}_k = \chi_k|_{(q,E) \mapsto (\bar{q}^{-1},\bar{E})}$$

with the polynomials \(p_n = p_n(q,E) \in \mathbb{Z}[q,q^{-1}][E]\) of degree \(n\) in \(E\) defined by

$$p_{n+1} = E p_n + (q^n - q^{-n})(q^{n-1} - q^{1-n})p_{n-2}, \quad p_0 = 1.$$ 

$$p_n(q,E)|_{n \to \infty} \sim q^{-n^2/3} \quad \Rightarrow \quad RC(\phi_{q,E}(z)) = \infty \quad \text{(radius of convergence)} \quad \text{and} \quad RC(\phi_{1/q,E}(z)) = 0.$$
Let $\mathcal{O}_{\mathbb{C} \neq 0}$ be the $\mathbb{C}$-vector space of holomorphic maps $f : \mathbb{C} \neq 0 \to \mathbb{C}$. For $c \in \mathbb{C}$, $p$, $r$, $\alpha \in \mathbb{C} \neq 0$ and $m \in \mathbb{Z}$, define the following vector subspaces of $\mathcal{O}_{\mathbb{C} \neq 0}$

- $F_{p,c} := \{ f \mid f(z/p^2) + (zp)^3 f(zp^2) = (1 - cz)f(z) \}$;
- $V_{p,\alpha,c} := \{ f \mid \alpha zf(z/p^2) + z^2 p\alpha^{-1} f(zp^2) = (1 - cz)f(z) \}$;
- $T^{m}_{p,r} := \{ f \mid rz^m f(zp) = f(z) \}$.

**Lemma**

(i) $|p|^m < 1 \Rightarrow \dim(T^{m}_{p,r}) = |m|$ (\(\theta\)-functions of order $|m|$);
(ii) $\dim(V_{\alpha,\alpha,E}) = 1$;
(iii) the multiplication of functions induces a linear map $V_{q,\alpha,E} \otimes T^{1}_{q^2, q^2\alpha} \to F_{q,E}$;
(iv) $\dim(F_{q,E}) = 3$ and $\phi_{q,E} \in F_{q,E}$. 

Rinat Kashaev  
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Proof of (ii) \( \dim(V_{q,\alpha,E}) = 1 \)

Consider a linear map

\[
A : \mathcal{O}_{\mathbb{C} \neq 0} \rightarrow \mathcal{O}_{\mathbb{C} \neq 0}, \quad (Af)(z) = P_+(f \psi_{q,E})(1/\sqrt{-z})
\]

where

\[
\psi_{q,E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q,E)q^{(1-n)n/2}}{(q^{-2}; q^{-2})_n} z^n = \psi_{1/q,E}(-z)
\]

(\(\infty\) radius of convergence) and \(P_+\) is the projection to the even part of a function:

\[
P_+(f)(z) = (f(z) + f(-z))/2.
\]

Then, the restriction \( A|_{T^1_{q,-\alpha}} \) is a linear isomorphism between \( T^1_{q,-\alpha} \) and \( V_{q,\alpha,E} \).
First order matrix difference equation for $F_{p,c}$

For any $f \in F_{p,c}$, we have

$$\hat{f}(z) = L(z)\hat{f}(zp^2), \quad \hat{f}(z) := \begin{pmatrix} f(\frac{z}{p^2}) \\ f(z) \end{pmatrix}, \quad L(z) := \begin{pmatrix} 1 - cz & -z^3p^3 \\ 1 & 0 \end{pmatrix}. $$

Defining

$$L_n(z) := L(z)L(zp^2) \cdots L(zp^{2n-2}) =: \begin{pmatrix} a_n(z) & b_n(z) \\ c_n(z) & d_n(z) \end{pmatrix}, \quad n \in \mathbb{Z}_{>0},$$

we have

$$L_{m+n}(z) = L_m(z)L_n(zp^{2m}), \quad \forall m, n \in \mathbb{Z}_{>0},$$

in particular,

$$L_{n+1}(z) = L(z)L_n(zp^2) = L_n(z)L(zp^{2n}), \quad \forall n \in \mathbb{Z}_{>0}.$$ 

Assuming $|p| < 1$ and taking the limit $n \to \infty$,

$$L_\infty(z) := \lim_{n \to \infty} L_n(z) = \begin{pmatrix} \phi_{p,c}(z/p^2) & 0 \\ \phi_{p,c}(z) & 0 \end{pmatrix}. $$
Adjoint functions

Define the (skew-symmetric bilinear) Wronskian pairing
\[ [\cdot, \cdot] : F_{q,E} \times F_{q,E} \to T_{q^2,q^3}^3, \quad [f, g](z) = f\left(\frac{z}{q^2}\right)g(z) - g\left(\frac{z}{q^2}\right)f(z), \]
and the adjoint function \( \tilde{f} : U([\phi_{q,E}, f]) \to \mathbb{C} \)
\[ \tilde{f}(z) := \frac{f(z)}{[\phi_{q,E}, f](z)}, \quad \forall f \in F_{q,E}, \quad U(g) := \mathbb{C}_{\neq 0} \setminus g^{-1}(0). \]

Adjoint functions are analytic substitutes for the series \( \phi_{1/q,E}(z) \).

Theorem

Let \( f \in F_{q,E} \) be such that \( U([\phi_{q,E}, f]) \neq \emptyset \). Then
\[ z \in U([\phi_{q,E}, f]) \Rightarrow zq^{2^Z} \subset U([\phi_{q,E}, f]), \quad \lim_{n \to \infty} \tilde{f}(zq^{2^n}) = 1, \]
and \( \tilde{f}(z) \) admits an asymptotic expansion at small \( z \) in the form of the series \( \phi_{1/q,E}(z) \).
The general Ansatz for $\Psi(x)$

$$\Psi(x) = \Psi_0(x) + \xi \Psi_1(x)$$

where $\xi \in \mathbb{C}$ and

$$\Psi_0(x) := \psi_0(x) \tilde{h}(e^{-2\pi bx}) \phi_{\bar{q}, \bar{E}}(e^{-2\pi b^{-1}x}), \quad h \in F_{q, E} \setminus \mathbb{C} \phi_{q, E},$$

$$\Psi_1(x) := \psi_1(x) \phi_{q, E}(e^{-2\pi bx}) \tilde{h}(e^{-2\pi b^{-1}x}), \quad \bar{h} \in F_{\bar{q}, \bar{E}} \setminus \mathbb{C} \phi_{\bar{q}, \bar{E}},$$

sharing a common pole set (Requirement(I)).

Then, for any $\zeta, \sigma \in \mathbb{C}$, there exist a (multivalued) function $E = E(q, \zeta, \sigma)$ and elements $f \in V_{q, q e^{-2\pi b \zeta}} E$, $\bar{f} \in V_{\bar{q}, \bar{q} e^{-2\pi b^{-1} \zeta}} \bar{E}$ with $\bar{E} := E|_{q \mapsto \bar{q}}$ such that

$$\Psi(x) = e^{-2\pi x \cos \theta} \frac{e^{\pi i x^2} f(z) \phi_{\bar{q}, \bar{E}}(\bar{z}) + \xi e^{-2\pi i (\zeta + 2 \sin \theta) x} \bar{f}(\bar{z}) \phi_{q, E}(z)}{\vartheta(z/s; q^2) \vartheta(z s e^{2\pi b \zeta}; q^2)}$$

where $z := e^{-2\pi bx}$, $\bar{z} := e^{-2\pi b^{-1}x}$, $s := e^{-2\pi b \sigma}$, $\bar{s} := e^{-2\pi b^{-1} \sigma}$, $\vartheta(u; p) := \sum_{n \in \mathbb{Z}} p^{(n-1)n/2} (-u)^n = (u, p/u, p; p)_\infty$. 
Pole cancellation in $\Psi(x)$

**Theorem**

Under the substitution

$$\xi = \xi(\theta, \zeta, \sigma) := -e^{\pi i \sigma (\sigma + 2\zeta)} \frac{f(s)\phi_{\tilde{q},\tilde{E}(\tilde{s})\tilde{s}}}{\tilde{f}(\tilde{s})\phi_{q,E(s)s}}.$$  

All the poles of $\Psi(x)$ at $x = \sigma + imb + ibn$, $m, n \in \mathbb{Z}$, are cancelled. Furthermore, the equation

$$\xi(\zeta, \theta, \sigma) = \xi(\zeta, \theta, \zeta - \sigma)$$

ensures that all the remaining poles of $\Psi(x)$ are cancelled as well (**Requirement(II)**).