On the spectral problem of a three term difference operator

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Motivation: topological strings

Topological String/Spectral Theory (TS/ST) correspondence

$$\text{Toric CY 3-fold } M \xrightarrow{\text{Mirror Symmetry}} \boldsymbol{\rho}_M \quad \text{(trace class operator)}$$

The spectrum of $\boldsymbol{\rho}_M$ is expected to be related to enumerative invariants of M through the topological string partition functions. Suggested by Aganagic–Dijkgraaf–Klemm–Mariño–Vafa (2006) and materialized by Grassi–Hatsuda–Mariño (2016).

Example: the local $\mathbb{P}^2$

$$\boldsymbol{\rho}_{\mathbb{P}^2}^{-1} = \boldsymbol{u} + \boldsymbol{v} + \mathrm{e}^{i\frac{\hbar}{2}}\, \boldsymbol{v}^{-1}\boldsymbol{u}^{-1}$$

with positive self-adjoint operators $\boldsymbol{u}$ and $\boldsymbol{v}$ in a (separable) Hilbert space satisfying the Heisenberg–Weyl commutation relation

$$\boldsymbol{u}\boldsymbol{v} = \mathrm{e}^{\mathrm{i}\hbar}\, \boldsymbol{v}\boldsymbol{u}, \quad \hbar \in \mathbb{R}_{>0}.$$

Implications of the TS/ST correspondence

Fredholm determinant

$$\det(1+\kappa\boldsymbol{\rho}_M) = 1 + \sum_{N=1}^{\infty} Z(N,\hbar)\kappa^N \quad \text{(convergent series)}$$

where the fermionic spectral traces $Z(N,\hbar) = \mathrm{e}^{F(N,\hbar)}$ provide a non-perturbative definition of the topological string partition functions.

$$\hbar = \lambda N, \quad N \to \infty, \quad \text{(t'Hooft limit)}$$

$$F(N,\hbar) \simeq \sum_{g=0}^{\infty} \mathcal{F}_g(\lambda)\hbar^{2-2g} \quad \text{(asymptotic series)}$$

with the standard topological string genus g free energies $\mathcal{F}_g(\lambda)$ in the conifold frame where λ is a flat coordinate for the CY moduli space vanishing at the conifold point.

The spectral problem

Define a positive self-adjoint operator in $L^2(\mathbb{R})$

$$\boldsymbol{H} := \mathrm{e}^{2\pi \mathrm{b} \boldsymbol{p}} + \mathrm{e}^{2\pi \mathrm{b} \boldsymbol{x}} + \mathrm{e}^{-2\pi \mathrm{b}(\boldsymbol{p}+\boldsymbol{x})}, \quad \mathrm{b} := \sqrt{\frac{\hbar}{2\pi}} \in \mathbb{R}_{>0}$$

with normalized Heisenberg's position and momentum operators

$$\langle x|\boldsymbol{x} = x\langle x|, \quad \langle x|\boldsymbol{p} = \frac{1}{2\pi i}\frac{\partial}{\partial x}\langle x|, \quad [\boldsymbol{p}, \boldsymbol{x}] = (2\pi i)^{-1}$$

Small $\hbar$ limit

$$\boldsymbol{H} = 3 + \hbar\sqrt{3}\left(\boldsymbol{a}^*\boldsymbol{a} + \frac{1}{2}\right) + \mathcal{O}(\hbar^{3/2}),$$

$$\boldsymbol{a} := \frac{\sqrt{2\pi}}{\sqrt[4]{3}}\left(\boldsymbol{x} + \boldsymbol{p}\,\mathrm{e}^{\frac{\pi i}{3}}\right), \quad [\boldsymbol{a}, \boldsymbol{a}^*] = 1$$

Power series expansion of eigenvalues $E_n(\hbar) = \sum_{k=0}^{\infty} E_{n,k}\hbar^{\frac{k}{2}}$,

$$E_{n,0} = 3, \quad E_{n,1} = 0, \quad E_{n,2} = \sqrt{3}\left(n + \frac{1}{2}\right), \dots$$

If $b = e^{i\theta}, 0 < \theta < \pi/2,$ then $\boldsymbol{H}$ is formally normal:

$$\mathcal{D}(\boldsymbol{H}) \subset \mathcal{D}(\boldsymbol{H}^*), \quad \|\boldsymbol{H}x\| = \|\boldsymbol{H}^*x\| \ \forall x \in \mathcal{D}(\boldsymbol{H}).$$

Lemma

Let $\{\boldsymbol{a}_j\}_{j\in J}$ be a finite set of densely defined operators such that $\boldsymbol{A} := \sum_{j\in J} \boldsymbol{a}_j$ is densely defined and, for any $j, k \in J$, the operator $\boldsymbol{a}_j + \boldsymbol{a}_k$ is formally normal. Then $\boldsymbol{A}$ is formally normal.

Proof. As $\boldsymbol{a}_j$ is formally normal for any $j \in J$, it follows that

$$\mathcal{D}(\boldsymbol{A}) = \cap_{j\in J}\mathcal{D}(\boldsymbol{a}_j) \subset \cap_{j\in J}\mathcal{D}(\boldsymbol{a}_j^*) = \mathcal{D}\Big(\sum_{j\in J} \boldsymbol{a}_j^*\Big) \subset \mathcal{D}(\boldsymbol{A}^*).$$

For any $j, k \in J$ and $x \in \mathcal{D}(\boldsymbol{a}_j + \boldsymbol{a}_k)$, one deduces that

$$\langle \boldsymbol{a}_j x | \boldsymbol{a}_k x\rangle - \langle \boldsymbol{a}_j^* x | \boldsymbol{a}_k^* x\rangle =: M_{j,k}(x) = -M_{k,j}(x).$$

For any $x \in \mathcal{D}(\boldsymbol{A})$, the equality $\|\boldsymbol{A}x\| = \|\boldsymbol{A}^*x\|$ follows from

$$\|\boldsymbol{A}x\|^2 - \|\boldsymbol{A}^*x\|^2 = \sum_{j,k\in J} M_{j,k}(x) = -\sum_{j,k\in J} M_{k,j}(x) = \|\boldsymbol{A}^*x\|^2 - \|\boldsymbol{A}x\|^2 \qquad \square$$

In our case $\boldsymbol{H} = \boldsymbol{a}_1 + \boldsymbol{a}_2 + \boldsymbol{a}_3$ with

$$\boldsymbol{a}_1 = \mathrm{e}^{2\pi \mathrm{b} \boldsymbol{p}}, \quad \boldsymbol{a}_2 = \mathrm{e}^{2\pi \mathrm{b} \boldsymbol{x}}, \quad \boldsymbol{a}_3 = \mathrm{e}^{-2\pi \mathrm{b}(\boldsymbol{p}+\boldsymbol{x})}$$

and $\boldsymbol{a}_1 + \boldsymbol{a}_2 = \boldsymbol{U}\, \mathrm{e}^{2\pi \mathrm{b} \boldsymbol{x}}\, \boldsymbol{U}^*$ with unitary operator

$$\boldsymbol{U} := \Phi_{\mathrm{b}}(\boldsymbol{p} - \boldsymbol{x}), \quad \Phi_{\mathrm{b}}(x) := \frac{(-q\, \mathrm{e}^{2\pi \mathrm{b} x}; q^2)_\infty}{(-\bar{q}\, \mathrm{e}^{2\pi \mathrm{b}^{-1} x}; \bar{q}^2)_\infty}$$

$q := \mathrm{e}^{\pi i \mathrm{b}^2}$, $\bar{q} := \mathrm{e}^{-\pi i \mathrm{b}^{-2}}$, and similarly for two other pairs.
Thus, $\boldsymbol{H}$ is at least formally normal and it is expected to admit a unique normal extension.

Principle of F-duality

The common spectral problem for $\boldsymbol{H}$ and $\boldsymbol{H}^*$ is equivalent to constructing an element $\langle x|\Psi\rangle := \Psi(x) \in L^2(\mathbb{R})$ admitting analytic continuation to a domain in $\mathbb{C}$ containing the strip $|\Im z| < \cos\theta$ and satisfying two difference equations

$$\Psi(x - i\mathrm{b}) + \mathrm{e}^{-\pi i \mathrm{b}^2}\,\mathrm{e}^{-2\pi \mathrm{b} x}\,\Psi(x + i\mathrm{b}) = (E - \mathrm{e}^{2\pi \mathrm{b} x})\Psi(x),$$

$$\Psi(x - i\mathrm{b}^{-1}) + \mathrm{e}^{-\pi i \mathrm{b}^{-2}}\,\mathrm{e}^{-2\pi \mathrm{b}^{-1} x}\,\Psi(x + i\mathrm{b}^{-1}) = (\bar{E} - \mathrm{e}^{2\pi \mathrm{b}^{-1} x})\Psi(x)$$

related to each other by the substitutions

$$(\mathrm{b}, E) \leftrightarrow (\mathrm{b}^{-1}, \bar{E}) \text{ (Faddeev's modular duality=F-duality).}$$

In the general case of Baxter's TQ-equations, an approach for constructing solutions in the strongly coupled regime is suggested by Sergeev (2005).
An approach through auxiliary non-linear integral equations is developed by Babelon–Kozlowski–Pasquier (2018).

F-dual asymptotics at $x \to +\infty$

There are two possibilities

$$\Psi(x)|_{x\to+\infty} \sim \psi_k(x) := \mathrm{e}^{\pi i(1-3k)x^2 - 2\pi x\cos\theta}, \quad k \in \{0,1\},$$

with exact solutions $\Psi_k(x) = \psi_k(x)\varphi_k(x)$,

$$\varphi_k(x-\epsilon_k) + \mathrm{e}^{-(2x+\epsilon_k)3\pi \mathrm{b}}\,\varphi_k(x+\epsilon_k) = (1 - E\,\mathrm{e}^{-2\pi \mathrm{b} x})\varphi_k(x),$$
$$\epsilon_k := (1-2k)i\mathrm{b},$$

$+$ the F-dual equations $(\mathrm{b}, E) \mapsto (\mathrm{b}^{-1}, \bar{E})$ and the boundary conditions

$$\lim_{x\to+\infty} \varphi_k(x) = 1.$$

The factorisation Ansatz

The F-dual substitutions

$$\varphi_k(x) = \chi_k(\mathrm{e}^{2\pi \mathrm{b} x})\bar{\chi}_k(\mathrm{e}^{2\pi \mathrm{b}^{-1} x}), \quad k \in \{0, 1\},$$

give rise to power series solutions ($q := \mathrm{e}^{\pi i \mathrm{b}^2}$)

$$\chi_k(z) = \phi_{q^{2k-1},E}(1/z), \quad \phi_{q,E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q,E)}{(q^{-2};q^{-2})_n} z^n,$$

$$\bar{\chi}_k = \chi_k|_{(q,E)\mapsto(\bar{q}^{-1},\bar{E})}$$

with the polynomials $p_n = p_n(q,E) \in \mathbb{Z}[q,q^{-1}][E]$ of degree n in E defined by

$$p_{n+1} = Ep_n + (q^n - q^{-n})(q^{n-1} - q^{1-n})p_{n-2}, \quad p_0 = 1.$$

$p_n(q,E)|_{n\to\infty} \sim q^{-n^2/3} \Rightarrow RC(\phi_{q,E}(z)) = \infty$ (radius of convergence) and $RC(\phi_{1/q,E}(z)) = 0$.

The vector spaces $F_{p,c}$, $V_{p,\alpha,c}$, $T^m_{p,r}$

Let $\mathcal{O}_{\mathbb{C}_{\neq 0}}$ be the $\mathbb{C}$-vector space of holomorphic maps $f\colon \mathbb{C}_{\neq 0} \to \mathbb{C}$. For $c \in \mathbb{C}$, $p, r, \alpha \in \mathbb{C}_{\neq 0}$ and $m \in \mathbb{Z}$, define the following vector subspaces of $\mathcal{O}_{\mathbb{C}_{\neq 0}}$

- $F_{p,c} := \{f \mid f(z/p^2) + (zp)^3 f(zp^2) = (1 - cz)f(z)\}$;
- $V_{p,\alpha,c} := \{f \mid \alpha z f(z/p^2) + z^2 p\alpha^{-1} f(zp^2) = (1 - cz)f(z)\}$;
- $T^m_{p,r} := \{f \mid rz^m f(zp) = f(z)\}$.

Lemma

(i) $|p|^m < 1 \Rightarrow \dim(T^m_{p,r}) = |m|$ *(θ-functions of order $|m|$);*

(ii) $\dim(V_{q,\alpha,E}) = 1$;

(iii) *the multiplication of functions induces a linear map* $V_{q,\alpha,E} \otimes T^1_{q^2,q^2\alpha} \to F_{q,E}$;

(iv) $\dim(F_{q,E}) = 3$ *and* $\phi_{q,E} \in F_{q,E}$.

Proof of (ii) $\dim(V_{q,\alpha,E}) = 1$

Cosider a linear map

$$A\colon \mathcal{O}_{\mathbb{C}_{\neq 0}} \to \mathcal{O}_{\mathbb{C}_{\neq 0}}, \quad (Af)(z) = P_+(f\psi_{q,E})(1/\sqrt{-z})$$

where

$$\psi_{q,E}(z) := \sum_{n=0}^{\infty} \frac{p_n(q,E)q^{(1-n)n/2}}{(q^{-2};q^{-2})_n} z^n = \psi_{1/q,E}(-z)$$

(∞ radius of convergence) and P_+ is the projection to the even part of a function:

$$P_+(f)(z) = (f(z) + f(-z))/2.$$

Then, the restriction $A|_{T^1_{q,-\alpha}}$ is a linear isomorphism between $T^1_{q,-\alpha}$ and $V_{q,\alpha,E}$.

First order matrix difference equation for $F_{p,c}$

For any $f \in F_{p,c}$, we have

$$\hat{f}(z) = L(z)\hat{f}(zp^2), \quad \hat{f}(z) := \begin{pmatrix} f(\frac{z}{p^2}) \\ f(z) \end{pmatrix}, \; L(z) := \begin{pmatrix} 1 - cz & -z^3p^3 \\ 1 & 0 \end{pmatrix}.$$

Defining

$$L_n(z) := L(z)L(zp^2)\cdots L(zp^{2n-2}) =: \begin{pmatrix} a_n(z) & b_n(z) \\ c_n(z) & d_n(z) \end{pmatrix}, \quad n \in \mathbb{Z}_{>0},$$

we have

$$L_{m+n}(z) = L_m(z)L_n(zp^{2m}), \quad \forall m, n \in \mathbb{Z}_{>0},$$

in particular,

$$L_{n+1}(z) = L(z)L_n(zp^2) = L_n(z)L(zp^{2n}), \quad \forall n \in \mathbb{Z}_{>0}.$$

Assuming $|p| < 1$ and taking the limit $n \to \infty$,

$$L_\infty(z) := \lim_{n\to\infty} L_n(z) = \begin{pmatrix} \phi_{p,c}(z/p^2) & 0 \\ \phi_{p,c}(z) & 0 \end{pmatrix}.$$

Adjoint functions

Define the (skew-symmetric bilinear) *Wronskian pairing*

$$[\cdot,\cdot]\colon F_{q,E}\times F_{q,E}\to T^3_{q^2,q^3},\quad [f,g](z)=f(\frac{z}{q^2})g(z)-g(\frac{z}{q^2})f(z),$$

and the *adjoint function* $\tilde f\colon U([\phi_{q,E},f])\to\mathbb{C}$

$$\tilde f(z):=\frac{f(z)}{[\phi_{q,E},f](z)},\quad \forall f\in F_{q,E},\quad U(g):=\mathbb{C}_{\neq 0}\setminus g^{-1}(0).$$

Adjoint functions are analytic substitutes for the series $\phi_{1/q,E}(z)$.

Theorem

Let $f\in F_{q,E}$ be such that $U([\phi_{q,E},f])\neq\emptyset$. Then

$$z\in U([\phi_{q,E},f])\Rightarrow zq^{2\mathbb{Z}}\subset U([\phi_{q,E},f]),\quad \lim_{n\to\infty}\tilde f(zq^{2n})=1,$$

and $\tilde f(z)$ admits an asymptotic expansion at small z in the form of the series $\phi_{1/q,E}(z)$.

The general Ansatz for $\Psi(x)$

$\Psi(x) = \Psi_0(x) + \xi\Psi_1(x)$ where $\xi \in \mathbb{C}$ and

$$\Psi_0(x) := \psi_0(x)\tilde{h}(\mathrm{e}^{-2\pi \mathrm{b} x})\phi_{\bar{q},\bar{E}}(\mathrm{e}^{-2\pi \mathrm{b}^{-1} x}), \quad h \in F_{q,E} \setminus \mathbb{C}\phi_{q,E},$$

$$\Psi_1(x) := \psi_1(x)\phi_{q,E}(\mathrm{e}^{-2\pi \mathrm{b} x})\tilde{\bar{h}}(\mathrm{e}^{-2\pi \mathrm{b}^{-1} x}), \quad \bar{h} \in F_{\bar{q},\bar{E}} \setminus \mathbb{C}\phi_{\bar{q},\bar{E}},$$

sharing a common pole set (*Requirement(I)*).
Then, for any $\zeta, \sigma \in \mathbb{C}$, there exist a (multivalued) function $E = E(q, \zeta, \sigma)$ and elements $f \in V_{q, q\,\mathrm{e}^{-2\pi \mathrm{b}\zeta}, E}$, $\bar{f} \in V_{\bar{q}, \bar{q}\,\mathrm{e}^{-2\pi \mathrm{b}^{-1}\zeta}, \bar{E}}$ with $\bar{E} := E|_{q \mapsto \bar{q}}$ such that

$$\Psi(x) = \mathrm{e}^{-2\pi x \cos\theta}\,\frac{\mathrm{e}^{\pi i x^2}\, f(z)\phi_{\bar{q},\bar{E}}(\bar{z}) + \xi\,\mathrm{e}^{-2\pi i(\zeta + 2\sin\theta)x}\,\bar{f}(\bar{z})\phi_{q,E}(z)}{\vartheta(z/s; q^2)\vartheta(zs\,\mathrm{e}^{2\pi \mathrm{b}\zeta}; q^2)}$$

where $z := \mathrm{e}^{-2\pi \mathrm{b} x}$, $\bar{z} := \mathrm{e}^{-2\pi \mathrm{b}^{-1} x}$, $s := \mathrm{e}^{-2\pi \mathrm{b}\sigma}$, $\bar{s} := \mathrm{e}^{-2\pi \mathrm{b}^{-1}\sigma}$,
$\vartheta(u; p) := \sum_{n\in\mathbb{Z}} p^{(n-1)n/2}(-u)^n = (u, p/u, p; p)_\infty$.

Pole cancellation in $\Psi(x)$

Theorem

Under the substitution

$$\xi = \xi(\theta, \zeta, \sigma) := -\,\mathrm{e}^{\pi i \sigma(\sigma+2\zeta)} \frac{f(s)\phi_{\bar{q},\bar{E}}(\bar{s})\bar{s}}{\bar{f}(\bar{s})\phi_{q,E}(s)s}.$$

all the poles of $\Psi(x)$ at $x = \sigma + i\mathsf{b}m + i\mathsf{b}n$, $m, n \in \mathbb{Z}$, are cancelled. Furthermore, the equation

$$\xi(\zeta, \theta, \sigma) = \xi(\zeta, \theta, \zeta - \sigma)$$

ensures that all the remaining poles of $\Psi(x)$ are cancelled as well (Requirement(II)).