Algebraic structure of the Hopf algebra of double posets

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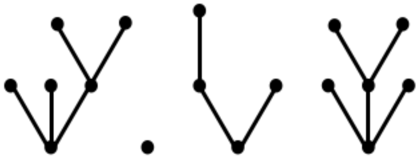
Content

The non-commutative Connes-Kreimer Hopf algebra NCK

The Hopf algebra NCK (Foissy, 2002) is freely generated by the set of all finite rooted planar trees.

Monomials of rooted planar trees $\longleftrightarrow$ Ordered forest

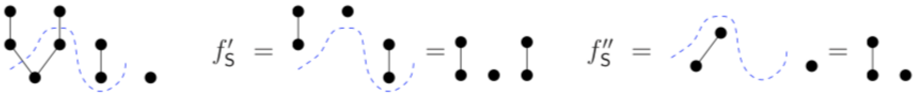
The algebra NCK is graded by the total number of nodes in a forest.



The non-commutative Connes-Kreimer Hopf algebra NCK

The coalgebra structure of NCK is given by *admissible cuts*:

$$\Delta(f) := \sum_{S \text{ admissible}} f'_S \otimes f''_S.$$



There is a map φ from ordered trees to binary trees which induces an isomorphism between NCK and the Hopf algebra of Loday-Ronco LR.

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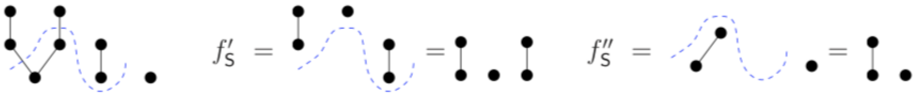


There is a map φ from ordered trees to binary trees which induces an isomorphism between NCK and the Hopf algebra of Loday-Ronco LR.

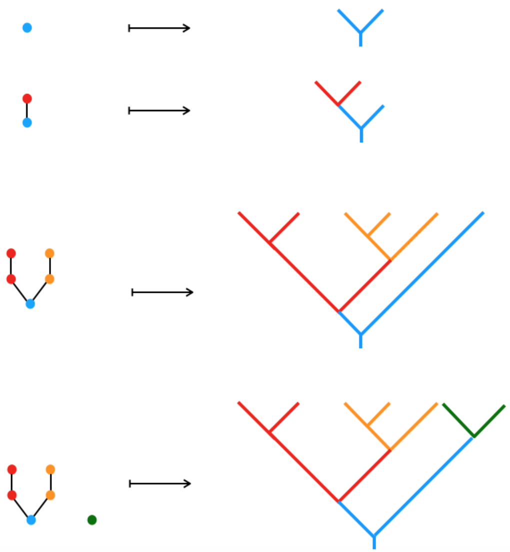
A canonical map from forests to permutations

Let $/$ and $\backslash$ the following operations on binary trees:

We construct φ recursively:

- $\varphi(\emptyset) := |$;
- $\varphi(B_+(f)) := \varphi(f)/Y$;
- if $f = (t_1, t_2, \ldots, t_n)$, then $\varphi(f) = \varphi(t_1)\backslash\varphi(t_2)\backslash\cdots\backslash\varphi(t_n)$.

A canonical map from forests to permutations



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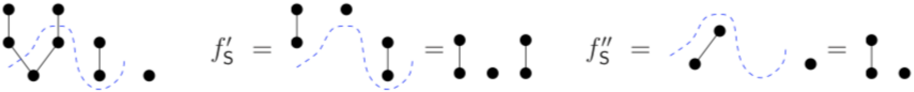

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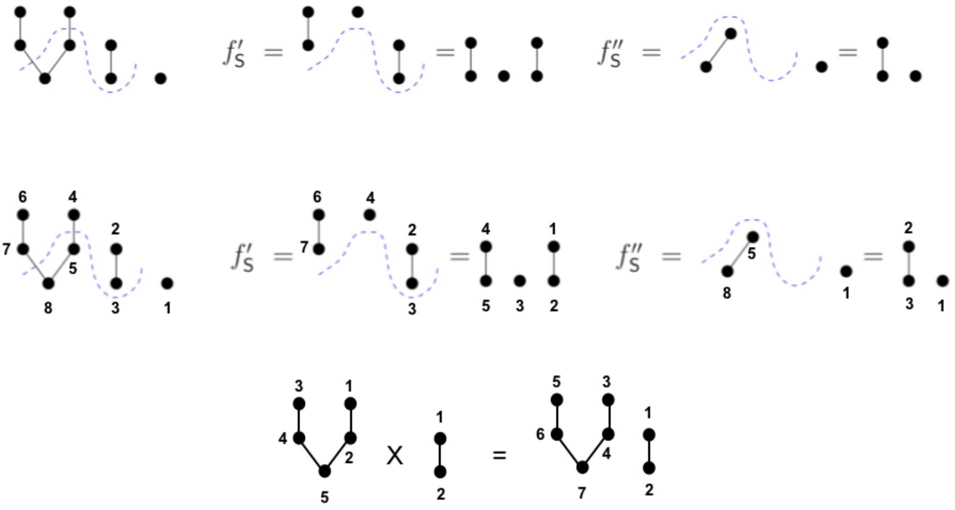
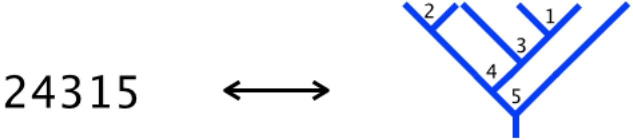


Permutation = binary tree with decreasing labelling

Let $\mathfrak{S}_n$ the symmetric group on $[n] := \{1, 2, \ldots, n\}$.
For $\sigma \in \mathfrak{S}_n$, we use the linear notation:

$$\sigma = \sigma(1)\sigma(2)\cdots\sigma(n).$$



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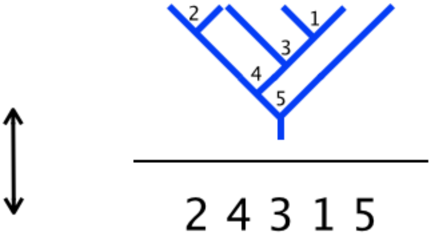


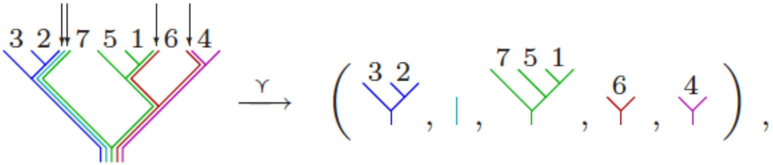
Operations on trees

Split an ordered tree $u \in \mathfrak{S}_n$ from one of his leaves produces an ordered forest $(u_0.u_1.\ldots,u_k)$, with labels in $[n]$. We denotes this process by

$$u \overset{\curlyvee}{\longmapsto} (u_0, u_1, \ldots, u_k).$$

For example:

Split ordered tree w to get an ordered forest, $w \xrightarrow{\curlyvee} (w_0, \ldots, w_p)$,

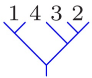


Operations on trees

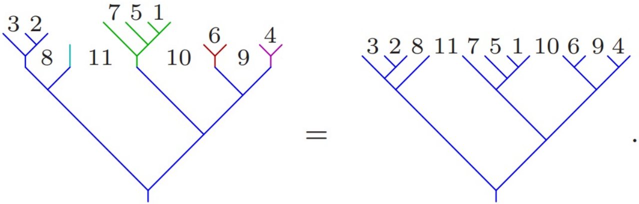
Graft an ordered forest $(u_0.u_1.\ldots,u_k)$, with labels in $[n]$, onto a tree $v \in \mathfrak{S}_k$ gives a tree

$$(u_0.u_1.\ldots,u_k)/v \in \mathfrak{S}_{n+k}.$$

For instance, the graft of the preceding ordered forest onto the tree

$v =$ 

produces



Malvenuto-Reutenauer Hopf algebra $\mathfrak{S}\mathrm{Sym}$

Is the Hopf algebra defined on $\bigoplus_{n\geq 0}\mathbb{K}[\mathfrak{S}_n]$, with basis $\{\mathsf{F}_w : w \in \mathfrak{S}\}$, where the product is the **shifted shuffle**, and the coproduct is the **destandardized deconcatenation**.

For $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_p$, we have:

$$\mathsf{F}_u \cdot \mathsf{F}_v = \sum_{u \overset{\curlyvee}{\longmapsto} (u_0, u_1, \ldots, u_p)} \mathsf{F}_{(u_0, u_1, \ldots, u_p)/v},$$

$$\Delta(\mathsf{F}_u) = \sum_{u \overset{\curlyvee}{\longmapsto} (u_0, u_1)} \mathsf{F}_{\mathsf{st}(u_0)} \otimes \mathsf{F}_{\mathsf{st}(u_1)}.$$

Malvenuto-Reutenauer Hopf algebra $\mathfrak{S}$Sym

If $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_p$, then

$$F_u \cdot F_v := \sum_{\substack{w \in \mathfrak{S}_{n+p} \\ \mathrm{st}(w \cap \{1,\ldots,n\}) = u \\ \mathrm{st}(w \cap \{n+1,\ldots,n+p\}) = v}} F_w,$$

where $w \cap I$ is the word obtained by erasing the letters in w which are not in I and st is the standardization operator.

For instance,

$$F_{12} \cdot F_{21} = F_{1243} + F_{1423} + F_{1432} + F_{4123} + F_{4132} + F_{4312}.$$

The set of permutations in the product $F_u \cdot F_v$ is called the **set of shuffles of u and v**.

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We have:

$$\Delta(F_{312}) = F_\lambda \otimes F_{312} + F_1 \otimes F_{12} + F_{21} \otimes F_1 + F_{312} \otimes F_\lambda.$$

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The graded dual $\mathfrak{S}\mathrm{Sym}^\star$

For every permutation u, let G_u the dual basis of the basis element F_u. The Hopf-algebraic structure of the graded dual of $\mathfrak{S}\mathrm{Sym}$ is described as follows.

- Product rule: if $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_p$, then

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Relevance of $\mathfrak{S}$Sym in renormalization

- As a Combinatorial Hopf Algebra (CHA), (almost) every CHA can be realized as a quotient or a sub-Hopf algebra of $\mathfrak{S}$Sym.
- $\mathfrak{S}$Sym is a sub-algebra of the convolution algebra of $(\mathrm{End}(\mathbb{K}\langle A), \sqcup\!\sqcup)$:

$$u \,\widetilde{\sqcup\!\sqcup}\, v := \sqcup\!\sqcup \circ (u \otimes v) \circ \delta$$

- D. Yang used $\mathfrak{S}$Sym to reinterpret the integration of Lipschitz one-forms along geometric rough paths developed by Lyons as an integration of time-varying exact one-forms along group-valued paths.

$\mathfrak{S}$Sym as a unital infinitesimal bialgebra

A **unital infinitesimal bialgebra** $(B, \bullet, \Delta)$ is a vector space B equipped with a unital associative product $\bullet$ and a counital coassociative coproduct Δ, such that they satisfies the following compatibility rule:

$$\Delta(x \bullet y) = (x \otimes 1) \bullet \Delta(y) + \Delta(x) \bullet (1 \otimes y) - x \otimes y.$$

This relation is called the **unital infinitesimal relation**.

$\mathfrak{S}$Sym as a unital infinitesimal bialgebra

We introduce two operations, / and \, on the set of permutations $\mathfrak{S}$. Given $u \in \mathfrak{S}_n$ and $v \in \mathfrak{S}_p$, let

$$u/v := u_1 \cdots u_n (v_1 + n) \cdots (v_p + n),$$

$$u \backslash v := (u_1 + p) \cdots (u_n + p) v_1 \cdots v_p.$$

In Malvenuto's talk, $/ = \square$ and $\backslash = \triangle$.

It is not difficult to show that $(\mathfrak{S}, /)$ and $(\mathfrak{S}, \backslash)$ are monoids, with same unit element λ.

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$\mathfrak{S}$Sym as a unital infinitesimal bialgebra

If $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \mathfrak{S}_n$, we define the permutation $\mathrm{rev}(\alpha)$ as

$$\mathrm{rev}(\alpha) := \alpha_n\cdots\alpha_2\alpha_1.$$

Both operations $/$ and $\backslash$ are related as follows:

$$\alpha\backslash\beta = \mathrm{rev}(\mathrm{rev}(\alpha)/\mathrm{rev}(\beta))$$

As the operation rev is an involution, it defines a monoid map $\mathrm{rev} : (\mathfrak{S}, \backslash) \to (\mathfrak{S}, /)$.
The inverse map on permutations acts as an endomorphism on $(\mathfrak{S}, /)$ and as an anti-endomorphism on $(\mathfrak{S}, \backslash)$:

Lemma

Let $\alpha, \beta \in \mathfrak{S}$. We have:

(a) $(\alpha/\beta)^{-1} = \alpha^{-1}/\beta^{-1}$;

(b) $(\alpha\backslash\beta)^{-1} = \beta^{-1}\backslash\alpha^{-1}$.

$\mathfrak{S}$Sym as a unital infinitesimal bialgebra

Theorem

1. *The Hopf algebra $\mathfrak{S}Sym$, together with the product $/$, is a 2-associative Hopf algebra.*
2. *The Hopf algebra $\mathfrak{S}Sym^{\star}$, together with the product $/$, is a 2-associative Hopf algebra, isomorphic to $(\mathfrak{S}Sym, /)$.*
3. *The Hopf algebras $\mathfrak{S}Sym$ and $\mathfrak{S}Sym^{\star}$, together with the product $\backslash$, are anti-isomorphic 2-associatives Hopf algebras.*

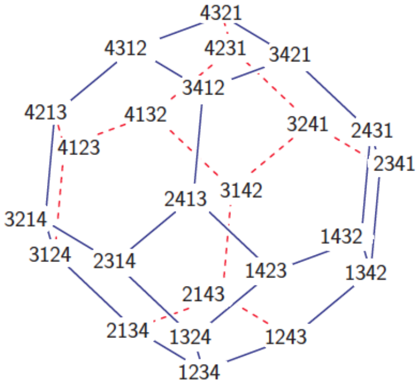
Second basis for $\mathfrak{S}$Sym

Let P a partially ordered set (poset).

Given $x, y \in P$, con $x < y$, the **Mobius function of P** is the map $\mu : P \times P \to P$ defined as:

- $\mu(x, x) = 1$;
- $\displaystyle\sum_{x \leq z \leq y} \mu(x, z) = 0.$

Second basis for $\mathfrak{S}$Sym: using the Permutohedron



Weak Bruhat order on $\mathfrak{S}_n$

Covering relation:

$$u \lessdot (i\ i+1)u,$$

if the letter i appears before $i+1$ inside u.

Product rule via weak Bruhat order

Left weak Bruhat order: $\leq_\ell$.

Right weak Bruhat order: $\leq_r$:

$$u \leq_r v \iff u^{-1} \leq_\ell v^{-1}.$$

Theorem (Loday-Ronco)

Let $u \in \mathfrak{S}_n$, $v \in \mathfrak{S}_p$. *We have*

$$G_u G_v = \sum_{u/v \leq_\ell w \leq_\ell u\backslash v} G_w \quad \textit{and} \quad F_u F_v = \sum_{u/v \leq_r w \leq_r v\backslash u} F_w.$$

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Product rule via weak Bruhat order

Let $u = 12, v = 21 \in \mathfrak{S}_2$. Then $u/v = 1243$, $u\backslash v = 3421$ and $v\backslash u = 4312$. We have:

$$G_{12}G_{21} = G_{1243} + G_{1342} + G_{1432} + G_{2341} + G_{2431} + G_{3421},$$

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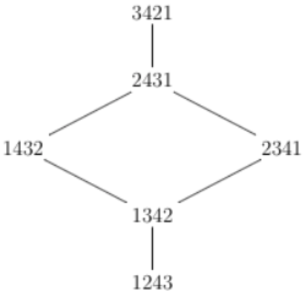
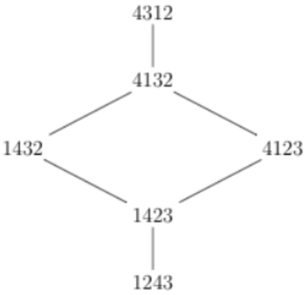
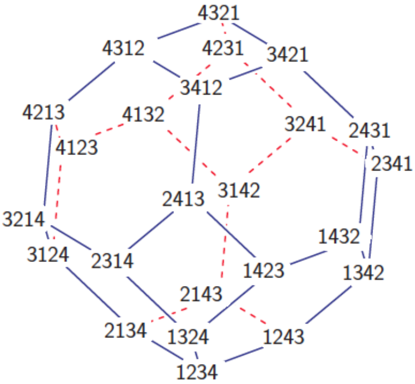


$[1243, 3421]_\ell$



$[1243, 4312]_r$

Second basis for $\mathfrak{S}$Sym: using the Permutohedron



(Left) weak Bruhat order on $\mathfrak{S}_n$

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if the letter i appears before $i+1$ inside u.

New basis (Aguiar-Sottile):

$$M_u := \sum_{u \leq v} \mu(u, v) F_v.$$

$$M_{3412} = F_{3412} - F_{4312} - F_{3421} + F_{4321}.$$

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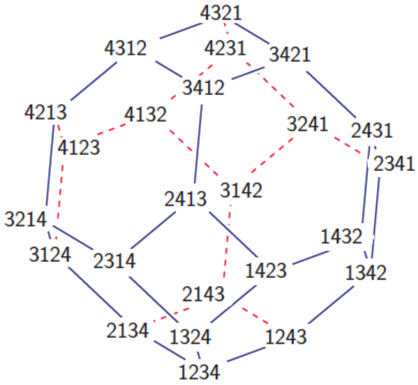

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Product and coproduct of monomials

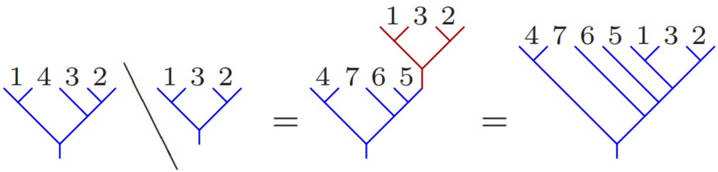
A formula for the product of the M-bases is not obvious:

$$\begin{aligned}
M_{312}\, M_1 &= (F_{312} - F_{321})\, F_1 \\
&= (F_{3124} + F_{3142} + F_{3412} + F_{4312}) \\
&\quad - (F_{3214} + F_{3241} + F_{3421} + F_{4321}) \\
&= M_{3124} + M_{3142} + M_{3412} + +2M_{4312} \\
&\quad 2M_{4132} + M_{4123} + M_{4231}.
\end{aligned}$$

The coproduct is easier.

Indecomposables trees and Prim($\mathfrak{S}$Sym)

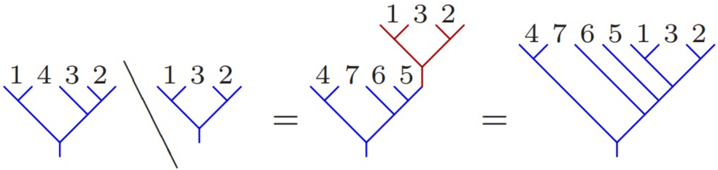
Prune w along its rightmost branch with all nodes above the cut smaller than all those below to get $w = u \backslash v$:



We say that w is **indecomposable** if only trivial prunings are possible.

Every $w \in \mathfrak{S}$ is uniquely pruned into indecomposables.

Indecomposables trees and Prim($\mathfrak{S}$Sym)



Theorem (Aguiar-Sottile)

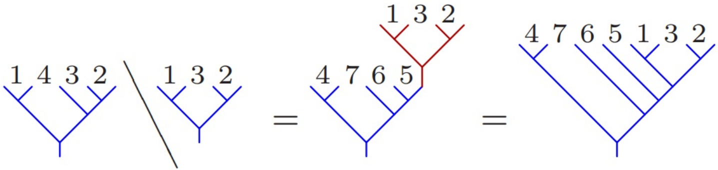
$$\Delta(M_w) = \sum_{w=u\backslash v} M_u \otimes M_v.$$

A basis for Prim($\mathfrak{S}$Sym) is then

$$\{M_w \ : \ w \text{ indecomposable } \}.$$

In particular, $\mathfrak{S}$Sym is cofree.

Indecomposables trees and Prim($\mathfrak{S}$Sym)



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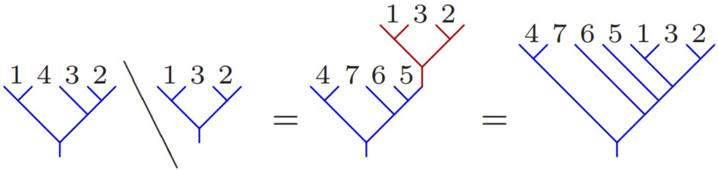

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Other bases for $\mathfrak{S}$Sym and $\mathfrak{S}$Sym*

$\mathfrak{S}$Sym-bases

- F_u
- $M_u := \sum_{u \leq_\ell v} \mu_\ell(u,v) F_v$
- $E_u := \sum_{u \leq_r v} F_v$

$\mathfrak{S}$Sym*-bases

- G_u $\quad (G_u = F_u^*)$
- $H_u := \sum_{v \leq_\ell u} G_u$ $\quad (H_u = M_u^*)$
- $N_u := \sum_{u \leq_r v} \mu_r(u,v) G_v$

Self-duality of $\mathfrak{S}$Sym

Theorem (Malvenuto, Reutenauer)

The map $\mathrm{F}_{\mathfrak{u}} \mapsto \mathrm{G}_{\mathfrak{u}^{-1}}$ is an isomorphism of Hopf algebras between $\mathfrak{S}Sym$ and $\mathfrak{S}Sym^{\star}$.

Theorem (V.)

The map $\mathrm{E}_{\mathfrak{u}} \mapsto \mathrm{H}_{rev(\mathfrak{u})}$ is an isomorphism of Hopf algebras between $\mathfrak{S}Sym$ and $\mathfrak{S}Sym^{\star}$.

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In particular, $\mathfrak{S}Sym$ is self-dual.

Self-duality of $\mathfrak{S}Sym$

Theorem (Malvenuto, Reutenauer)

The map $\mathrm{F}_{\mathfrak{u}} \mapsto \mathrm{G}_{\mathfrak{u}^{-1}}$ is an isomorphism of Hopf algebras between $\mathfrak{S}Sym$ and $\mathfrak{S}Sym^{\star}$.

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Primitive space from the 2-associative Hopf algebra

If V is a vector space, the tensor module $\bigoplus_{k \geq 0} V^{\otimes k}$ is endowed with a natural structure of unital infinitesimal bialgebra, , denoted by $\mathrm{Inf}(V)$, considering the concatenation $\odot$ and the deconcatenation $\Delta_{\odot}$:

$$(u_1 \cdots u_r) \odot (v_1 \cdots v_s) := u_1 \cdots u_r v_1 \cdots v_s,$$

$$\Delta_{\odot}(u_1 u_2 \cdots u_k) = \sum_{i=0}^{k} (u_1 u_2 \cdots u_i) \otimes (u_{i+1} u_{i+2} \cdots u_k)$$

This is an important example of unital infinitesimal bialgebra.

Theorem (Loday, Ronco)

Any connected unital infinitesimal bialgebra B *is isomorphic to* $\mathrm{Inf}(\mathit{Prim}(B))$.

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The isomorphism is constructed using the following linear operator: if $\bullet$ and Δ are the product and the coproduct of the unital infinitesimal bialgebra B, let

$$e := \sum_{n \geq 0} (-1)^n (\mathrm{id}_B - \iota\varepsilon)^{*(n)} \in \mathrm{End}(B),$$

where $*$ is the convolution product constructed from the product $\bullet$ and the coproduct Δ of B. In other words, $e(c) = 0$ if $c \in \mathbb{K}$ and

$$e|_{B_+} = \mathrm{id}_B - \bullet \circ \Delta_+ + \bullet^2 \circ \Delta_+^2 - \bullet^3 \circ \Delta_+^3 + \cdots .$$

Primitive space from the 2-associative Hopf algebra

From here, an isomorphism between B_+ and $\mathrm{Inf}(\mathrm{Prim}(B))_+$ is given by

$$x \mapsto \sum_{n \geq 1} e^{\otimes n} \Delta_+^{(n-1)}(x).$$

The next proposition allows to construct a basis for the primitive space of a unital infinitesimal bialgebra $(B, \bullet, \Delta)$ from special elements of the monoid $(B, \bullet)$. If $(M, \bullet)$ is a monoid, with unit element 1_M, we say that $x \in M$ is **$\bullet$-indecomposable** if $x \neq 1_M$ and $x = y \bullet z$ implies $y = 1_M$ or $z = 1_M$; otherwise, we say that x is **$\bullet$-decomposable**. We let $\mathrm{Dec}(M, \bullet)$ and $\mathrm{Ind}(M, \bullet)$ the set of decomposables and indecomposables elements of the monoid $(M, \bullet)$, respectively.

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Primitive space from the 2-associative Hopf algebra

Proposition

The operator e *satisfies the following properties:*

1. $Im(e) = Prim(B)$;
2. $Ker(e) = \mathbb{K}\, Dec(B, \bullet)$;
3. e *is an idempotent.*

Corollary

Let $(B, \bullet, \Delta)$ *a unital infinitesimal bialgebra. The set*

$$\{e(x) \; : \; x \in Ind(B, \bullet)\}$$

is a basis of $Prim(B)$.

Primitive space of $\mathfrak{S}$Sym, part II

Let $e_{/}$ (resp. $e_{\backslash}$) the operator associated to the 2-associative Hopf algebra $(\mathfrak{S}\mathrm{Sym}, /)$ (resp. $(\mathfrak{S}\mathrm{Sym}, \backslash)$ (see (37)). The sets

$$\{e_{/}(F_\alpha) : \alpha \in \mathrm{Ind}(\mathfrak{S}, /\,)\} \quad \text{and} \quad \{e_{\backslash}(F_\alpha) : \alpha \in \mathrm{Ind}(\mathfrak{S}, \backslash\,)\}$$

are bases of $\mathrm{Prim}(\mathfrak{S}\mathrm{Sym})$.

By definition of $e_{/}$ (resp. $e_{\backslash}$), the element $e_{/}(F_\alpha)$, for $\alpha \in \mathrm{Ind}(\mathfrak{S}, /\,)$ (resp. the element $e_{\backslash}(F_\alpha)$, for $\alpha \in \mathrm{Ind}(\mathfrak{S}, \backslash)$), is an alternating sum with possibly many cancellations.

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Primitive space of $\mathfrak{S}$Sym, part II

$$e := \sum_{n \geq 0} (-1)^n (\mathrm{id}_B - \iota\varepsilon)^{*(n)}$$

$$\begin{aligned}
e_{/}(F_{3421}) &= F_{3421} - F_{1/321} - F_{12/21} - F_{231/1} + F_{1/1/21} \\
&\quad + F_{1/21/1} + F_{12/1/1} - F_{1/1/1/1} \\
&= F_{3421} - F_{1432} - F_{1243} - F_{2314} + F_{1243} + F_{1324} + F_{1234} - F_{1234} \\
&= F_{3421} - F_{1432} - F_{2314} + F_{1324}
\end{aligned}$$

$$\begin{aligned}
e_{\backslash}(F_{1432}) &= F_{1432} - F_{1\backslash 321} - F_{12\backslash 21} - F_{132\backslash 1} + F_{1\backslash 1\backslash 21} \\
&\quad + F_{1\backslash 21\backslash 1} + F_{12\backslash 1\backslash 1} + F_{1\backslash 1\backslash 1\backslash 1} \\
&= F_{1432} - F_{4321} - F_{3421} - F_{2431} + F_{4321} + F_{4321} + F_{3421} - F_{4321} \\
&= F_{1432} - F_{2431}.
\end{aligned}$$

Primitive space of $\mathfrak{S}$Sym, part II

If $n \in \mathbb{N}$, let $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \mathfrak{S}_n$ and $q = q_1q_2\cdots q_r \vdash n$. For $1 \leq k \leq r$, we denotes by α_{q_k} the following subword of α formed by the consecutives letters

$$\alpha_{q_k} := \alpha_{q_1+q_2+\cdots+q_{k-1}+1}\,\alpha_{q_1+q_2+\cdots+q_{k-1}+2}\,\cdots\,\alpha_{q_1+q_2+\cdots+q_{k-1}+q_k}.$$

For example, if $q = q_1q_2q_3 = 313 \vdash 7$ and $\alpha = 2756143 \in \mathfrak{S}_7$, then:

$$\alpha_{q_1} = 275 \quad , \quad \alpha_{q_2} = 6 \quad , \quad \alpha_{q_3} = 143.$$

Primitive space of $\mathfrak{S}$Sym, part II

Let $\alpha \in \mathfrak{S}_n$ and $\bullet$ an associative and unitary product on $\mathfrak{S}$. Let $q = q_1 q_2 \cdots q_r \vdash n$ a composition. We say that α is a q-**locally free of relative order**, or q-**locally free** (relating to $\bullet$) if, for every $1 \leq k \leq r$, we have

1. $\mathrm{st}(\alpha_{q_k}) \in \mathrm{Ind}(\mathfrak{S}, \bullet)$;
2. $\mathrm{st}(\alpha_{q_k} \alpha_{q_{k+1}}) \neq \mathrm{st}(\alpha_{q_k}) \bullet \mathrm{st}(\alpha_{q_{k+1}})$.

We put $\mathrm{LF}_q(\mathfrak{S}, \bullet)$ the set of q-locally free permutations, considering the monoid structure $(\mathfrak{S}, \bullet)$.

Primitive space of $\mathfrak{S}$Sym, part II

Let consider, for example, the fixed monoid $(\mathfrak{S}, /)$ and the permutation $\alpha = 45321$. Then, α is 1211-locally free, as:

$$231 = \mathrm{st}(453) \neq \mathrm{st}(4)/\mathrm{st}(53) = 1/21 = 132,$$

$$321 = \mathrm{st}(532) \neq \mathrm{st}(53)/\mathrm{st}(2) = 21/1 = 213,$$

$$21 = \mathrm{st}(21) \neq \mathrm{st}(2)/\mathrm{st}(1) = 1/1 = 12,$$

and each $\mathrm{st}(\alpha_k) \in \mathrm{Ind}(\mathfrak{S}, /)$. The permutation α is also 32-locally free:

$$45321 \neq \mathrm{st}(453)/\mathrm{st}(21) = 231/21 = 23154,$$

with each $\mathrm{st}(\alpha_k) \in \mathrm{Ind}(\mathfrak{S}, /)$. However, α is not 1112-locally free: if $q = q_1 q_2 q_3 q_4 = 1112$, then

$$\mathrm{st}(\alpha_{q_1}\alpha_{q_2}) = \mathrm{st}(45) = 12 = \mathrm{st}(4)/\mathrm{st}(5).$$

The permutation α is neither 212-locally free: if $q = q_1 q_2 q_3 = 212$, then $\mathrm{st}(\alpha_{q_1}) = \mathrm{st}(45) = 12$ is not indecomposable.

Primitive space of $\mathfrak{S}$Sym, part II

Theorem (C. Benedetti, D. Artenstein, A. Gonzalez, R. Gonzalez, J. Gutierrez, M. Ronco, D. Tamayo, Y. Vargas)

Let $(\mathfrak{S}Sym, \bullet)$ a 2-associative Hopf algebra structure on $\mathfrak{S}Sym$. Let $e_\bullet$ the idempotent associated to $(\mathfrak{S}Sym, \bullet)$. If $\alpha \in Ind(\mathfrak{S}_n, \bullet)$, the primitive element $e_\bullet(F_\alpha)$ is given by the following cancellation-free and grouping-free formula:

$$e_\bullet(F_\alpha) = \sum_{\substack{q = q_1 \cdots q_r \vdash n \\ \alpha \in LF_q(\mathfrak{S}, \bullet)}} (-1)^r F_{st(\alpha_{q_1}) \bullet \cdots \bullet st(\alpha_{q_r})}.$$

Relation between $e_{\backslash}(F_\alpha)$ and M_α

Every M_α is a primitive element if $\alpha \in \mathrm{Ind}(\mathfrak{S}, \backslash)$. Thus, $e_{\backslash}(M_\alpha) = M_\alpha$ and

$$\begin{aligned} M_\alpha &= e_{\backslash}(\mathscr{M}_\alpha) \\ &= \sum_{\alpha \leq_\ell \beta} \mu(\alpha, \beta)\, e_{\backslash}(F_\beta). \end{aligned}$$

As $e_{\backslash}(F_\beta) = 0$ if β is $\backslash$-decomposable, the sum above is given by $\backslash$-irreducibles permutations β greater than α. This condition induces a subposet of the left weak Bruhat order. By Mobius inversion, we obtain the following result.

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Relation between $e_{\backslash}(F_\alpha)$ and M_α

Proposition

Let $\alpha \in Ind(\mathfrak{S}, \backslash)$. The primitive elements $e_{\backslash}(F_\alpha)$ and M_α are related by

$$e_{\backslash}(F_\alpha) = \sum_{\substack{\alpha \leq_\ell \beta \\ \beta \in Ind(\mathfrak{S}, \backslash)}} M_\beta .$$

In particular, if α is a maximum $\backslash$-indecomposable element for the left weak Bruhat order, then $e_{\backslash}(F_\alpha) = M_\alpha$ and

$$\mu(\alpha, \beta) = \begin{cases} (-1)^r & \text{if } \alpha \in LF_q(\mathfrak{S}, \backslash) \text{ and } \beta = st(\alpha_{q_1})\backslash \cdots \backslash st(\alpha_{q_r}), \\ & \text{for some composition } q = q_1 q_2 \cdots q_r; \\ 0 & \text{else.} \end{cases}$$

Thanks!