

**Group–Theoretical Structure of
the Entangled States of
N Identical Particles**

**Suranjana Rai
Jagdish Rai**

Vienna, Preprint ESI 904 (2000)

July 3, 2000

Supported by Federal Ministry of Science and Transport, Austria
Available via <http://www.esi.ac.at>

Group-theoretical Structure of the Entangled States of N Identical Particles

Suranjana Rai and Jagdish Rai

RAITECH

M - 12/28 DLF CITY

GURGAON 122002 INDIA

2000 June 23

Abstract

We provide a group-theoretical classification of the entangled states of N identical particles. The connection between quantum entanglement and the exchange symmetry of the states of N identical particles is made explicit using the duality between the permutation group and the simple unitary group. Each particle has n -levels and spans the n -dimensional Hilbert space. We shall call the general state of the particle as a qunit. The direct product of the N qunit space is given a decomposition in terms of states with definite permutation symmetry. The nature of fundamental entanglement of a state can be related to the classes of partitions of the integer N . The maximally entangled states are generated from linear combinations of the less entangled states of the direct product space. We also discuss the nature of maximal entanglement and its measures.

1 Introduction

We give a fundamental group theoretical result, which explains quantum entanglement in terms of the exchange symmetry of any number N of identical

particles, as well as the exchange symmetry of the levels of those particles. We claim that there is a requirement of symmetry for entanglement to occur. We give symmetrization procedures to generate entangled states starting from non-entangled states. We see that maximum symmetry leads to maximum entanglement, i.e. states which are maximally symmetric with respect to the number of particles as well as to the number of levels, also have the maximum possible entanglement [1].

We proceed in two steps. Firstly, the particle symmetrization procedure gives rise to ordered subspaces of various types of entanglement. In the second step, level symmetrization leads to combinations of these ordered states which are conjugate to each other in level excitations. We thus get the maximally entangled states we seek, the states with maximum symmetry. The power of this procedure is that all possible types of entanglement, i.e. all N particle entanglements, and their various combinations, can be generated and classified. The number of maximally entangled states is equal to the dimension of the Hilbert space, and they are orthonormal. We are thus able to span the entire Hilbert space of the particles using these ordered maximally entangled states as basis states. Indeed, the ordering can be taken to be a measure of the entanglement.

We discuss the meaning and nature of maximum entanglement. In the ordered maximally entangled basis, by maximal entanglement we mean maximum possible entanglement for each different type of entanglement. A useful measure of maximal entanglement is the criterion of concurrence [2]. We illustrate these concepts with the help of examples. Another quantity of interest, is the N particle maximum entanglement which is measured by the maximum one-particle entropy obtained by tracing over the rest of the particles. We have also generated N -particle maximally entangled states by taking proper linear combinations of the different types of entanglement. It is also possible to use these states as a basis to span the Hilbert space. We also obtain the $N-1$, $N-2$, etc. particle entanglements in terms of entropy criteria.

2 Entanglement of N -identical particles

We now discuss the general case of a system of N identical particles. Each particle has n levels. By analogy with a qubit, we call it a qunit. In this work we consider the case of pure states only. A qunit is written as

$$|i_1, i_2, i_3, \dots, i_N\rangle$$

where each $i_1, i_2, i_3, \dots, i_N$ take values from 1 to n . This state spans an n dimensional Hilbert space H for one particle. For N particles, the space H^N spanned has a dimension of n^N . We are interested in obtaining the structure of the direct product space of N particles in terms of entanglement. We use the methods of group theory, to decompose the n^N dimensional space of the direct product of the of the spaces of each particle into a direct sum of spaces [3,4]. Each constituent of the direct sum has a definite permutational symmetry. This is quite naturally expected since the system contains identical particles and permutations of the particles leave the system invariant. Each of these constituents of the direct sum forms a representation of the permutation group of N particles, also known as the symmetric group S_N . For the case of N , n -level atoms, the state space can be written as

$$SU(n) \times SU(n) \times \dots \times SU(n) \text{ (N copies)}, (2.1)$$

The group $SU(n)$ consists of all $n \times n$ unitary matrices. Each $SU(n)$ group describes the states of a single n -level atom. As there are N copies in the direct product, it is possible to decompose the above direct product as

$$S_N \times SU(n), (2.2)$$

We wish to generate states from this space that have a definite permutational symmetry and are maximally entangled for that set. The full space can be given a decomposition using the various representations of the permutation group as described above. For this purpose it is necessary to consider the representations of the symmetric group S_N of all $N!$ permutations, given in detail in References 3 and 4. We choose a basis of H^N whose elements transform simultaneously as would a basis for an irreducible representation of $SU(n)$ and a basis for an irreducible representation of S_N . The decomposition is possible using the well known result

$$H^N = \amalg S^\lambda \times T^\lambda,$$

where λ is a partition of N treated as a non-increasing ordered k -tuple of positive integers with sum N . Also S^λ is a representation of the group S_N and T^λ is a representation of H . The states of the direct product space are taken

to be the basis states of the various irreducible representations occurring in the direct product space H^N . It is interesting to note that the frequency of the S^λ is equal to the dimensionality of the T^λ and the frequency of the T^λ is equal to the dimensionality of S^λ . A simple formula for obtaining the dimensionality of the T^λ can be obtained from the possible ways of filling the Young's diagram. Now we classify the n^N states based on the symmetry which corresponds to a definite Young's diagram. We call these the collective states of the system of N identical particles. This procedure produces a hierarchy of states that are ordered according to the degree of entanglement in subspaces of definite symmetry. We explain this using several examples for two and three particles in Section 4. For a collection of two-level atoms the states can be ordered with the quantum numbers j and m . The state with $m = j$, is unentangled and the degree of entanglement increases as the m value decreases from j to zero or half, and then starts decreasing again corresponding to a bell shaped curve. The procedure for generating maximally entangled states from states of lower entanglement is to combine conjugate states i.e., states with fixed Casimir operator value but opposite sign for values of the diagonal generators [3,4]. The conjugate states are states related by spin flip operators [2] and are connected by local operations [5]. In this way, one can generate all possible states of the system with a definite degree of entanglement (Table given below).

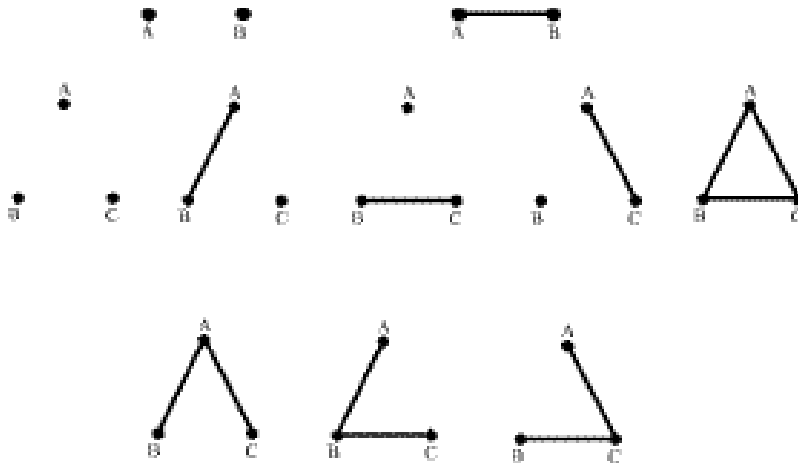
We find that the classification obtained on symmetry group lines is much more powerful than other methods as we are able to generate all possible types of entanglement for a system of many particles. It is also possible to generalize our methods to non-identical particles. Our procedure generates states that are connected by non-local operations. There has been extensive work done to understand entanglement by local unitary operations [5]. However, non-locality is at the heart of entanglement. We feel one should not be limited to local operations only. In fact, non-local operations have already been found to be useful e.g. in bound entanglement. Our procedure in fact is powerful enough to introduce both local and non-local operations. The general case, involves non-local or joint unitary operations where the unitary operator cannot be decomposed in the form given by Eq. 2.1. It is necessary to have joint unitary operations as we are increasing entanglement by our procedures. Entanglement does not increase in local operations. However, in the specific case, where the particles are distinguishable, like the case where they are at different locations, it is not necessary to symmetrize. Then,

there is no need for the permutation operator, the total Hilbert space can be decomposed into the space of local unitary operations each represented by $SU(n)$.

The structure of H^N has been considered by several authors. Werner [6] has used the idea of symmetric subspaces of H^N and applied it to the optimal cloning of pure states. J. Eisert et. al., have found a class of mixed states with known distillable entanglement [7], using techniques that employ decomposition of the product space into direct sum space. Cirac et. al., also consider similar decompositions for the $SU(2)$ group in the context of the two-level atom space [8]. In our paper, we provide an understanding of the necessity of symmetrization, and a group theoretical justification of the mathematical basis of the above works.

3 Nature of Entanglement

The methods of group theory give a very simple and elegant description of the nature of entanglement for a system of N identical particles. The different types of entanglements can be related to the various partitions of the integer N . These partitions form a class [3,4]. For example in the case of the group S_2 , there are two classes, identity, (e) and the two-cycle permutation (12). Essentially for two particles, there are two possibilities - either the particles are entangled or not. There is a very easy geometric way to describe this as in the top line of the figure. For three particles, we have three classes for the group S_3 : (e), $\{(12),(23),(13)\}$ and $\{(123), (132)\}$. Geometrically , this can be shown as



In the above diagrams, the line attaching the two particles shows entanglement while the two particles unconnected show unentangled particles. The second line of the figure shows the fundamental entanglements which represent the classes. The figures at the bottom show the possible combinations of the fundamental entanglements which have also been discussed in Section 4.

This description can easily be generalized to a system of N particles. It is well known that the number of representations is equal to the number of classes (Schur's lemma). According to this description, each class can be put in correspondence with a unique representation of the group. In this way, the various types of entanglement which are related to the representations, are connected to the various classes.

4 Examples

We now consider some specific examples of the bipartite and tripartite system to illustrate our procedure.

Example 1. Bipartite two-level system - 2 qubits

For a bipartite two-level system, the state space consists of $SU(2) \times SU(2)$. The four possible states of the system are $|11\rangle, |12\rangle, |21\rangle$ and $|22\rangle$. This space is four-dimensional ($2^N = 4$). The well-known symmetrization procedure which gives the singlet and triplet states is as follows.

$|j, m\rangle$ Symmetric

$$\begin{aligned} |1, 1\rangle &= |11\rangle \\ |1, 0\rangle &= 1/\sqrt{2}(|12\rangle + |21\rangle) \\ |1, -1\rangle &= |22\rangle \end{aligned}$$

$|j, m\rangle$ Antisymmetric

$$|0, 0\rangle = 1/\sqrt{2}(|12\rangle - |21\rangle)$$

This decomposition is on the lines of $S_2 \times SU(2)$. The group S_2 has two irreducible representations. The three symmetric states form the basis of one irreducible representations. The antisymmetric states correspond to the other representation. With this break up, the decomposition is complete.

The four-dimensional product space of $SU(2) \times SU(2)$ has been broken into the direct sum of $S_2 \times SU(2)$ as $2 \times 2 = 3 + 1$.

In this classification, there are two unentangled states and two maximally entangled states. If we also symmetrize with respect to the levels then we need to add/subtract to the $|11\rangle$ state the $|22\rangle$ state and normalize. The $|22\rangle$ state could be called the conjugate state to the $|11\rangle$ state and vice versa. This generates two additional states which are maximally entangled. Thus we have generated the full set of all four maximally entangled states:

$$\begin{aligned} & 1/\sqrt{2}(|11\rangle + |22\rangle) \\ & 1/\sqrt{2}(|11\rangle - |22\rangle) \\ & 1/\sqrt{2}(|12\rangle + |21\rangle) \\ & 1/\sqrt{2}(|12\rangle - |21\rangle) \end{aligned}$$

Example 2. Tripartite two-level system - 3 qubits

We now extend this procedure to the case of the tripartite system. The state space consists of $SU(2) \times SU(2) \times SU(2)$ and is of dimension $2 \times 2 \times 2 = 8$. The decomposition proceeds on the lines of $S_3 \times SU(2)$. The group S_3 has three irreducible representations which correspond to the three types of states in the state space. First, we have the symmetric representation which has the 4 symmetric wave functions as the basis. Then we have two states of mixed symmetry which correspond to two degenerate representations, each of dimension 2. There are no completely antisymmetric states for a system of three particles each with two levels. This is essentially a statement of the Pauli exclusion principle in terms of the group representations. These are the only possible states for this system ($2 \times 2 \times 2 = 4 + 2 + 2 + 0$).

In this case, the classification based on symmetry proceeds as follows. There are two sets of states which are fully symmetric and of mixed symmetry.

$|j, m\rangle$ Symmetric

$$\begin{aligned} |3/2, 3/2\rangle &= |111\rangle \\ |3/2, 1/2\rangle &= 1/\sqrt{3}(|112\rangle + |121\rangle + |211\rangle) \\ |3/2, -1/2\rangle &= 1/\sqrt{3}(|221\rangle + |212\rangle + |122\rangle) \\ |3/2, -3/2\rangle &= |222\rangle \end{aligned}$$

$|j, m; d\rangle$ Mixed symmetry

$$\begin{aligned} |1/2, 1/2, 1\rangle &= 1/\sqrt{6}(2|211\rangle - |112\rangle - |121\rangle) \\ |1/2, -1/2, 1\rangle &= 1/\sqrt{6}(|212\rangle + |221\rangle - 2|122\rangle) \end{aligned}$$

and

$$\begin{aligned} |1/2, 1/2, 2\rangle &= 1/\sqrt{2}(|112\rangle - |121\rangle) \\ |1/2, -1/2, 2\rangle &= 1/\sqrt{2}(|221\rangle - |212\rangle) \end{aligned}$$

In the symmetric subspace, $|111\rangle$ and $|222\rangle$ are unentangled and are conjugate to each other. The states with $|j, m\rangle; |3/2, 1/2\rangle$ and $|3/2, -1/2\rangle$ are also conjugate to each other but have partial entanglement (with non-zero one particle entropy). The conjugate state is generated by the action of the flip operators on the original state and are obtained by interchanging 1 and 2. To produce maximally entangled states by level symmetrization we add and subtract these states. The states are labeled by $|j, m; d\rangle$ where $j = N/2, N/2 - 1, 1/2$ or 0 , and m is the magnetic quantum number and d denotes the representation. For the three particle case of mixed symmetry there are two degenerate representations. In general the state $|j, m\rangle$ is conjugate to $|j, -m\rangle$ and their addition and subtraction produces a maximally entangled state. It is maximally entangled in the sense that the concurrence is one. It is well known that concurrence in itself can be used as a measure of entanglement [2]. For example, the Bell diagonal states in the case of a bipartite system have a concurrence equal to one and are maximally entangled. The unentangled pure states have a concurrence of zero. The degree of entanglement in a manifold of states of a certain specific symmetry increases from zero to higher entanglement in the middle of the ladder of states and then decreases. The states with lower m have higher degree of entanglement. It is interesting to note, however, that the concurrence for each conjugate by itself is zero. There is no fully antisymmetric state for the three spins and the procedure stops here generating eight possible maximally entangled states (with proper normalizations):

$$\begin{aligned} &|3/2, 3/2\rangle + |3/2, -3/2\rangle, \\ &|3/2, 3/2\rangle - |3/2, -3/2\rangle, \\ &|3/2, 1/2\rangle + |3/2, -1/2\rangle, \\ &|3/2, 1/2\rangle - |3/2, -1/2\rangle, \end{aligned}$$

$$\begin{aligned}
& |1/2, 1/2; 1\rangle_+ |1/2, -1/2; 1\rangle \\
& |1/2, 1/2; 1\rangle_- |1/2, -1/2; 1\rangle \\
& |1/2, 1/2; 2\rangle_+ |1/2, -1/2; 2\rangle \\
& |1/2, 1/2; 2\rangle_- |1/2, -1/2; 2\rangle
\end{aligned}$$

We now discuss the nature of entanglement of these states. From the first pair of conjugate states of the three particle states we get the GHZ state:

$$|GHZ\rangle = |3/2, 3/2\rangle + (\pm) |3/2, -3/2\rangle = |111\rangle + (\pm) |222\rangle, \quad (4.2.1)$$

For the GHZ state the single particle entropy is maximum and thus it is truly three particle maximal entangled.

From the next pair of states we get the following state that we call the Z-state

$$\begin{aligned}
|Z\rangle &= 1/\sqrt{3}(|112\rangle_+ + |121\rangle_+ + |211\rangle_+), \\
|Z\rangle_+ + |\bar{Z}\rangle &= |3/2, 1/2\rangle_+ + |3/2, -1/2\rangle \\
&= \{(|1\rangle_A + |2\rangle_A) |\psi^+\rangle_{BC}\} + \{|2\rangle_A |11\rangle_{BC} + |1\rangle_A |22\rangle_{BC}\} \quad (4.2.2)
\end{aligned}$$

The above state consists of two parts. The first part shows two particle maximal entanglement in B and C (Bell diagonal state) while A is unentangled. The two-particle entanglement, could be between any two of the particles, shown by suitable algebraic manipulation. The second part is a state of the GHZ type. This shows that there is both two particle and three particle entanglement.

We also define

$$\begin{aligned}
|Y\rangle &= 1/\sqrt{6}(2|211\rangle_- - |112\rangle_- - |121\rangle_-) \\
|X\rangle &= 1/\sqrt{2}(|112\rangle_- - |121\rangle_-)
\end{aligned}$$

Combining the conjugates

$$|Y\rangle_+ + |\bar{Y}\rangle = |\psi^-\rangle_{AC}(|1\rangle_B + |2\rangle_B) + |\psi^-\rangle_{AB}(|1\rangle_C + |2\rangle_C) \quad (4.2.3)$$

The above state is a sum of two two-particle maximally entangled states with one particle separable from each. Again, any one particle can be separated out depending on the representation we start from.

$$|X\rangle + |\bar{X}\rangle = (|1\rangle_A + |2\rangle_A) |\psi^-\rangle_{BC} \quad (4.2.4)$$

This state is a product state of a one particle unentangled and a two-particle maximal entangled state (Bell diagonal).

So we see from Eqs. (4.2.1 - 4.2.4), that we obtain all types of maximal entanglement of their own kind.

The number of maximally entangled states is exactly equal to the dimension of the space and they are mutually orthogonal. The maximally entangled states could form a useful basis. It is very interesting to note, that further linear combinations of these states give rise to fundamental entanglements.

We first take the case of the states, from $(|Y\rangle + |\bar{Y}\rangle) + 1/\sqrt{3}(|X\rangle + |\bar{X}\rangle)$, Eqs (4.2.3-4.2.4). We see that these states are a sum of three, two-particle maximally entangled terms. We have verified that this combination is three-particle maximally entangled i.e. its single particle entropy is maximum. Thus it is in a way comparable to the GHZ state which cannot be broken down into two-particle entanglements.

Next, we take the states $|Z\rangle + |\bar{Z}\rangle - |GHZ\rangle$, Eqs (4.2.1-4.2.2). We find that it is the product state of a two-particle maximal entangled (Bell diagonal) state and a one particle unentangled state. Here we see that by combining three-particle entangled states we can get two-particle entanglement [9].

Example 3. N-partite two-level system

It is straight forward to extend this procedure to the case of N spins. The state space corresponds to

$$SU(2) \times SU(2) \times \dots \times SU(2) \quad (N \text{ copies}),$$

Each $SU(2)$ group describes the states of the two-level atoms. As there are N copies in the direct product, it is possible to decompose the above direct product as

$$S_N \times SU(2).$$

There are 2^N maximally entangled states which can be generated using our procedure. We combine the states $|j, m; d\rangle + |j, -m; d\rangle$, where $j = N/2$,

$N/2 - 1, \dots, 1/2$ or 0 . Each possible value of j , corresponds to a set of states with a definite symmetry in the exchange of particles. The maximum value of $j = N/2$, corresponds to the fully symmetric states which are $2j + 1 = N + 1$ in number. The next value of $j = N/2 - 1$, gives states which are symmetric in the exchange of $N - 1$ particles and antisymmetric in the exchange of two particles. This procedure is continued to obtain a complete decomposition of the states of the entire system.

The states for the N qubit system, $|j, m\rangle$ are shown in the table below.

$j = N/2$;

$|N/2, N/2\rangle, |N/2, N/2 - 1\rangle, \dots, |N/2, -N/2\rangle$

$j = N/2 - 1$;

$|N/2 - 1, N/2 - 1\rangle, |N/2 - 1, N/2 - 2\rangle, \dots, |N/2 - 1, -(N/2 - 1)\rangle$

till ... $j = 1/2$ or 0 .

The angular momentum states $|j, m\rangle$ are well known through Dicke's work on superradiance [10]. They form a complete and orthogonal basis. The way we have generated the entangled states can be related to Dicke's interpretation of superradiance. The conjugate state corresponds to a value of m which is negative of the original m . Entanglement is maximum for low values of m which is well known to correspond to the superradiant state which occurs for the value of m around zero. The entangled state thus could be interpreted as the state where all the atoms of the radiating system are maximally interacting with each other. During superradiant emission the entanglement is transferred to the radiation called swapping of entanglement from the atom to the photon.

Acknowledgments

We would like to thank N. Mukunda, P. Zoller, I. Cirac, G. Vidal, N. Cerf and A. Zeilinger for discussions. This work was supported in part by the Erwin Schrodinger Institute, Vienna.

References

1. J. Rai and S. Rai, e-print quant-ph/003055.
2. William K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).
3. Jagdish Rai, C. L. Mehta and N. Mukunda, J. Math. Phys. **29**, 510 (1988).
4. Jagdish Rai, C. L. Mehta and N. Mukunda, J. Math. Phys. **29**, 2443 (1988).
5. N. Linden and S. Popescu, e-print quant-ph/9711016.
6. R. F. Werner, Physical Review A, 58, 1827 (1998). See also, D. Bruss, A. Ekert and C. Macchiavello, Phys. Rev. Lett. 81, 2598 (1998).
7. J. Eisert et. al., Phys. Rev. Lett., 84, 1611 (2000).
8. J. I. Cirac, A. K. Ekert and C. Macchiavello, Phys. Rev. Lett., **82**, 4344 (1999).
9. N. Gisin and H. Bechmann-Pasquinucci, e-print quant-ph/9804045.
10. R. H. Dicke, Phys. Rev. A 93, 99 (1954).