

The Bergman Kernel for Convex Tubes

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Vienna, Preprint ESI 808 (1999)

December 27, 1999

Supported by Federal Ministry of Science and Transport, Austria
Available via <http://www.esi.ac.at>

THE BERGMAN KERNEL FOR CONVEX TUBES

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ABSTRACT. Let $\Omega \subset \mathbb{R}^n$ be a bounded, convex and open set with real analytic boundary. Let $T_\Omega \subset \mathbb{C}^n$ be the tube with base Ω , and let \mathcal{B} be the Bergman kernel of T_Ω . If Ω is strongly convex, then \mathcal{B} is analytic away from the boundary diagonal. In the weakly convex case this is no longer true. In this situation, we relate the off diagonal points where analyticity fails to the characteristic lines. These lines are contained in the boundary of T_Ω , and are projections to the base of the Treves curves. These curves are symplectic invariants which are determined by the CR structure of the boundary of T_Ω . Note that Treves curves exist only when Ω has at least one weakly convex boundary point.

1. INTRODUCTION

Let $U \subset \mathbb{C}^n$ be open. Let $L^2(U)$ denote the Hilbert space of complex valued functions defined on U , which are square integrable with respect to Lebesgue measure. Let $H(U) = \{f \in L^2(U) : \bar{\partial}f = 0\}$, the closed subspace of holomorphic functions on U . We denote by \mathcal{B}

$$\mathcal{B} : L^2(U) \rightarrow H(U)$$

the orthogonal projection, which is known as the Bergman projection. If $\{\varphi_j\}$ denotes an orthonormal basis for $H(U)$, then it is well known that \mathcal{B} has kernel, which we also denote by \mathcal{B} ,

$$\mathcal{B}(z, w) = \sum \varphi_j(z) \overline{\varphi_j(w)}, \quad z, w \in U.$$

The above series is uniformly convergent on compact subsets of $U \times U$. \mathcal{B} is holomorphic in z and anti-holomorphic in w . In particular \mathcal{B} is real analytic on $U \times U$.

In case U is strictly pseudoconvex, the boundary behavior of \mathcal{B} is well understood. If $z^0 \in \partial U$, the boundary of U , then it follows that

$$\lim_{z \rightarrow z^0} \mathcal{B}(z, z) = +\infty.$$

See Hörmander [17] for the precise statement.

1991 *Mathematics Subject Classification.* Primary: 32H10, 32A25; Secondary: 35H05.

Key words and phrases. Bergman kernel, several complex variables, weakly pseudoconvex domain, tube, analytic wave front set, symplectic form.

The first author is grateful to the Schrödinger Institute and the Courant Institute of Mathematical Sciences, where a portion of this work was completed.

The second author was partially supported by NSF grant DMS-9104569.

We now assume, in addition to strict pseudoconvexity, that the boundary of U is real analytic. If we have $z^0, w^0 \in \partial U$, with $z^0 \neq w^0$, then it follows that \mathcal{B} extends to a full neighborhood of $(z^0, w^0) \in \mathbb{C}^n \times \mathbb{C}^n$ as a function holomorphic in z and antiholomorphic in w . In particular \mathcal{B} is real analytic near (z^0, w^0) . This follows from the analytic hypoellipticity of \square_b , a consequence of results of Treves [30], Tartakoff [28]. Also see Kashiwara [20] for a different point of view.

Our main interest here is the weakly pseudoconvex case where ∂U is real analytic. Here off-diagonal singularities may occur. For example, Christ and Geller [5] have shown that the Bergman kernel for the domain

$$U = \{(z_1, z_2) \in \mathbb{C}^2 : \Im z_2 > (\Re z_1)^m\}$$

is not analytic at certain points away from the boundary diagonal, when m is even and $m \geq 4$.

We now continue the study begun in [10]. Our goal here is to present recent results we have obtained concerning the Bergman kernel for tubes. We assume these tubes are convex with bounded base and analytic boundary. We show that off-diagonal singularities are described by the characteristic lines. These lines are contained in the boundary and are projections to the base of the Treves curves. These curves are symplectic invariants which are determined by the CR structure of the boundary. Treves curves exist exactly when the base of the tube has at least one weakly convex boundary point.

Treves introduced these curves in [30], where he conjectured that the existence of such curves should prevent analytic hypoellipticity, for certain partial differential operators with double characteristics. Recently Treves has extended his conjecture, [31]. The reader should note that in the case of tubes, the two conjectures are essentially the same.

We have been motivated by several important results on analytic regularity. These include, besides those already mentioned, Chen [2], Christ [3], [4], Derridj [6], Derridj–Tartakoff [7], Geller [12], Grigis–Sjöstrand [13], Hanges – Himonas [14], Helffer [15], Métivier [23], Sjöstrand [26], Tartakoff [27], Trepreau [29]. There is also recent related work by Kamimoto [18] and Kamimoto, Ki and Kim [19]. The reader may consult our survey, [11] for more references. Some of the results discussed here were announced at Saint–Jean–de–Monts, June 1998, [9].

In section 2 we state our results. In section 3 we discuss the notions of convexity that we need. In section 4 we discuss the Treves curves and characteristic lines for tubes. In section 5 we discuss the formula of Boutet de Monvel, [1]. In section 6 we prove Theorems 3, 4 and half of Theorem 1. In section 7 we prove half of Theorem 2. In sections 8 and 9 we prove the second half of Theorems 1 and 2. In section 10 we prove a version of the stationary phase formula which is specific to our needs. In section 11 we prove that the zeroes of \mathcal{N} (introduced in section 9) which are closest to the origin are simple zeroes.

2. STATEMENTS OF RESULTS

Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex with real analytic boundary. Let $T_\Omega = \{z \in \mathbb{C}^n : \Im z \in \Omega\}$ be the tube with base Ω . The following results depend on the notion of characteristic line, which is defined in Section 4, in the Remarks following Proposition 1.

Note that in \mathbb{C}^2 the characteristic lines are particularly easy to describe. Indeed, if $z^0 = x^0 + iy^0 \in \partial T_\Omega \subset \mathbb{C}^2$, then

$$z(s) = x^0 + s\mathbf{v} + iy^0, s \in \mathbb{R}$$

is the characteristic line through z^0 if and only if y^0 is a weakly convex boundary point of Ω and the vector \mathbf{v} is tangent to $\partial\Omega$ at the point y^0 . These lines are projections to the base of the Treves curves, which are invariants of the symplectic structure associated to the natural CR structure on ∂T_Ω . All this is discussed in detail in Section 4.

Assume that $z^0 \in \partial T_\Omega \subset \mathbb{C}^2$ and let L be a vector field defined near z^0 that generates the CR structure. Let \mathfrak{g} be the Lie algebra generated by L and \bar{L} under the commutator bracket. Since $\partial\Omega$ is bounded and analytic, it follows that there exists $X \in \mathfrak{g}$ such that L, \bar{L} and X are linearly independent at z^0 . We say that z^0 is a point of type m if the smallest possible commutator length for X is m . Note that m is even and $m \geq 2$; z^0 is a strictly pseudoconvex boundary point if and only if $m = 2$. Also observe that if z^0 and w^0 can be connected by a characteristic line, then z^0 and w^0 have the same type $m \geq 4$.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded and convex with real analytic boundary. Let $z^0, w^0 \in \partial T_\Omega$. Then \mathcal{B} , the Bergman kernel for T_Ω , extends as an analytic function to a full neighborhood of (z^0, w^0) if and only if z^0 and w^0 do not lie on the same characteristic line.*

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded and convex with real analytic boundary. Let $z^0, w^0 \in \partial T_\Omega$, with $z^0 \neq w^0$. Assume that z^0 and w^0 lie on the same characteristic line. Assume that the type of z^0 (or w^0) is m . Then \mathcal{B} , the Bergman kernel for T_Ω , extends as a smooth function, of Gevrey class m , to a full neighborhood of (z^0, w^0) . Furthermore, this is the best Gevrey class possible.*

In particular, these results tell us that \mathcal{B} has off-diagonal singularities if and only if Ω has at least one weakly convex boundary point. As far as we know, the above results are new, even when the base of the tube is as simple as

$$\Omega = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^m < 1\}$$

for m even, $m \geq 4$.

The above results in \mathbb{C}^2 are sharp. In higher dimensions our results are far from complete, however it is not difficult to prove the following when $n \geq 2$.

Theorem 3. *Let $z^0, w^0 \in \mathbb{C}^n$ and assume that $y^0 = \Im z^0$ and $v^0 = \Im w^0$ are two distinct boundary points of Ω . Then the Bergman kernel of T_Ω , $\mathcal{B}(z, w)$, can be extended as a real analytic function to a full neighborhood of (z^0, w^0) .*

Theorem 4. *Let $z^0, w^0 \in \mathbb{C}^n$. Assume that either $z^0 \in T_\Omega$ and $w^0 \in \partial T_\Omega$ or that $z^0 \in \partial T_\Omega$ and $w^0 \in T_\Omega$. Then the Bergman kernel of T_Ω , $\mathcal{B}(z, w)$, can be extended as a real analytic function to a full neighborhood of (z^0, w^0) .*

Our methods also apply if Ω is allowed to be unbounded. However, certain difficulties arise. Indeed, if $\partial\Omega$ contains a straight line, then it follows that $\mathcal{B} = 0$. See Lemma 4. In this case Treves curves may exist, yet \mathcal{B} has no singularities. Note that when Ω is bounded, T_Ω is biholomorphic to a bounded domain, and certainly $\mathcal{B} \neq 0$. The case of unbounded Ω will be treated elsewhere.

3. GEOMETRIC PRELIMINARIES

In this section we discuss the notions of convexity that we need. Our main interest is when the boundary of Ω is real analytic. We always assume in this section that Ω has smooth boundary, unless we explicitly state the assumption of real analyticity.

We begin by discussing Ω , the base of the tube T_Ω . Let $U \subset \mathbb{R}^n$ be open. Let $r : U \rightarrow \mathbb{R}$ be smooth. The base Ω is defined as follows:

$$\Omega = \{y \in U : r(y) < 0\}.$$

We assume that $dr(y) \neq 0$ whenever $r(y) = 0$. Furthermore, we assume that

$$\bar{\Omega} \subset\subset U, \tag{1}$$

that is, we assume that the closure of Ω is a compact subset of U .

Throughout we will assume that Ω is *convex*. This means that for each $y \in \partial\Omega$, the boundary of Ω , we have

$$\sum_{j,k=1}^n \frac{\partial^2 r}{\partial y_j \partial y_k}(y) a_j a_k \geq 0 \tag{2}$$

whenever $a_j \in \mathbb{R}$ and

$$\sum_{j=1}^n a_j \frac{\partial r}{\partial y_j}(y) = 0. \tag{3}$$

Note that it follows from (2) and (3) that Ω is *geometrically convex*. This means that if $y, y' \in \Omega$, then the segment connecting y to y' is contained in Ω . See for example [22], Proposition 3.1.8, page 102. If strict inequality holds in (2) whenever $(a_1, \dots, a_n) \neq 0$ satisfies (3), we say that $y \in \partial\Omega$ is a *strongly convex* boundary point.

Let $y, \xi \in \mathbb{R}^n$, with $\xi \neq 0$. We denote by \mathcal{P}_y^ξ the affine hyperplane which passes through y and has normal ξ . That is we define

$$\mathcal{P}_y^\xi = \{x \in \mathbb{R}^n : \langle x - y, \xi \rangle = 0\}.$$

We say that \mathcal{P}_y^ξ is a *supporting hyperplane* for Ω if $y \in \partial\Omega$ and $\Omega \cap \mathcal{P}_y^\xi = \emptyset$. We have the following technical lemmas.

Lemma 1. *Let $y \in \partial\Omega$. Then there exists $\xi \in \mathbb{R}^n$ with $\xi \neq 0$ such that \mathcal{P}_y^ξ is a supporting hyperplane for Ω .*

Proof. This is well known, see for example Theorem 2.1.10, page 44 of [16]. \square

Lemma 2. *Let $y \in \partial\Omega$. Assume that \mathcal{P}_y^ξ is a supporting hyperplane for Ω . Then there exists $\lambda \in \mathbb{R}, \lambda \neq 0$ such that $\xi = \lambda dr(y)$.*

Proof. Let $x \in \mathcal{P}_y^\xi \cap U$. Since we have $\Omega \cap \mathcal{P}_y^\xi = \emptyset$, it follows that $r(x) \geq 0$. Since $r(y) = 0$, it follows that r has a minimum at y along the line with direction $x - y$. Hence $\langle x - y, dr(y) \rangle = 0$ and the lemma follows. \square

Lemma 3. *Let $r : U \rightarrow \mathbb{R}$ be real analytic and assume that $y \in \partial\Omega$. If \mathcal{P}_y^ξ is a supporting hyperplane for Ω , then $\bar{\Omega} \cap \mathcal{P}_y^\xi = \{y\}$.*

Proof. Assume that there exists $x \neq y$ such that $x \in \bar{\Omega} \cap \mathcal{P}_y^\xi$. Let I be the segment connecting x to y . Observe that $\bar{\Omega} \cap \mathcal{P}_y^\xi$ is geometrically convex. Since $\Omega \cap \mathcal{P}_y^\xi = \emptyset$, it follows that $I \subset \partial\Omega$. Since r is analytic, it must vanish on the line containing I . Hence $\partial\Omega$ is not compact in U . Contradiction. \square

The next Lemma indicates some of the difficulties one faces if Ω is allowed to be unbounded.

Lemma 4. *Let $\Omega \subset \mathbb{R}^n$ be convex and open with smooth boundary. Assume that $\partial\Omega$ contains a straight line. If \mathcal{B} denotes the Bergman kernel for T_Ω , then it follows that $\mathcal{B} = 0$.*

Proof. We begin by observing that our assumption implies that Ω contains a straight line. First note that if S denotes a line segment connecting an interior point of Ω to a boundary point of Ω , then all interior points of S belong to Ω . This follows from the fact that Ω is convex with smooth boundary.

Let L be a straight line contained in $\partial\Omega$. Assume without loss of generality that $0 \in L$. We denote points of L by sy^0 , where $y^0 \in \mathbb{R}^n$ is a fixed direction vector and $s \in \mathbb{R}$ is a parameter. Let $y^1 \in \Omega$. Then $\frac{y^1}{2} + \frac{sy^0}{2}, s \in \mathbb{R}$ is a straight line contained in Ω .

Assume now that L is a straight line contained in Ω . We will denote $y \in \mathbb{R}^n$ by $y = (y_1, y')$ where $y_1 \in \mathbb{R}$ and $y' \in \mathbb{R}^{n-1}$. Assume without loss of generality that points of L are given by $(s, 0), s \in \mathbb{R}$. Let $\Omega_0 = \{y' : (0, y') \in \Omega\}$. We will show that $\Omega = \mathbb{R} \times \Omega_0$.

Assume first that $s \in \mathbb{R}$ and that $y' \in \Omega_0$. Since Ω_0 is open, there exists t near 1, $0 < t < 1$ such that $\frac{y'}{t} \in \Omega_0$. Also $(\frac{s}{1-t}, 0) \in L$. Hence we have $t(0, \frac{y'}{t}) + (1-t)(\frac{s}{1-t}, 0) \in \Omega$. Hence $(s, y') \in \Omega$.

Now let $(s, y') \in \Omega$. Choose t near 1, $0 < t < 1$ such that $(s, \frac{y'}{t}) \in \Omega$. We have $(\frac{-ts}{1-t}, 0) \in L$. Hence $(0, y') \in \Omega$, and we have shown that $\Omega = \mathbb{R} \times \Omega_0$.

From this we see that $T_\Omega = \mathbb{C} \times T_{\Omega_0}$. Now we see that if f is a holomorphic function on T_Ω which is square integrable, then $f = 0$. Indeed, this follows from Fubini's theorem, and the fact that any L^2 entire function of one variable must be zero. \square

4. SYMPLECTIC GEOMETRY AND TREVES CURVES

Our goal in this section is the calculation of the Treves curves for the tube T_Ω . These curves are determined by the symplectic geometry associated to the CR structure of ∂T_Ω . We begin with a general definition.

Let (M, ω) be an analytic symplectic manifold with symplectic form ω . If $\Sigma \subset M$ is a submanifold with $p \in \Sigma$, we denote by $T_p \Sigma$ the tangent space to Σ at p . We denote by $(T_p \Sigma)^\perp$ the space orthogonal to $T_p \Sigma$ with respect to ω . Let $(0, 1) \subset \mathbb{R}$ denote the open unit interval. We have the following :

Definition 1. Let $\Sigma \subset M$ be an analytic submanifold and let $\gamma : (0, 1) \rightarrow \Sigma$ be a non-constant analytic curve. We call γ a *Treves curve* for Σ if

$$\frac{d\gamma}{dt}(t) \in (T_{\gamma(t)}\Sigma)^\perp \quad (4)$$

for all $t \in (0, 1)$.

We now discuss the characteristic set of the CR structure of ∂T_Ω . Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ be natural coordinates. We think of \mathbb{C}^n as the space $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the complex structure generated by the functions $z_j = x_j + iy_j$, $j = 1, \dots, n$. This then induces coordinates $(x, y, \xi, \eta) \in T^*\mathbb{C}^n$. Since $T_\Omega = \{z \in \mathbb{C}^n : r(y) < 0\}$, we have $T^*(\partial T_\Omega) \subset T^*(\mathbb{C}^n)$ is defined by two equations; that is we have

$$T^*(\partial T_\Omega) = \{(x, y, \xi, \eta) \in T^*(\mathbb{C}^n) : r(y) = 0 \text{ and } \sum_{j=1}^n \eta_j \frac{\partial r}{\partial y_j}(y) = 0\}. \quad (5)$$

We now study the CR structure on ∂T_Ω . We will work near a point $z = x + iy \in \mathbb{C}^n$ such that $r(y) = 0$ and $\frac{\partial r}{\partial y_k}(y) \neq 0$ for some k , $1 \leq k \leq n$. The following $n-1$ vector fields form a basis for the natural CR structure on the boundary of T_Ω near z

$$L_j = \frac{\partial r}{\partial \bar{z}_k} \frac{\partial}{\partial \bar{z}_j} - \frac{\partial r}{\partial \bar{z}_j} \frac{\partial}{\partial \bar{z}_k}, j \neq k. \quad (6)$$

Then $\Sigma \subset T^*(\mathbb{C}^n)$, the characteristic set of the CR structure, is defined by $2n$ equations. Indeed, we have $(x, y, \xi, \eta) \in \Sigma$ if and only if (x, y, ξ, η) satisfies the two equations of (5) along with the following $2n-2$ equations:

$$\xi_j \frac{\partial r}{\partial y_k} - \xi_k \frac{\partial r}{\partial y_j} = 0, \quad j \neq k \quad (7)$$

$$\eta_j \frac{\partial r}{\partial y_k} - \eta_k \frac{\partial r}{\partial y_j} = 0, \quad j \neq k. \quad (8)$$

It follows immediately from (7) and (8) that we have the following

Lemma 5. *Let Σ be the characteristic set for the natural CR structure induced on the boundary of T_Ω . Then we have*

$$\Sigma = \{(x, y, \xi, \eta) \in T^*(\mathbb{C}^n) : r(y) = 0, \eta = 0, \frac{\xi}{|\xi|} = \pm \frac{dr(y)}{|dr(y)|}\}.$$

We will now study the Treves curves for Σ . Let $\rho^0 = (x^0, y^0, \xi^0, \eta^0) \in \Sigma$ and let $I \subset \mathbb{R}$ be an open interval containing the origin. Assume that

$$\gamma : I \rightarrow \Sigma$$

is a Treves curve such that $\gamma(0) = \rho^0$. If $s \in I$, we write

$$\gamma(s) = (x(s), y(s), \xi(s), \eta(s)). \quad (9)$$

We have the following

Proposition 1. *Assume that $\partial\Omega$ is real analytic. Suppose that γ is a Treves curve for Σ as in (9). Then we have*

$$y(s) = y^0, \quad \xi(s) = \xi^0, \quad \eta(s) = 0 \quad (10)$$

for all $s \in I$. Furthermore we have

$$\left\langle \frac{dx}{ds}(s), dr(y^0) \right\rangle = 0 \quad (11)$$

and

$$\sum_{j=1}^n \frac{\partial^2 r}{\partial y_j \partial y_l}(y^0) \frac{dx_j}{ds}(s) = 0 \quad (12)$$

for all $s \in I$ and $l = 1, \dots, n$. Conversely, any non-constant curve $\gamma : I \rightarrow \Sigma$ satisfying (10), (11) and (12) is a Treves curve for Σ .

Remark 1. If $(x^0, y^0, \xi^0, 0)$ and $(x^1, y^0, \xi^0, 0)$ lie on the same Treves curve for Σ , then define $x(s) = sx^1 + (1-s)x^0$, $s \in \mathbb{R}$. It follows that the straight line $(x(s), y^0, \xi^0, 0)$, $s \in \mathbb{R}$ is a Treves curve for Σ . This follows from (11) and (12). Now we define the **characteristic lines** to be the projection to the base of these straight lines.

Remark 2. Note that by definition, Treves curves are not constant. Hence it follows from the Proposition that if a Treves curve passes through ρ^0 , we must have $x(s)$ not constant. As a consequence we see that y^0 must be a weakly convex boundary point of Ω . So we see that if $\partial\Omega$ is strongly convex, it follows that Σ contains no Treves curves. This is a special case of the general fact that the characteristic set of $\bar{\partial}_b$ is symplectic for any strictly pseudoconvex domain.

Proof. It follows from Lemma 5 that $\eta(s) = 0$ for all $s \in I$. Note that $T\Sigma^-$ is spanned by the Hamilton fields of the defining functions of Σ . According to Lemma 5 these defining functions are independent of x , hence their Hamilton fields are independent of $\frac{\partial}{\partial \xi}$. Thus it follows that $\xi(s) = \xi^0$ for all $s \in I$. We have $r(y(s)) = 0$ for all $s \in I$. Hence we have $\langle \frac{dy}{ds}, dr \rangle = 0$. By Lemma 5 we know that dr is a multiple of ξ . Thus it follows that $\langle y(s) - y^0, \xi^0 \rangle = 0$ for all $s \in I$. Now by Lemma 3 it follows that $y(s) = y^0$ for all $s \in I$. So we see that y, ξ and η must be constant functions of s . We now investigate the behavior of x , which we will see is more complicated.

We assume that $r_k(y^0) \neq 0$ for some $k, 1 \leq k \leq n$. We denote derivatives of r by subscripts. Near $(x^0, y^0, \xi^0, \eta^0)$ we see that Σ is defined by the equations $r = 0; \eta_l = 0, l = 1, \dots, n; p_j = 0, j \neq k$. Here we define $p_j = r_k \xi_j - r_j \xi_k$. Since γ is a Treves curve for Σ , it follows that we have

$$\frac{d\gamma}{ds} = aH_r + \sum_{j \neq k} \alpha_j H_{p_j}$$

for some functions a, α_j . We have used the notation H to denote the Hamilton field. Note that

$$H_r = - \sum_{j=1}^n r_j \partial_{\eta_j},$$

and on Σ we have

$$H_{p_j} = r_k \partial_{x_j} - r_j \partial_{x_k} - \frac{\xi_k}{r_k} \sum_{l=1}^n (r_{kl} r_j - r_{jl} r_k) \partial_{\eta_l}.$$

It follows that

$$\frac{dx_k}{ds} = - \sum_{j \neq k} \alpha_j r_j \tag{13}$$

and that

$$\frac{dx_j}{ds} = r_k \alpha_j, \quad j \neq k. \tag{14}$$

Hence we have

$$\langle \frac{dx}{ds}(s), dr(y^0) \rangle = 0, \quad s \in I.$$

Since we know that η vanishes identically, it follows that we have

$$-ar_l - \frac{\xi_k}{r_k} r_{kl} \sum_{j \neq k} \alpha_j r_j + \xi_k \sum_{j \neq k} \alpha_j r_{jl} = 0, \quad l = 1, \dots, n.$$

It follows now from (13) and (14) that

$$ar_l = \frac{\xi_k}{r_k} \sum_{j=1}^n \frac{dx_j}{ds} r_{jl}, \quad l = 1, \dots, n.$$

Summing over l we have

$$0 = a \left\langle \frac{dx}{ds}, dr \right\rangle = \frac{\xi_k}{r_k} \sum_{l,j=1}^n r_{jl} \frac{dx_j}{ds} \frac{dx_l}{ds}.$$

Since the matrix (r_{jl}) is non-negative and symmetric it follows that

$$\sum_{j=1}^n r_{jl} \frac{dx_j}{ds} = 0, \quad l = 1, \dots, n.$$

Now assume that $\gamma : I \rightarrow \Sigma$ is any curve satisfying (10), (11) and (12). Assuming as before that $r_k(y^0) \neq 0$, it follows that we have

$$\sum_{j \neq k} \frac{dx_j}{ds} H_{p_j} = r_k \sum_{j=1}^n \frac{dx_j}{ds} \partial_{x_j}$$

and the converse follows. □

5. THE FORMULA

We give here a brief discussion of a result of Boutet de Monvel, [1]. See also Koranyi [21], Vinberg [32], Faraut and Koranyi [8]. Let $\Omega \subset \mathbb{R}^n$ be open. If Ω is bounded and convex, then we have the following formula for the Bergman kernel of T_Ω . Note that no assumptions on $\partial\Omega$ are necessary for the validity of this formula.

If we denote by \mathcal{B} the Bergman kernel of T_Ω , then we have for $z, w \in T_\Omega$

$$\mathcal{B}(z, w) = \int_{\mathbb{R}^n} e^{i\langle z - \bar{w}, \xi \rangle} A(\xi)^{-1} \frac{d\xi}{(2\pi)^n} \quad (15)$$

where we define

$$A(\xi) = \int_{\Omega} e^{-2\langle \xi, y \rangle} dy. \quad (16)$$

If the boundary of Ω is of class C^2 we may use Green's theorem to obtain

$$A(\xi) = \frac{1}{2|\xi|} \int_{\partial\Omega} e^{-2\langle \xi, y \rangle} \left\langle -\frac{dr}{|dr|}, \frac{\xi}{|\xi|} \right\rangle d\sigma(y) \quad (17)$$

where r is a defining function for Ω and $d\sigma(y)$ denotes the surface area on $\partial\Omega$.

6. SOME REGULARITY RESULTS

Our goal in this section is to prove half of Theorem 1. That is, we will prove that if $z^0, w^0 \in \partial T_\Omega \subset \mathbb{C}^2$ and z^0 and w^0 do not lie on the same characteristic line, then \mathcal{B} , the Bergman kernel for T_Ω , extends as an analytic function to a full neighborhood of (z^0, w^0) . This result follows from Theorems 3, 7 and 8 which are proved below. Theorem 3 is a consequence of the following result.

Lemma 6. *Let $y^1, y^2 \in \partial\Omega$, $y^1 \neq y^2$. Then $ty^1 + (1-t)y^2 \in \Omega$ for all $t \in (0, 1)$.*

Proof. Since $\bar{\Omega}$ is convex $ty^1 + (1-t)y^2 \in \bar{\Omega}$, that is $r(ty^1 + (1-t)y^2) \leq 0$ for all $t \in [0, 1]$. If there is a point $y^3 = t_0y^1 + (1-t_0)y^2 \in \partial\Omega$ for some $t_0 \in (0, 1)$, then the function $t \mapsto r(ty^1 + (1-t)y^2)$ has a local maximum at t_0 so

$$\frac{d}{dt}(r(ty^1 + (1-t)y^2))\Big|_{t=t_0} = \sum_{j=1}^n \frac{\partial r}{\partial y_j}(y^3)(y_j^1 - y_j^2) = 0.$$

Since $y^1 - y^3 = (1-t_0)(y^1 - y^2)$ and $y^2 - y^3 = -t_0(y^1 - y^2)$ the supporting hyperplane at y^3 , $\mathcal{P}_{y^3}^{dr(y^3)} = \{y \in \mathbb{R}^n : \langle y - y^3, dr(y^3) \rangle = 0\}$ contains both y^1 and y^2 . So $y^1, y^2 \in \bar{\Omega} \cap \mathcal{P}_{y^3}^{dr(y^3)}$ and using Lemma 3 we get $y^1 = y^2 = y^3$ which contradicts the assumption of Lemma 6. \square

We are now in a position to prove Theorem 3.

Proof. Let $z = x + iy$ and $w = u + iv$. Since

$$\mathcal{B}(z, w) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\langle x-u, \xi \rangle} \frac{1}{\int_{\Omega} e^{-2\langle \xi, \sigma - \frac{y+v}{2} \rangle} d\sigma} d\xi$$

it is enough to prove that

$$\int_{\Omega} e^{-2\langle \xi, \sigma - \frac{y+v}{2} \rangle} d\sigma = \int_{-\frac{y+v}{2} + \Omega} e^{-2\langle \xi, \sigma \rangle} d\sigma$$

is exponentially increasing in ξ as long as (y, v) stays in a small neighborhood of (y^0, v^0) . Let $\sigma^0 = \frac{y^0 + v^0}{2}$. From Lemma 6 it follows that $\sigma^0 \in \Omega$. Select a small open ball $B_{2\delta}(\sigma^0)$ in Ω . By continuity $\frac{y+v}{2} \in B_\delta(\sigma^0)$ provided that (y, v) is close enough to (y^0, v^0) . Then $\frac{y+v}{2} + B_\delta(0) \subset B_{2\delta}(\sigma^0) \subset \Omega$ so $-\frac{y+v}{2} + \Omega \supset B_\delta(0)$ and we have

$$\int_{-\frac{y+v}{2} + \Omega} e^{-2\langle \xi, \sigma \rangle} d\sigma \geq \int_{B_\delta(0)} e^{-2\langle \xi, \sigma \rangle} d\sigma = \delta^n \int_{|\sigma| < 1} e^{-2\langle \xi, \delta\sigma \rangle} d\sigma.$$

Let G be the set $G \equiv \{\sigma \in \mathbb{R}^n : -\xi \cdot \sigma \geq \frac{1}{2}|\sigma||\xi|, \frac{1}{2} < |\sigma| < 1\}$. Note that G depends on ξ , but that the measure of G is independent of ξ , as long as $\xi \neq 0$. Then the last integral is greater than

$$\delta^n \int_G e^{-2\langle \xi, \delta\sigma \rangle} d\sigma \geq \delta^n \int_G e^{\delta|\sigma||\xi|} d\sigma \geq \delta^n \int_G e^{\frac{1}{2}\delta|\xi|} d\sigma = c(y^0, v^0, \Omega) \delta^n e^{\frac{1}{2}\delta|\xi|}.$$

This completes the proof of Theorem 3. \square

It is easy to see that the arguments used in the proofs of Lemma 6 and Theorem 3 can also be used to prove Theorem 4. Hence we leave the proof of Theorem 4 to the reader.

Next we present some localizing results.

Theorem 5. *Let $z^0, w^0 \in \mathbb{C}^n$, with $\Im z^0 = \Im w^0 = y^0 \in \partial\Omega$. Let $V \subset \mathbb{R}^n$ be a closed conic set that does not contain $-dr(y^0)$. Then the function*

$$\frac{1}{(2\pi)^n} \int_V e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A(\xi)} d\xi$$

is real analytic for (z, w) near (z^0, w^0) .

Proof. We write $z = x + iy$ and $w = u + iv$. It is enough to prove that

$$\int_{\Omega} e^{-2\langle \xi, \sigma - \frac{y+v}{2} \rangle} d\sigma$$

is exponentially increasing in $|\xi|$, for $\xi \in V$ and (y, v) near (y^0, y^0) . Let $\xi^0 \in V$, with $|\xi^0| = 1$. Since ξ^0 does not have the same direction as $-dr(y^0)$, it follows that

$$\{\sigma \in \Omega : \langle \xi^0, \sigma - y^0 \rangle < 0\} \neq \emptyset.$$

Hence there exist $\epsilon > 0$ and $\delta > 0$ such that if $|\frac{\xi}{|\xi|} - \xi^0| < \delta$ and $|y - y^0| < \delta$ and $|v - y^0| < \delta$ then we have

$$G = \{\sigma \in \Omega : \langle \xi, \sigma - \frac{y+v}{2} \rangle < -\epsilon|\xi|\} \neq \emptyset.$$

In particular, G has positive measure since G is open. Now we have

$$\int_{\Omega} e^{-2\langle \xi, \sigma - \frac{y+v}{2} \rangle} d\sigma \geq \int_G e^{-2\langle \xi, \sigma - \frac{y+v}{2} \rangle} d\sigma \geq \int_G e^{2\epsilon|\xi|} d\sigma \geq C(\xi^0, y^0) e^{2\epsilon|\xi|}.$$

Now the theorem follows by the compactness of the unit ball in ξ space. \square

Theorem 5 allows us to localize in the ξ variable. Now we will discuss localizing on $\partial\Omega$. We define

$$S_{\delta, y^0} = \{y \in \partial\Omega : |y - y^0| \leq \delta\}.$$

and

$$A_{\delta, y^0}(\xi) = \frac{1}{2|\xi|} \int_{S_{\delta, y^0}} e^{-2\langle \xi, y \rangle} \left\langle -\frac{dr}{|dr|}, \frac{\xi}{|\xi|} \right\rangle d\sigma(y).$$

We have the following

Theorem 6. *Let $z^0, w^0 \in \mathbb{C}^n$, with $\Im z^0 = \Im w^0 = y^0 \in \partial\Omega$. Let $\Gamma \subset \mathbb{R}^n$ be a small conic neighborhood of $-dr(y^0)$. Then the function*

$$\mathcal{B}(z, w) - \frac{1}{(2\pi)^n} \int_{\Gamma} e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A_{\delta, y^0}(\xi)} d\xi$$

can be extended as a real analytic function to a full neighborhood of (z^0, w^0) .

Proof. We will begin by choosing convenient coordinates. Note that the formula (15) is invariant under translations and real rotations. Hence we may assume that $y^0 = 0$. We may also assume that we have $\delta > 0$ and φ real valued and real analytic near $|y'| \leq \delta$ such that r has the form $r(y) = \varphi(y') - y_n$ with $d\varphi(0) = 0$. Here $y' = (y_1, \dots, y_{n-1})$. Hence $dr(0) = (0, \dots, 0, -1)$. So we may assume that

$$S_{\delta, y^0} = \{y \in \partial\Omega : y_n = \varphi(y'), |y'| \leq \delta\}.$$

Given $M > 0$, we define Γ as follows:

$$\Gamma = \{\xi \in \mathbb{R}^n : \xi_n > M|\xi'|\}. \quad (18)$$

Note that if $M > 0$ is large, Γ will be a small conic neighborhood of $-dr(0)$. We define $V = \mathbb{R}^n \setminus \Gamma$ and for the rest of the proof we write A_δ in place of A_{δ, y^0} .

Using formula (15) we see that

$$\begin{aligned} \mathcal{B}(z, w) - \frac{1}{(2\pi)^n} \int_{\Gamma} e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A_\delta(\xi)} d\xi = \\ \frac{1}{(2\pi)^n} \int_V e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A(\xi)} d\xi + \frac{1}{(2\pi)^n} \int_{\Gamma} e^{i\langle z - \bar{w}, \xi \rangle} \left(\frac{1}{A(\xi)} - \frac{1}{A_\delta(\xi)} \right) d\xi. \end{aligned}$$

By applying Theorem 5 and using the fact that $\Im(z - \bar{w})$ is near 0, we see that the theorem will be proved if the quantity

$$\frac{1}{A} - \frac{1}{A_\delta} = \frac{A_\delta - A}{AA_\delta}$$

is exponentially decreasing for $\xi \in \Gamma$, with M sufficiently large. This follows from the next two lemmas. \square

Lemma 7. *Let $\delta > 0$ be given. Then there exist $M > 0$ and $\epsilon > 0$ such that if $\xi_n \geq M|\xi'|\}$ then*

$$|A(\xi) - A_\delta(\xi)| \leq \frac{1}{\epsilon} e^{-\epsilon|\xi|}.$$

Proof. Since $\partial\Omega$ is convex, bounded and analytic, it follows that there exists $C_\delta > 0$ such that if $y \in \partial\Omega \setminus S_{\delta, y^0}$ then $y_n \geq C_\delta$.

Now consider

$$\langle y, \xi \rangle = \langle y', \xi' \rangle + y_n \xi_n \geq -|y'| |\xi'| + C_\delta \xi_n \geq -\frac{|y'| \xi_n}{M} + C_\delta \xi_n \geq \left(C_\delta - \frac{K}{M}\right) \xi_n.$$

Here K is the diameter of Ω . The lemma follows once M is chosen large enough. \square

Our next result uses stationary phase methods. Note that the critical point (which may be degenerate) for the function $\langle y', \xi' \rangle + \varphi(y') \xi_n$ is given by the equation $\varphi'(y'_*) = -\frac{\xi'}{\xi_n}$. Observe that φ' is a one to one map, since φ is convex and analytic, and because $\partial\Omega$ contains no straight lines. Hence we denote by y'_* the unique critical point, which is a function of the parameter ξ .

The next lemma gives us a bound from below for A_δ and hence for A also, via Lemma 7.

Lemma 8. *Let $\delta > 0$ be given. Then there exist $M > 0$ and $C > 0$ such that if $\xi_n \geq M|\xi'|$ then*

$$A_\delta(\xi) \geq \frac{C}{\xi_n^{(n+1)/2}} e^{-2\xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle)} \geq \frac{C}{\xi_n^{(n+1)/2}}.$$

Proof. We clearly have $C > 0$ such that

$$A_\delta(\xi) \geq \frac{C}{\xi_n} \int_{|y'| \leq \delta} e^{-2(\langle y', \xi' \rangle + \varphi(y')\xi_n)} dy'.$$

The Taylor expansion about the critical point y'_* gives

$$\begin{aligned} \langle y', \xi' \rangle + \varphi(y')\xi_n &= \langle y'_*, \xi' \rangle + \varphi(y'_*)\xi_n + \xi_n R = \\ &= \xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle + R), \end{aligned}$$

with $|R| \leq C|y' - y'_*|^2$. Here $C > 0$ depends on φ but not on ξ .

Since φ is convex, it follows that we have

$$\varphi(y') \geq \varphi(y'_*) + \langle \varphi'(y'_*), y' - y'_* \rangle,$$

for $|y'| \leq \delta$. In particular, letting $y' = 0$ we obtain

$$\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle \leq 0.$$

Now it follows that

$$\langle y', \xi' \rangle + \varphi(y')\xi_n \leq \xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle + C|y' - y'_*|^2) \leq C\xi_n|y' - y'_*|^2,$$

for $\xi_n > 0$. We have

$$\begin{aligned} & \int_{|y'| \leq \delta} e^{-2(\langle y', \xi' \rangle + \varphi(y')\xi_n)} dy' \\ & \geq e^{-2\xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle)} \int_{|y'| \leq \delta} e^{-C\xi_n|y' - y'_*|^2} dy' \\ & \geq e^{-2\xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle)} \int_T e^{-C\xi_n|y' - y'_*|^2} dy' \\ & \geq \frac{C}{\xi_n^{(n-1)/2}} e^{-2\xi_n(\varphi(y'_*) - \langle \varphi'(y'_*), y'_* \rangle)} \\ & \geq \frac{C}{\xi_n^{(n-1)/2}}, \end{aligned} \tag{19}$$

where $T = \{y' : |y' - y'_*| \leq 1/\sqrt{\xi_n}\}$. □

Theorem 7. *Let $z^0, w^0 \in \mathbb{C}^2$, with $\Im z^0 = \Im w^0 = y^0$. Let y^0 be a weakly convex boundary point of Ω . Then if z^0 and w^0 do not lie on the same characteristic line, then \mathcal{B} , the Bergman kernel for T_Ω , can be extended as a real analytic function to a full neighborhood of (z^0, w^0) .*

Proof. We may assume that $y^0 = (0, 0) \in \mathbb{R}^2$. In addition we may assume that near $(0, 0)$, the boundary of Ω is defined by $\{r < 0\}$, where $r(y) = \varphi(y_1) - y_2$. Here φ is real valued and real analytic for $|y_1| \leq \delta$, some $\delta > 0$. Also we may assume that $\varphi(0) = 0 = \varphi'(0)$. Since $(0, 0)$ is a weakly convex point, it follows that there is an even integer $m \geq 4$ and a real analytic function u defined near $|y_1| \leq \delta$ such that $u(0) > 0$ and

$$\varphi(y_1) = u(y_1)y_1^m.$$

We have $z^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ and $w^0 = (u_1^0, u_2^0) \in \mathbb{R}^2$. Note that $(1, 0)$ is the direction of the tangent vector to $\partial\Omega$ at the point $(0, 0)$. Hence $(1, 0)$ is the direction of the characteristic line through z^0 or w^0 . Thus, if z^0 and w^0 do not lie on the same characteristic line, we must have $x_2^0 \neq u_2^0$.

Since $dr(0) = (0, -1)$, we will focus on

$$Q(z, w) = \int_{\Gamma} e^{i\langle z - \bar{w}, \xi \rangle} A_{\delta}(\xi)^{-1} \frac{d\xi}{(2\pi)^2} \quad (20)$$

where

$$\Gamma = \{\xi \in \mathbb{R}^2 : M|\xi_1| \leq \xi_2\}.$$

By Theorem 6, it suffices to show that Q is analytic near (z^0, w^0) .

In the present case, we have

$$A_{\delta}(\xi) = \frac{1}{2|\xi|} \int_{-\delta}^{\delta} e^{-2(\xi_1 s + \xi_2 \varphi(s))} a(s, \xi) ds \quad (21)$$

where

$$a(s, \xi) = \frac{\xi_2 - \varphi'(s)\xi_1}{|\xi|}. \quad (22)$$

We make the change of variable

$$\xi_1 = \rho\eta, \xi_2 = \rho^m/\alpha$$

where we define $\alpha = u(0)$. We obtain

$$Q(z, w) = \frac{m}{\alpha} \int_0^{\infty} \int_{|\eta| \leq \rho^{m-1}/\alpha M} e^{i((z_1 - \bar{w}_1)\rho\eta + (z_2 - \bar{w}_2)\rho^m/\alpha)} \rho^m A_{\delta}(\rho\eta, \rho^m/\alpha)^{-1} d\eta d\rho.$$

Now we will exploit the fact that $x_2^0 \neq u_2^0$. Let $\epsilon > 0$ be given. We make the change of contour

$$\rho \rightarrow (1 + i\epsilon(x_2^0 - u_2^0))^{1/m} \rho = \rho_{\epsilon}.$$

We now show that given $\epsilon > 0$, there exists an $M > 0$, which gives us good control of the phase. Indeed, when $(z, w) = (z^0, w^0)$, we have

$$\Im((z_1 - \bar{w}_1)\rho_{\epsilon}\eta + (z_2 - \bar{w}_2)\rho_{\epsilon}^m/\alpha) \geq \epsilon(x_2^0 - u_2^0)^2 \rho^m/\alpha - \epsilon C \rho^m/M > C' \rho^m$$

once $M > 0$ is chosen large enough.

Stationary phase arguments (as in the proofs of Theorem 9 or Lemma 15) give the following estimate for ρ large:

$$|A_\delta(\rho_\epsilon \eta, \rho_\epsilon^m / \alpha)| \geq \frac{C}{\rho^{m+1}} (1 + |\eta|)^{\frac{2-m}{2(m-1)}} e^{C|\eta|^{\frac{m}{m-1}}}.$$

It follows that Q is analytic near (z^0, w^0) by differentiating under the integral sign. \square

The next result probably can be proved using the methods of Geller [12] on strictly pseudoconvex domains. However, his results are proved only for bounded domains.

Theorem 8. *Let $z^0, w^0 \in \mathbb{C}^2$, with $\Im z^0 = \Im w^0 = y^0$. Let y^0 be a strongly convex boundary point of Ω . If $z^0 \neq w^0$, then \mathcal{B} , the Bergman kernel for T_Ω , can be extended as a real analytic function to a full neighborhood of (z^0, w^0) .*

Proof. We use the same localization as in the proof of Theorem 7. In particular we assume that $y^0 = (0, 0) \in \mathbb{R}^2$, and that $z^0 = (x_1^0, x_2^0) \in \mathbb{R}^2$ and $w^0 = (u_1^0, u_2^0) \in \mathbb{R}^2$. Strong convexity here means that $m = 2$. If we have $x_2^0 \neq u_2^0$, then we may proceed as in the proof of Theorem 7. From now on we may assume that $x_2^0 = u_2^0$, and that $x_1^0 \neq u_1^0$.

We first analyse $A_\delta(\xi)$ by stationary phase. Recall that $\varphi(s) = u(s)s^2$, with $u(0) > 0$. The critical points, s_* , of the phase

$$\xi_1 s + \xi_2 \varphi(s)$$

are given by the equation

$$\varphi'(s_*) = -\xi_1 / \xi_2.$$

In what follows the quantity ξ_1 / ξ_2 will always vary in a small complex neighborhood of zero. On the other hand, $\xi_2 > 0$ will be large. It follows that

$$-2(\xi_1 s_* + \xi_2 \varphi(s_*)) = 2(\xi_1^2 / \xi_2) \tilde{u}(s_*).$$

We define \tilde{u} as follows

$$\tilde{u}(s) = \frac{u(s) + s u'(s)}{(2u(s) + s u'(s))^2}.$$

As long as ξ_1 / ξ_2 is small, we may assume that $\Re \tilde{u}(s_*)$ is strictly positive, while $\Im \tilde{u}(s_*)$ is small. This follows since $u(0) > 0$, u is analytic and real valued.

Stationary phase arguments yield

$$A_\delta(\xi) = \tilde{a}(\xi) e^{2(\xi_1^2 / \xi_2) \tilde{u}(s_*)},$$

where \tilde{a} is holomorphic in ξ . Also, for $\xi_2 > 0$, large and ξ_1 / ξ_2 near zero and complex we have $C > 0$ such that

$$|\tilde{a}(\xi)| \geq C \xi_2^{-3/2}.$$

We study Q as in the proof of Theorem 7, keeping in mind that $m = 2$.

$$Q(z, w) =$$

$$\frac{2}{\alpha} \int_0^\infty \int_{|\eta| \leq \rho / \alpha M} e^{i((z_1 - \bar{w}_1)\rho\eta + (z_2 - \bar{w}_2)\rho^2 / \alpha)} \rho^2 (\tilde{a}(\rho\eta, \rho^2 / \alpha))^{-1} e^{-2\alpha\eta^2 \tilde{u}(s_*)} d\eta d\rho.$$

Next, we take advantage of the fact that $x_1^0 \neq u_1^0$. We make the change of contour

$$\eta \rightarrow \eta + i\epsilon t\rho,$$

where $t = x_1^0 - u_1^0 \neq 0$. Note that A_δ can have no zeroes. Indeed, there are none when $u(s)$ is identically equal to one and $m = 2$. This follows from known results concerning the function \mathcal{N} . This is explained in section 7. In particular, see the argument preceding equation (44). Hence there is no problem with this large contour deformation.

We now must estimate the exponential, when $z = z^0$ and $w = w^0$. For the moment we write $\tilde{u}(s_\star) = a + ib$. That is, we must estimate

$$\begin{aligned} & \Re(it\rho(\eta + i\epsilon t\rho) - 2\alpha(\eta + i\epsilon t\rho)^2\tilde{u}(s_\star)) \\ &= -\epsilon t^2\rho^2(1 - 2\alpha\epsilon a - 2\alpha\epsilon a(b/a)^2) - 2a\alpha(\eta - \epsilon t\rho(b/a))^2 \\ &\leq -C\rho^2 - 2a\alpha(\eta - \epsilon t\rho(b/a))^2, \end{aligned}$$

provided that $\epsilon > 0$ is small enough. Now the result follows by differentiating under the integral sign. Observe that each derivative contributes one factor of ρ^2 , hence the bound $e^{-C\rho^2}$ yields the analyticity. \square

7. GEVREY REGULARITY

In this section we prove the first part of Theorem 2. The fact that m is the best Gevrey class possible is proved at the end of Section 9.

Theorem 9. *Assume that the two distinct points $z^0, w^0 \in \partial T_\Omega \subset \mathbb{C}^2$ lie on the same characteristic line. Let m be the type of the point z^0 . Then the Bergman kernel of the tube T_Ω , $\mathcal{B}(z, w)$, can be extended as a smooth function of Gevrey class m , to a full neighborhood of the point (z^0, w^0) .*

Proof. The assumption that $z^0, w^0 \in \partial T_\Omega$ lie on the same characteristic line means that we have $t \in \mathbb{R} \setminus \{0\}$, $e \in \mathbb{R}^2$, $|e| = 1$ such that $\Re z^0 = \Re w^0 + te$, $e \perp \nabla r(y^0)$, $e \in \text{Ker } r''(y^0)$. After translating and rotating the domain Ω we may assume that $y^0 = 0$, $\nabla r(y^0) = (0, -1)$ and $e = (1, 0)$. So we have $x^0 \equiv \Re z^0 = \Re w^0 + t(1, 0)$. Near $(0, 0) \in \partial\Omega$ the boundary of Ω is given by $y_2 = \varphi(y_1)$. The function φ is nonnegative, convex, real analytic and $\varphi(0) = \varphi'(0) = \dots = \varphi^{(m-1)}(0) = 0$, $\varphi^{(m)}(0) = am! > 0$, and $m > 2$ is even. We may also assume that φ satisfies the estimates

$$\frac{a}{2} \binom{m}{k} y^m \leq \frac{\varphi^{(k)}(y)y^k}{k!} \leq \frac{3a}{2} \binom{m}{k} y^m \quad (23)$$

for $k = 0, \dots, m$ and

$$\varphi^{(4)}(y) \geq 0 \quad (24)$$

in the interval $[-\delta, \delta]$ for some $\delta > 0$.

Let $B_{\epsilon_0}(z^0, w^0) \subset \mathbb{C}^4$ be the open ball of radius $\epsilon_0 > 0$ centered at $(z^0, w^0) \in \mathbb{C}^2 \times \mathbb{C}^2$. We remark that it is sufficient to prove the Gevrey estimate in the set

$$B_{\epsilon_0}(z^0, w^0) \cap \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2; \Im z \in \overline{\Omega}, \Im w \in \overline{\Omega}\},$$

that is, for the points $(z, w) \equiv (x + iy, u + iv)$ near (z^0, w^0) satisfying the conditions

$$y_2 \geq \varphi(y_1), \quad v_2 \geq \varphi(v_1) \quad (25)$$

for all $y_1, v_1 \in (-\delta, \delta)$.

After taking derivatives, there exists $C > 0$ such that

$$C \partial_{x_1}^{\alpha_1} \partial_{y_1}^{\alpha_2} \partial_{u_1}^{\alpha_3} \partial_{v_1}^{\alpha_4} \partial_{x_2}^{\beta_1} \partial_{y_2}^{\beta_2} \partial_{u_2}^{\beta_3} \partial_{v_2}^{\beta_4} B(z, w) = \int_{\mathbf{R}^2} \xi_1^{|\alpha|} \xi_2^{|\beta|} e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A(\xi)} d\xi.$$

Let Γ_M be the cone $\Gamma_M = \{\xi \in \mathbf{R}^2; \xi_2 > M|\xi_1|\}$. Then it follows from the localization theorem, Theorem 5, that it is enough to prove the Gevrey estimate for the integral

$$\int_{\Gamma_M} \xi_1^{|\alpha|} \xi_2^{|\beta|} e^{i\langle z - \bar{w}, \xi \rangle} \frac{1}{A(\xi)} d\xi. \quad (26)$$

Consider the integral

$$A_{\delta, b}(\xi) = \int_{-\delta}^{\delta} \int_{\varphi(y_1)}^b e^{-2(\xi_1 y_1 + \xi_2 y_2)} dy_2 dy_1$$

for some $b > 0$. We may assume that $\delta > 0$ is so small that $\{y \in \mathbb{R}^2; |y_1| < \delta, \varphi(y_1) < y_2 < b\} \subset \Omega$. Then an argument similar to Theorem 6 shows that we can replace A in (26) by $A_{\delta, b}$. After changing the coordinates to $\xi_1 = \rho\eta$, $\xi_2 = \frac{\rho^m}{a}$ we need to estimate the integral

$$J \equiv \frac{m}{a^{|\beta|+1}} \int_{\tilde{\Gamma}_M} e^{i(\rho\eta(z_1 - \bar{w}_1) + \frac{\rho^m}{a}(z_2 - \bar{w}_2))} \frac{\rho^{|\alpha|+m|\beta|+m} \eta^{|\alpha|}}{A_{\delta, b}(\rho\eta, \rho^m/a)} d\rho d\eta \quad (27)$$

when (z, w) is near $(z^0, w^0) \equiv (x^0, u^0)$. The domain of integration is $\tilde{\Gamma}_M = \{(\eta, \rho) \in \mathbb{R}^2; \rho > (aM|\eta|)^{\frac{1}{m-1}}\}$. Taking advantage of the fact that $x_1^0 \neq u_1^0$, we deform the contour of the η -integration in J as $\eta \mapsto \eta + 2i\varepsilon(x_1^0 - u_1^0)$. The new exponential factor is e^{iE} with $E \equiv (\eta + 2i\varepsilon(x_1^0 - u_1^0))\rho(z_1 - \bar{w}_1) + \frac{\rho^m}{a}(z_2 - \bar{w}_2)$. We have the lower bound for

$$\begin{aligned} \Im E &= 2\varepsilon(x_1^0 - u_1^0)(x_1 - u_1)\rho + \rho\eta(y_1 + v_1) + \frac{\rho^m}{a}(y_2 + v_2) \\ &\geq \varepsilon(x_1^0 - u_1^0)^2\rho + \rho\eta(y_1 + v_1) + \frac{\rho^m}{a}(y_2 + v_2) \end{aligned} \quad (28)$$

provided (x_1, u_1) is close enough to (x_1^0, u_1^0) .

Here we use assumption (25) and the convexity of the function φ to obtain

$$y_2 + v_2 \geq \varphi(y_1) + \varphi(v_1) \geq 2\varphi\left(\frac{y_1 + v_1}{2}\right).$$

So the exponential factor in J satisfies the upper bound

$$|e^{iE}| \leq e^{-\varepsilon(x_1^0 - u_1^0)^2 \rho - 2(\rho \eta \frac{y_1 + v_1}{2} + \frac{\rho^m}{a} \varphi(\frac{y_1 + v_1}{2}))}.$$

Consider the function

$$(-\delta, \delta) \ni y \mapsto \rho \eta y + \frac{\rho^m}{a} \varphi(y). \quad (29)$$

It follows from the convexity and analyticity of φ that φ' is strictly increasing and therefore the function (29) has a unique critical point $y_* = y_*(\rho, \eta)$. The critical point satisfies the equation $\varphi'(y_*) = -\frac{a\eta}{\rho^{m-1}}$. If $(\rho, \eta) \in \tilde{\Gamma}_M$ then $-\frac{a\eta}{\rho^{m-1}}$ lies in the set $\varphi'((-\delta/2, \delta/2))$ therefore $y_* \in (-\delta/2, \delta/2)$ provided that the constant M is large enough.

Since the function (29) is convex, its minimum occurs at $y_* = y_*(\rho, \eta)$. This yields the estimate

$$|e^{iE}| \leq e^{-\varepsilon(x_1^0 - u_1^0)^2 \rho - 2(\rho \eta y_* + \frac{\rho^m}{a} \varphi(y_*))}. \quad (30)$$

Next step in the proof is to obtain a lower bound for $|A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \rho^m/a)|$. We have

$$\begin{aligned} A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \rho^m/a) &= \\ &= \int_{-\delta}^{\delta} \int_{\varphi(y_1)}^b e^{-2(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0))y_1 + \frac{\rho^m}{a}y_2)} dy_2 dy_1 \\ &= \frac{a}{2\rho^m} \int_{-\delta}^{\delta} e^{-2(\rho \eta y_1 + \frac{\rho^m}{a} \varphi(y_1))} e^{-4i\varepsilon \rho(x_1^0 - u_1^0)y_1} [1 - e^{\frac{2\rho^m}{a}(\varphi(y_1) - b)}] dy_1 \\ &= \frac{a}{2\rho^m} \int_{-\delta}^{\delta} e^{-f(y_1)} g(y_1) dy_1, \end{aligned} \quad (31)$$

where the functions f and g are defined as $f(y) = 2(\rho \eta y + \frac{\rho^m}{a} \varphi(y))$, and

$$g(y) = e^{-4i\varepsilon \rho(x_1^0 - u_1^0)y} [1 - e^{\frac{2\rho^m}{a}(\varphi(y) - b)}].$$

To estimate the last integral we use the stationary phase method as in Lemma 12. We start with decomposing the last integral into the sum of the four integrals

$$J_1 = e^{-f(y_*)} \int_{-\delta}^{\delta} e^{-(f(y) - f(y_*))} (g(y) - g(y_*)) dy, \quad (32)$$

$$J_2 = g(y_*) e^{-f(y_*)} \int_{-\delta}^{\delta} e^{-(f(y) - f(y_*))} - e^{-\frac{f''(y_*)}{2}(y - y_*)^2} dy, \quad (33)$$

$$J_3 = g(y_*) e^{-f(y_*)} \int_{-\infty}^{\infty} e^{-\frac{f''(y_*)}{2}(y - y_*)^2} dy, \quad (34)$$

$$J_4 = -g(y_*) e^{-f(y_*)} \int_{|y| \geq \delta} e^{-\frac{f''(y_*)}{2}(y - y_*)^2} dy. \quad (35)$$

The integral J_3 can be evaluated explicitly

$$J_3 = \sqrt{\frac{2\pi}{f''(y_*)}} g(y_*) e^{-f(y_*)}. \quad (36)$$

The fourth integral

$$J_4 = -g(y_*) e^{-f(y_*)} \int_{|x+y_*| \geq \delta} e^{-\frac{f''(y_*)}{2} x^2} dx$$

can be estimated easily. If $|x + y_*| \geq \delta$ then $x^2 \geq |y_*|^2$ because $y_* \in [-\delta/2, \delta/2]$. So we can use $\frac{f''(y_*)}{2} x^2 \geq \frac{f''(y_*)}{4} x^2 + \frac{f''(y_*)}{4} |y_*|^2$ to obtain

$$|J_4| \leq \sqrt{\frac{4\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} e^{-\frac{1}{4} f''(y_*) |y_*|^2}.$$

In the exponent we have $f''(y_*) |y_*|^2 = 2 \frac{\rho^m}{a} \varphi''(y_*) |y_*|^2 \geq m(m-1) \rho^m |y_*|^m$, using (23) with $k = 2$.

The critical point can be estimated by using (23) with $k = 1$. Since $|\varphi'(y_*)| \leq \frac{3am}{2} |y_*|^{m-1}$ we get

$$|y_*| \geq \left(\frac{2}{3m}\right)^{\frac{1}{m-1}} \frac{|\eta|^{\frac{1}{m-1}}}{\rho}. \quad (37)$$

So the exponent $f''(y_*) |y_*|^2 \geq c_1(m) |\eta|^{\frac{m}{m-1}}$. Therefore we obtain

$$|J_4| \leq \frac{1}{6} \sqrt{\frac{2\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} = \frac{1}{6} |J_3| \quad (38)$$

for $|\eta| \geq \eta_0(a, m)$.

To estimate the integrals J_1, J_2 we need the following lemma.

Lemma 9. *The function $f(y, \xi) = 2(\xi_1 y + \xi_2 \varphi(y))$ satisfies the inequality*

$$f(y) \geq f(y_*) + a \xi_2 y_*^{m-2} (y - y_*)^2 \quad (39)$$

for all $\xi \in \Gamma_M$ and $y \in [-\delta, \delta]$.

Proof. Case I: $\xi_1 > 0$. It follows from (23) that $y_* < 0$ in this case. First we estimate $f(y)$ in the interval $[y_*, \delta]$. We have

$$f(y) = f(y_*) + \sum_{k=2}^{m-1} \frac{f^{(k)}(y_*)}{k!} (y - y_*)^k + \frac{f^{(m)}(\tilde{y})}{m!} (y - y_*)^m$$

for some \tilde{y} between y_* and y . The inequality (23) implies that

$$\frac{f^{(m)}(\tilde{y})}{m!} (y - y_*)^m \geq a \xi_2 (y - y_*)^m = a \xi_2 y_*^m \left(\frac{y}{y_*} - 1\right)^m$$

and

$$\frac{f^{(k)}(y_*)}{k!}(y - y_*)^k = \frac{f^{(k)}(y_*)y_*^k}{k!}\left(\frac{y}{y_*} - 1\right)^k \geq a\xi_2 \binom{m}{k} y_*^m \left(\frac{y}{y_*} - 1\right)^k.$$

So

$$f(y) \geq f(y_*) + a\xi_2 y_*^m \sum_{k=2}^m \binom{m}{k} \left(\frac{y}{y_*} - 1\right)^k = f(y_*) + a\xi_2 y_*^m \beta\left(\frac{y}{y_*} - 1\right)$$

where the function β is defined as $\beta(t) = (t+1)^m - mt - 1$. It is elementary to see that $\beta(t) \geq t^2$ for all $t \in \mathbf{R}$. This proves the inequality (39) for $y \in [y_*, \delta]$.

In the interval $[-\delta, y_*)$ it is enough to use fourth order Taylor expansion

$$f(y) = f(y_*) + \frac{f''(y_*)}{2}(y - y_*)^2 + \frac{f^{(3)}(y_*)}{3!}(y - y_*)^3 + \frac{f^{(4)}(\tilde{y})}{4!}(y - y_*)^4.$$

The third term is nonnegative when $y \leq y_*$ because it follows from $\xi_1 < 0$ and (23) that $f^{(3)}(y_*) < 0$. The fourth term is always nonnegative because $\varphi^{(4)} \geq 0$. So using (23) again we get

$$f(y) \geq f(y_*) + \frac{f''(y_*)}{2}(y - y_*)^2 \geq f(y_*) + a \binom{m}{2} \xi_2 y_*^{m-2} (y - y_*)^2$$

for $y \in [-\delta, y_*]$. This completes Case I.

Case II: $\xi_1 < 0$. We introduce the function $\psi(x) = \varphi(-x)$ and define the function g as $g(x) = 2(-\xi_1 x + \xi_2 \psi(x))$ for $x \in [-\delta, \delta]$. Let the critical point of g be $x_* = x_*(-\xi_1, \xi_2, \psi)$. If $M > 0$ is large enough then $x_* \in (-\delta/2, \delta/2)$ for all $\xi \in \Gamma_M$. Since the properties of ψ and g are similar to that of φ and f , Case I can be applied to g , so $g(x) \geq g(x_*) + a\xi_2 x_*^{m-2} (x - x_*)^2$ for all $x \in [-\delta, \delta]$. Replacing x by $-x$ we get

$$g(-x) \geq g(x_*) + a\xi_2 x_*^{m-2} (-x - x_*)^2. \quad (40)$$

Notice that $g(-x) = f(x)$. Since the critical point $x_*(-\xi_1, \xi_2, \psi)$ satisfies $\varphi'(x_*) = -\frac{\xi_1}{\xi_2}$, we obtain that $-x_*(-\xi_1, \xi_2) = y_*(\xi_1, \xi_2)$. Therefore the right-hand side of (40) is equal to $f(y_*) + ay_*^{m-2} (x - y_*)^2$. This proves Case II.

When $\xi_1 = 0$, the inequality (39) is trivially satisfied, and the proof of Lemma 9 is complete. \square

Applying Lemma 9 in estimating the integral J_1 we get

$$\begin{aligned} |J_1| &\leq \max_{[-\delta, \delta]} |g'(y)| e^{-f(y_*)} \int_{-\delta}^{\delta} |y - y_*| e^{-\rho^m y_*^{m-2} (y - y_*)^2} dy \\ &\leq \max_{[-\delta, \delta]} |g'(y)| e^{-f(y_*)} \int_{-\infty}^{\infty} |s| e^{-\rho^m y_*^{m-2} s^2} ds \\ &= \frac{c_1}{\rho^m y_*^{m-2}} \max_{[-\delta, \delta]} |g'(y)| e^{-f(y_*)}. \end{aligned}$$

We may assume that δ is selected so small that $\varphi(y) - b \leq -\frac{b}{2}$ for all $y \in [-\delta, \delta]$. Then we have

$$|g'(y)| \leq 8\varepsilon\rho|x_1^0 - u_1^0| + \frac{2}{a} \max_{[-\delta, \delta]} |\varphi'(y)| \rho^m e^{-\frac{b}{a}\rho^m}.$$

So if ε is small enough and $\eta_0 = \eta_0(a, m, \delta, b)$ is large enough then $|g'(y)| \leq C\varepsilon\rho$ for all $y \in [-\delta, \delta]$, $(\eta, \rho) \in \tilde{\Gamma}_M$, and $|\eta| \geq \eta_0$. Moreover we know from (23), (37) that

$$\frac{\sqrt{f''(y_*)}}{\rho^{m-1}y_*^{m-2}} \leq c_2(a, m)(\rho^{m-2}y_*^{m-2})^{-1/2} \leq c_3(a, m)(|\eta|^{\frac{m-2}{m-1}})^{-1/2}$$

tends to zero if $(\eta, \rho) \in \tilde{\Gamma}_M$, and $|\eta| \rightarrow \infty$. We also know that $|g(y_*)| = |1 - e^{\frac{2\rho^m}{a}(\varphi(y_*)-b)}| \geq 1 - e^{-\frac{b}{a}}$ because $\rho \geq (aM\eta_0)^{\frac{1}{m-1}} \geq 1$. Therefore we get

$$|J_1| \leq \frac{1}{6} \sqrt{\frac{2\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} = \frac{1}{6} |J_3| \quad (41)$$

for $|\eta| \geq \eta_0(a, m, M, b)$.

To estimate the second integral J_2 we introduce the function $h(t) = h(t, y, \eta, \rho) = t(f(y) - f(y_*)) + (1-t)\frac{1}{2}f''(y_*)(y - y_*)^2$. Then

$$J_2 = g(y_*)e^{-f(y_*)} \int_{-\delta}^{\delta} e^{-h(t)} - e^{-h(0)} dy = g(y_*)e^{-f(y_*)} \int_{-\delta}^{\delta} \int_0^1 -h'(t)e^{-h(t)} dt dy.$$

Here $h'(t) = f(y) - f(y_*) - \frac{f''(y_*)}{2}(y - y_*)^2 = \sum_{k=3}^{m-1} \frac{f^{(k)}(y_*)}{k!}(y - y_*)^k + \frac{f^{(m)}(\tilde{y})}{m!}(y - y_*)^m$ for some \tilde{y} between y and y_* . It follows from (23) that $\frac{|f^{(k)}(y_*)|}{k!} = \frac{2\rho^m |\varphi^{(k)}(y_*)|}{a} \leq 3\rho^m \binom{m}{k} |y_*|^{m-k}$, for $k = 3, \dots, m-1$, and $\frac{|f^{(m)}(\tilde{y})|}{m!} = \frac{2\rho^m |\varphi^{(m)}(\tilde{y})|}{a} \leq 3\rho^m$. So $|h'(t)| \leq 3\rho^m \sum_{k=3}^m \binom{m}{k} |y_*|^{m-k} |y - y_*|^k$. To estimate the exponent $h(t)$ we use Lemma 9 and $f''(y_*) \geq m(m-1)\rho^m y_*^{m-2}$ to obtain $h(t) \geq t\rho^m y_*^{m-2}(y - y_*)^2 + (1-t)\frac{m(m-1)}{2}\rho^m y_*^{m-2}(y - y_*)^2 \geq \rho^m y_*^{m-2}(y - y_*)^2$. These inequalities lead to

$$\begin{aligned} |J_2| &\leq |g(y_*)| e^{-f(y_*)} 3\rho^m \sum_{k=3}^m \binom{m}{k} |y_*|^{m-k} \int_{-\delta}^{\delta} |y - y_*|^k e^{-\rho^m y_*^{m-2}(y - y_*)^2} dy \\ &\leq |g(y_*)| e^{-f(y_*)} 3 \sum_{k=3}^m c_k \binom{m}{k} \frac{\rho^m |y_*|^{m-k}}{(\rho^m |y_*|^{m-2})^{\frac{k+1}{2}}} \\ &\leq c_4(a, m) \sqrt{\frac{2\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} \sum_{k=3}^m c_k \binom{m}{k} (\rho^m |y_*|^m)^{1-\frac{k}{2}}. \end{aligned}$$

In the last step we took the advantage of $\frac{1}{\sqrt{\rho^m |y_*|^{m-2}}} \leq c_4(a, m) \sqrt{\frac{2\pi}{f''(y_*)}}$. Since $\rho^m |y_*|^m \geq c_5(a, m) |\eta|^{\frac{m}{m-1}}$ we see that

$$|J_2| \leq \frac{1}{6} \sqrt{\frac{2\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} = \frac{1}{6} |J_3| \quad (42)$$

provided $|\eta| \geq \eta_0(a, m, M, b)$.

Now from (38), (41), and (42) we see that

$$\left| \frac{2\rho^m}{a} A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a}) - J_3 \right| \leq |J_1| + |J_2| + |J_4| \leq \frac{1}{2} |J_3|.$$

Therefore we can conclude that for small ε

$$\frac{2\rho^m}{a} |A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a})| \geq \frac{1}{2} |J_3| = \frac{1}{2} \sqrt{\frac{2\pi}{f''(y_*)}} |g(y_*)| e^{-f(y_*)} \quad (43)$$

for all $(\eta, \rho) \in \tilde{\Gamma}_M$, $|\eta| \geq \eta_0(a, m, M, b)$.

Our next step is to obtain a lower bound for $|A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \rho^m/a)|$ in the region $\{(\eta, \rho) \in \mathbb{R}^2; |\eta| \leq \eta_0, \rho > \rho_0 > 0\}$. We write $\varphi(y) = ay^m u(y)$, then $\lim_{y \rightarrow 0} u(y) = 1$. Substituting $y = -\frac{x}{\rho}$ into (31) we get

$$\begin{aligned} & \lim_{\rho \rightarrow \infty} \rho^m A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \rho^m/a) \\ &= \lim_{\rho \rightarrow \infty} \frac{a}{2} \int_{-\delta\rho}^{\delta\rho} e^{2((\eta + 2i\varepsilon(x_1^0 - u_1^0))x - x^m u(-\frac{x}{\rho}))} [1 - e^{\frac{2\rho^m}{a}(\varphi(-\frac{x}{\rho}) - b)}] dx \\ &= \frac{a}{2} \mathcal{N}(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \end{aligned}$$

where the function \mathcal{N} is defined in (54). It easy to see that the last limit is uniform in $\eta \in [-\eta_0, \eta_0]$. Since the roots of \mathcal{N} are all imaginary, there is a constant $C_2 = C_2(\varepsilon, a, m, \delta, b) > 0$ such that $|\mathcal{N}(\eta + 2i\varepsilon(x_1^0 - u_1^0))| \geq \frac{4}{a} C_2$ for all $|\eta| \leq \eta_0$, provided that ε is small enough. So there is a constant $\rho_0 = \rho_0(a, m, \delta, m) > 1$ such that

$$|A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \rho^m/a)| \geq \frac{C_2}{\rho^m}$$

for all $\rho \geq \rho_0$. Moreover, notice that the exponential factor $e^{-f(y_*)}$ is bounded when η is bounded. Indeed, using the Hölder inequality we get

$$\begin{aligned} -f(y_*) &\leq 2\rho\eta|y_*| - \rho^m|y_*|^m \leq \rho|y_*|^m + c_1(m)|\eta|^{\frac{m}{m-1}} - \rho|y_*|^m \\ &\leq c_1(m)|\eta|^{\frac{m}{m-1}} = c_2(m, \eta_0) \end{aligned}$$

that is, $e^{-f(y_*)} \leq c_3(m, \eta_0)$. So

$$|A_{\delta,b}(\rho(\eta + 2i\varepsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a})|^{-1} \leq c_4(m, \eta_0) \rho^m e^{f(y_*)} \quad (44)$$

in the region $\{(\eta, \rho) \in \mathbb{R}^2; |\eta| \leq \eta_0; \rho \geq \rho_0 > 0\}$.

In the region $\{(\eta, \rho) \in \mathbb{R}^2; |\eta| \leq \eta_0, 0 < \rho \leq \rho_0\}$, it follows from (31) that $A_{\delta,b}(\rho\eta, \rho^m/a)$ has a positive minimum. So there is a positive constant $C = C(\epsilon, \delta, m, M, b) > 0$ that $|A_{\delta,b}(\rho(\eta + 2i\epsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a})| \geq C$, provided that $\epsilon > 0$ is small enough. Using the boundedness of $e^{-f(y_*)}$ again, we obtain the estimate (44) with a different constant $c_5(m, \eta_0)$ in the region $|\eta| \leq \eta_0, \rho \leq \rho_0$.

Combining these estimates with (43) we obtain

$$|A_{\delta,b}(\rho(\eta + 2i\epsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a})|^{-1} \leq C(1 + \rho^{\frac{3m}{2}})e^{f(y_*)} \quad (45)$$

for all $(\eta, \rho) \in \tilde{\Gamma}_M$. In the last estimate we have used that $|g(y_*)| \geq 1 - e^{-\frac{b}{a}}$, and $f''(y_*) \leq 3m(m-1)\rho^2|\eta|^{\frac{m-2}{m-1}}$. Returning to the integral J in (27) we get

$$\begin{aligned} |J| &\leq \frac{m}{a^{|\beta|+1}} \int_{\tilde{\Gamma}_M} \rho^{|\alpha|+m|\beta|+m} |\eta + 2i\epsilon(x_1^0 - u_1^0)|^{|\alpha|} \times \\ &\quad e^{-\epsilon(x_1^0 - u_1^0)^2 \rho - f(y_*)} |A_{\delta,b}(\rho(\eta + 2i\epsilon(x_1^0 - u_1^0)), \frac{\rho^m}{a})|^{-1} d\rho d\eta \\ &\leq C \frac{m}{a^{|\beta|+1}} \int_0^\infty \int_{-\frac{\rho^{m-1}}{aM}}^{\frac{\rho^{m-1}}{aM}} \rho^{|\alpha|+m|\beta|+m} (1 + |\eta|)^{|\alpha|} (1 + \rho^{\frac{3m}{2}}) e^{-\epsilon(x_1^0 - u_1^0)^2 \rho} d\eta d\rho \\ &\leq \frac{2mC}{a^{|\beta|+2}M} \int_0^\infty \rho^{|\alpha|+m|\beta|+2m-1} (1 + \rho^{m-1})^{|\alpha|} (1 + \rho^{\frac{3m}{2}}) e^{-\epsilon(x_1^0 - u_1^0)^2 \rho} d\rho \\ &\leq C^{|\alpha|+|\beta|+1} \Gamma(m(|\alpha| + |\beta|) + 1). \end{aligned}$$

Here the constant C is independent of α and β . This completes the proof of the Gevrey estimate. \square

8. ANALYTIC SINGULARITIES AWAY FROM THE BOUNDARY DIAGONAL

Our goal in this section, and the next section, is to prove the second half of Theorems 1 and 2. To be precise let $z^0, w^0 \in \partial T_\Omega$. Assume that $z^0 \neq w^0$. We will show that if z^0 and w^0 lie on the same characteristic line, then \mathcal{B} has no analytic extension past (z^0, w^0) . However, as we have already seen, a smooth extension of Gevrey class m exists, if the type of z^0 or w^0 is m . We will show in the next section that this is the best Gevrey class possible.

We begin by observing that under the above assumption, there exists $y^0 \in \partial\Omega$, a weakly convex boundary point, such that

$$\Im z^0 = y^0 = \Im w^0.$$

Our assumption also guarantees the existence of a vector $a \in \mathbb{R}^n, |a| = 1$ and $t_0 \in \mathbb{R}, t_0 \neq 0$ such that

$$\Re z^0 = \Re w^0 + t_0 a.$$

The vector a also satisfies

$$\langle a, dr(y^0) \rangle = 0 \text{ and } r''(y^0)a = 0.$$

All this follows from Proposition 1 and Remark 1. We now introduce the function $U(t)$ as follows :

$$U(t) = \mathcal{B}(z^0, w^0 + tdr(y^0)). \quad (46)$$

We will show that U is not an analytic function of t near $t = 0$. It follows as a consequence of this that \mathcal{B} has no analytic extension to any neighborhood of (z^0, w^0) .

We will begin by choosing convenient coordinates. Note that the formula (15) is invariant under translations and real rotations. Hence we may assume that $y^0 = 0$.

We may also assume that we have $\delta > 0$ and φ real valued and real analytic near $|y'| \leq \delta$ such that r has the form $r(y) = \varphi(y') - y_n$ with $d\varphi(0) = 0$. Here $y' = (y_1, \dots, y_{n-1})$. Hence $dr(0) = (0, \dots, 0, -1)$.

We may also assume that $a = (a', 0)$, $|a'| = 1$, $a' \in \mathbb{R}^{n-1}$ with $\varphi''(0)a' = 0$.

So we see that if $\xi \in \mathbb{R}^n$ we have

$$\langle z^0 - (\bar{w}^0 + tdr(y^0)), \xi \rangle = t_0 \langle a', \xi' \rangle + t\xi_n.$$

Hence it follows that we may assume that $0 \in \partial\Omega$ and that for $t \in \mathbb{R}$, t near 0, we have

$$U(t) = \int_{\mathbb{R}^n} e^{i(t_0 \langle a', \xi' \rangle + t\xi_n)} A(\xi)^{-1} \frac{d\xi}{(2\pi)^n} \quad (47)$$

where $t_0 \in \mathbb{R}$, $t_0 \neq 0$ and $a' \in \mathbb{R}^{n-1}$, $|a'| = 1$.

Now that we have chosen convenient coordinates, we will localize. The main idea is that the important singularities of U arise near the interior normal of Ω . Given $M > 0$, we define Γ as follows:

$$\Gamma = \{\xi \in \mathbb{R}^n : \xi_n \geq M|\xi'|\}, \quad (48)$$

where $\xi' = (\xi_1, \dots, \xi_{n-1})$. Note that Γ is a conic neighborhood of the vector $\mathbf{n} = (0, \dots, 0, 1)$, which is the interior normal to Ω at the origin.

We define

$$S_\delta = \{y \in \partial\Omega : y_n = \varphi(y'), |y'| \leq \delta\},$$

and

$$A_\delta(\xi) = \frac{1}{2|\xi|} \int_{S_\delta} e^{-2\langle \xi, y \rangle} \left\langle -\frac{dr}{|dr|}, \frac{\xi}{|\xi|} \right\rangle d\sigma(y).$$

Now we introduce

$$U_{\Gamma, \delta}(t) = \int_{\Gamma} e^{i(t_0 \langle a', \xi' \rangle + t\xi_n)} A_\delta(\xi)^{-1} \frac{d\xi}{(2\pi)^n}.$$

We have the following

Lemma 10. *Let $\delta > 0$ be given. Then there exist $M > 0$ such that*

$$U - U_{\Gamma, \delta}$$

is an analytic function of t , for t near 0.

Proof. We apply Theorem 6 to (46) and (47). □

We now focus our attention on $U_{\Gamma,\delta}$. Our goal is to show that this function is not analytic near $t = 0$.

9. THE TWO DIMENSIONAL CASE

We now begin our study of the two variable case. We assume that $\Omega \subset \mathbb{R}^2$ is open and convex with real analytic boundary. We may also assume that $0 \in \partial\Omega$ and that we have $\delta > 0$ and φ real valued and real analytic near $|y_1| \leq \delta$ such that r has the form $r(y) = \varphi(y_1) - y_2$ with $\varphi'(0) = 0$. Hence $dr(0) = (0, -1)$. Our previous work allows us to focus on

$$U_{\Gamma,\delta}(t) = \int_{\Gamma} e^{i(t_0\xi_1+t\xi_2)} A_{\delta}(\xi)^{-1} \frac{d\xi}{(2\pi)^2} \quad (49)$$

where

$$\Gamma = \{\xi \in \mathbb{R}^2 : M|\xi_1| \leq \xi_2\}.$$

In the present case, we have

$$A_{\delta}(\xi) = \frac{1}{2|\xi|} \int_{-\delta}^{\delta} e^{-2(\xi_1 s + \xi_2 \varphi(s))} a(s, \xi) ds \quad (50)$$

where

$$a(s, \xi) = \frac{\xi_2 - \varphi'(s)\xi_1}{|\xi|}. \quad (51)$$

Since we assume weak convexity, we have $\varphi''(0) = 0$. We may assume that we have, for $|s| \leq \delta$, a strictly positive analytic function u such that

$$\varphi(s) = u(s)s^m$$

where $m \geq 4$ is an even integer.

Rather than studying $U_{\Gamma,\delta}$, we introduce

$$F(t) = \int_{\Gamma} e^{i(t_0\xi_1+t\xi_2)} \frac{\xi_2^{p+2}}{2m|\xi|^2} A_{\delta}(\xi)^{-1} d\xi. \quad (52)$$

Note that F is the image of $U_{\Gamma,\delta}$ under the elliptic pseudodifferential operator $\frac{2\pi^2}{m} |D_t|^{p+2} \Delta^{-1}$. Here Δ is the Laplacian in the (t_0, t) variables. Hence it suffices to prove that F is not analytic at $t = 0$.

We are free to choose the real number p . We choose $p = -(\frac{2m+1}{m})$. Clearly we may assume without loss of generality that $t_0 = 1$ and that $u(0) = 1$.

We make the following change of variables in the formula for $D_t^k F$:

$$\eta = \frac{\xi_1}{(\xi_2)^{1/m}}$$

and

$$\rho = (\xi_2)^{1/m}.$$

We obtain

$$(D_t^k F)(0) = \int_0^\infty \int_{M|\eta| \leq \rho^{m-1}} e^{i\rho\eta} \rho^{mk} G_1(\rho, \eta)^{-1} d\eta d\rho, \quad (53)$$

where we define

$$G_1(\rho, \eta) = \int_{-\rho\delta}^{+\rho\delta} e^{-2(\eta t + t^m u(t/\rho))} b(t, \rho, \eta) dt$$

and

$$b(t, \rho, \eta) = 1 - \frac{\varphi'(t/\rho)\eta}{\rho^{m-1}}.$$

We will show that F is not analytic at 0 by estimating $(D_t^k F)(0)$. To do so, we introduce

$$G_2(\rho, \eta) = \int_{-|\rho|\delta}^{+|\rho|\delta} e^{-2(\eta t + t^m u(t/\rho))} dt$$

which is defined for all $\rho \neq 0, \eta \in \mathbb{R}$. Indeed we have the identity

$$G_2(\rho, \eta) = G_2(-\rho, -\eta)$$

for all $\rho \neq 0, \eta \in \mathbb{R}$.

We now introduce I_1, I_2 and I_3 . We define

$$\begin{aligned} I_1 &= \int_0^\infty \int_{-\infty}^{+\infty} e^{i\rho\eta} \rho^{mk} G_2(\rho, \eta)^{-1} d\eta d\rho, \\ I_2 &= \int_0^\infty \int_{M|\eta| \geq \rho^{m-1}} e^{i\rho\eta} \rho^{mk} G_2(\rho, \eta)^{-1} d\eta d\rho, \\ I_3 &= \int_0^\infty \int_{M|\eta| \leq \rho^{m-1}} e^{i\rho\eta} \rho^{mk} (G_1(\rho, \eta)^{-1} - G_2(\rho, \eta)^{-1}) d\eta d\rho. \end{aligned}$$

Note that

$$(D_t^k F)(0) = I_1 - I_2 + I_3.$$

The estimate for I_1

The argument is based on that presented in [10]. In that paper, the study is based on the function

$$\mathcal{N}(\eta) = \int_{-\infty}^{+\infty} e^{2(\eta s - s^m)} ds = \int_{-\infty}^{+\infty} e^{-2(\eta s + s^m)} ds, \quad (54)$$

where $m \geq 2$ is even.

We know that \mathcal{N} is an entire, even function of η . We also know that \mathcal{N} has zeroes only when $m \geq 4$. In this case, all zeroes are on the imaginary axis. This all follows from classical work of Pólya, [24]. Since $\mathcal{N}(0) \neq 0$, it follows that there exists $R > 0$ such that $\pm iR$ are the two zeroes of \mathcal{N} closest to the origin. We also know that $\pm iR$

are simple roots. This is a nontrivial fact which we prove in Section 11. We have the following:

Lemma 11. *Let $\pm iR$ be the two zeroes of \mathcal{N} closest to the origin. Then there exists $C > 0$, such that for all k , we have*

$$|\partial_\eta^{2k}(\frac{1}{\mathcal{N}})(0)| \geq \frac{C(2k)!}{R^{2k}}.$$

Proof. There exists f entire and even such that $\mathcal{N}(z) = (z^2 + R^2)f(z)$, for all $z \in \mathbb{C}$. It follows that there exists an $R_1 > R$ such that $f(z) \neq 0$ for all z such that $|z| \leq R_1$. Since $\frac{1}{f}$ is holomorphic near the closed disk $|z| \leq R_1$, there exists a $C > 0$, depending on f , such that

$$|\partial_z^{2k}(\frac{1}{f})(0)| \leq \frac{C(2k)!}{R_1^{2k}}$$

for all k . Now let $P_{2d}(z) = \sum_{j=0}^d a_{2j} z^{2j}$ be the Taylor polynomial of $\frac{1}{f}$, centered at the origin, of order $2d$. We will choose d conveniently.

Note that

$$\partial_z^{2k}(\frac{1}{z^2 + R^2})(0) = \frac{(-1)^k (2k)!}{R^{2k+2}}.$$

Hence it follows that if $k \geq d$, we have

$$\partial_z^{2k}(\frac{P_{2d}(z)}{z^2 + R^2})(0) = \frac{(-1)^k (2k)! P_{2d}(iR)}{R^{2k+2}}.$$

Now it follows that if $k \geq d$ we have

$$|\partial_z^{2k}(\frac{1}{z^2 + R^2}(\frac{1}{f} - P_{2d})))(0)| \leq \frac{C(2k)!}{R^{2k+2}} \sum_{j=d+1}^{\infty} (\frac{R}{R_1})^{2j}.$$

Combining the above we see that if $k \geq d$ we have

$$|\partial_z^{2k}(\frac{1}{\mathcal{N}})(0)| \geq \frac{(2k)!}{R^{2k+2}} (|P_{2d}(iR)| - C \sum_{j=d+1}^{\infty} (\frac{R}{R_1})^{2j}).$$

Now the Lemma follows, since

$$\lim_{d \rightarrow \infty} (|P_{2d}(iR)| - C \sum_{j=d+1}^{\infty} (\frac{R}{R_1})^{2j}) = |\frac{1}{f(iR)}| > 0.$$

□

Lemma 12. *For every integer k we have*

$$I_1 = \pi D_\eta^{mk}(\mathcal{N}^{-1})(0).$$

Proof. Since G_2 is an even function of (ρ, η) , we have after integrating by parts,

$$I_1 = (1/2) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\rho\eta} D_\eta^{mk}(G_2^{-1})(\rho, \eta) d\eta d\rho.$$

Let f be a smooth function with compact support such that $f(0) = 1$. Then we have

$$\begin{aligned} I_1 &= (1/2) \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} f(\epsilon\rho) \int_{-\infty}^{+\infty} e^{i\rho\eta} D_\eta^{mk}(G_2^{-1})(\rho, \eta) d\eta d\rho \\ &= (1/2) \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} f(\rho) \int_{-\infty}^{+\infty} e^{i\rho\eta} D_\eta^{mk}(G_2^{-1})(\frac{\rho}{\epsilon}, \epsilon\eta) d\eta d\rho \\ &= (1/2) \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\rho) e^{i\rho\eta} D_\eta^{mk}(G_2^{-1})(\frac{\rho}{\epsilon}, \epsilon\eta) d\rho d\eta \\ &= (1/2) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\rho) e^{i\rho\eta} \lim_{\epsilon \rightarrow 0} D_\eta^{mk}(G_2^{-1})(\frac{\rho}{\epsilon}, \epsilon\eta) d\rho d\eta \\ &= (1/2) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(\rho) e^{i\rho\eta} D_\eta^{mk}(\mathcal{N}^{-1})(0) d\rho d\eta \\ &= \pi D_\eta^{mk}(\mathcal{N}^{-1})(0). \end{aligned}$$

□

Combining Lemmas 11 and 12 we have the following estimate for I_1 : There exists an integer N and a real number $C > 0$ such that if $k \geq N$ we have

$$|I_1| \geq C \frac{(mk)!}{R^{mk}}. \quad (55)$$

The estimate for I_2

We begin by estimating $G_2(\rho, \eta)$ for $\eta < 0$, the arguments being similar for $\eta > 0$. Define $f(t) = -2(\eta t + t^m u(t/\rho))$. Since we assume $u \leq 3/2$ we have $f(t) \geq 2|\eta|t - 3t^m$. Note that the function $|\eta|t - 3t^m$ has a critical point at $t_\star = (\frac{|\eta|}{3m})^{\frac{1}{m-1}}$. We divide the argument into two cases.

First assume that $t_\star \leq \rho\delta$. Assume that $t_\star/2 \leq t \leq t_\star$, with $t_\star \geq 2$. Since $|\eta|t - 3t^m$ is increasing on this interval, we have

$$f(t) \geq |\eta| + |\eta|t_\star/2 - 3(t_\star/2)^m \geq |\eta| + 3\left(\frac{|\eta|}{3m}\right)^{\frac{m}{m-1}} \left(\frac{m}{2} - \frac{1}{2^m}\right).$$

Hence we have

$$G_2(\rho, \eta) \geq \frac{t_\star}{2} e^{|\eta| + C\left(\frac{|\eta|}{3m}\right)^{\frac{m}{m-1}}} \geq C\rho e^{|\eta| + C\rho^m},$$

as long as $M|\eta| \geq \rho^{m-1}$.

Now assume that $t_\star \geq \rho\delta$. This implies that the function $|\eta|t - 3t^m$ is increasing on the interval $0 \leq t \leq \rho\delta$. If we assume that $\rho\delta/2 \leq t \leq \rho\delta$, with $t \geq 1$, we have

$$f(t) \geq |\eta| + |\eta|\rho\delta/2 - 3(\rho\delta/2)^m.$$

The fact that $t_* \geq \rho\delta$ implies that $|\eta| \geq 3m(\rho\delta)^{m-1}$. Hence it follows that

$$f(t) \geq |\eta| + 3(\rho\delta)^m \left(\frac{m}{2} - \frac{1}{2^m} \right).$$

We obtain a similar estimate as in the first case:

$$G_2(\rho, \eta) \geq \frac{\rho\delta}{2} e^{|\eta| + C\rho^m}.$$

We now can estimate I_2 . We have

$$|I_2| \leq C \int_0^\infty \rho^{mk} e^{-C\rho^m} d\rho \int_{-\infty}^{+\infty} e^{-|\eta|} d\eta.$$

Hence there exists $C > 0$ such that for all k we have

$$|I_2| \leq C^k k! \tag{56}$$

The estimate for I_3

In order to estimate I_3 , we must study $G_1(\rho, \zeta)^{-1} - G_2(\rho, \zeta)^{-1}$ where $\zeta = \eta + i\gamma$ is a complex variable. A contour deformation in the integral defining I_3 is necessary. We will need to make use of our stationary phase results of section 10.

First consider

$$(G_2 - G_1)(\rho, \zeta) = \frac{\zeta}{\rho^{2m-2}} \int_{-\rho\delta}^{+\rho\delta} e^{-2(\zeta t + t^m u(t/\rho))} O(t^{m-1}) dt. \tag{57}$$

Lemma 15 shows that there exists an $M_0 > 0$ and an $\epsilon_0 > 0$ such that if $M > M_0$, $0 < \epsilon < \epsilon_0$, $M|\eta| \leq \rho^{m-1}$, $|\gamma| \leq \epsilon|\eta|$ then we have

$$|(G_2 - G_1)(\rho, \zeta)| \leq \frac{C}{\rho^{2m-2}} |\eta|^{\frac{3m-2}{2m-2}} e^{2(m-1)|\frac{\eta}{m}|^{\frac{m}{m-1}} (1+O(\epsilon))(1+O(\delta))}.$$

Hence it follows that there exists an $M_0 > 0$ and $C > 0$ such that if $M > M_0$, $C|\gamma| \leq |\eta|$, and $|\eta| \leq \frac{\rho^{m-1}}{M}$, then we have

$$|G_1(\rho, \zeta)^{-1} - G_2(\rho, \zeta)^{-1}| \leq \frac{C}{\rho^{2m-2}} |\eta|^{\frac{5m-6}{2m-2}} e^{-\frac{1}{C}|\eta|^{\frac{m}{m-1}}}. \tag{58}$$

This estimate is valid for large η . However, we must also estimate $G_1(\rho, \zeta)^{-1} - G_2(\rho, \zeta)^{-1}$ near $\eta = 0$; in fact near $\zeta = \pm iR$, which are the zeroes of \mathcal{N} nearest the origin.

There exists $C > 0$ such that

$$|\mathcal{N}(\zeta)| \geq C|\zeta \mp iR|$$

for ζ near $\pm iR$. This estimate follows from the following highly non-trivial result.

Lemma 13. *The points $\pm iR$ are simple zeroes of \mathcal{N} .*

The proof of this Lemma is postponed until the final section.

Next we need an estimate for $|G_j(\rho, \zeta)|$, $j = 1, 2$ when $\zeta = \eta + i\gamma$ is near $\pm iR$. We think of G_j as a family of entire functions of ζ , indexed by ρ . Any possible zeroes for these functions will be near $\pm iR$, as we shall now see.

Observe that we have for $j = 1, 2$

$$\lim_{\rho \rightarrow \infty} G_j(\rho, \zeta) = \mathcal{N}(\zeta).$$

Indeed, the convergence is uniform on compact sets, since the family of functions $G_j(\rho, \zeta)$ is equicontinuous. (Note that the first derivative, with respect to ζ , is bounded independent of ρ .)

Since \mathcal{N} has a simple zero at iR , it follows that for ρ sufficiently large, G_j has a simple root $\zeta^j(\rho)$ such that $\lim_{\rho \rightarrow \infty} \zeta^j(\rho) = iR$. Hence there exists a $C > 0$ such that for ρ large and ζ near iR we have

$$|G_j(\rho, \zeta)| \geq C|\zeta - \zeta^j(\rho)|. \quad (59)$$

We will also need information on the rate at which $\zeta^j(\rho)$ approaches iR . We begin by differentiating the identity

$$G_j(\rho, \zeta^j(\rho)) = 0$$

with respect to ρ . We obtain

$$\frac{d\zeta^j}{d\rho} = -\frac{\partial_\rho G_j}{\partial_\zeta G_j}.$$

Note that for ρ large and ζ near iR , we have $|\partial_\zeta G_j| \geq C > 0$, since iR is a simple root of \mathcal{N} . Now if we differentiate the integral defining G_j , with respect to ρ , we see that there exists $C > 0$ such that

$$\left| \frac{d\zeta^j}{d\rho}(\rho) \right| \leq \frac{C}{\rho^2}$$

for ρ large and $j = 1, 2$. Now by the fundamental theorem of calculus we have $C > 0$ such that for ρ large we have

$$|\zeta^j(\rho) - iR| \leq \frac{C}{\rho}. \quad (60)$$

We are now in a position to begin to estimate I_3 . We will make the shift of contour

$$\eta \rightarrow \eta + i\gamma \quad (61)$$

where γ will be real and depend on $\rho > 0$ and k (the number of derivatives), but will be bounded, independently of k , $\rho > 0$. We perform the change of contour (61) in I_3 to obtain

$$I_3 = \int_0^\infty e^{-\gamma\rho} \rho^{mk} J(\rho) d\rho, \quad (62)$$

where we define $J(\rho)$ as follows:

$$J(\rho) = \int_{|\eta| \leq \frac{\rho^{m-1}}{M}} e^{i\rho\eta} (G_1^{-1} - G_2^{-1})(\rho, \eta + i\gamma) d\eta. \quad (63)$$

We now discuss the change of contour in (63). We must specify γ . First we introduce a sequence of real numbers

$$0 < \mu_k < 1 \quad (64)$$

such that

$$\lim_{k \rightarrow \infty} \mu_k = 1. \quad (65)$$

We now define γ as follows:

$$\gamma = \min\{\Im\zeta^1(\rho) - (1 - \mu_k)R, \Im\zeta^2(\rho) - (1 - \mu_k)R\}. \quad (66)$$

Note that γ depends on both k and ρ . Note also that for all k and ρ we have

$$0 < \gamma < \Im\zeta^j(\rho), j = 1, 2. \quad (67)$$

This inequality tells us that the shift of contour (61) avoids the zeroes of G_1 and G_2 . Furthermore

$$\lim_{\rho \rightarrow \infty} \gamma = \mu_k R. \quad (68)$$

It now follows from (66) and (60) that there exists $C > 0$, independent of k and ρ such that

$$\gamma \geq \mu_k R - \frac{C}{\rho}. \quad (69)$$

We now begin to estimate J . For $C > 0$ we define J_1 and J_2 as follows:

$$J_1(\rho) = \int_{C \leq |\eta| \leq \frac{\rho^{m-1}}{M}} e^{i\rho\eta} (G_1^{-1} - G_2^{-1})(\rho, \eta + i\gamma) d\eta, \quad (70)$$

$$J_2(\rho) = \int_{|\eta| \leq C} e^{i\rho\eta} (G_1^{-1} - G_2^{-1})(\rho, \eta + i\gamma) d\eta. \quad (71)$$

Using estimate (58) we see that we have

$$|J_1(\rho)| \leq \frac{C}{\rho^{2m-2}}. \quad (72)$$

We now estimate J_2 . Using estimates (59) and (57) we see that we have

$$|J_2(\rho)| \leq \frac{C}{\rho^{2m-2}} \int_{|\eta| \leq C} \frac{d\eta}{|\eta + i\gamma - \zeta^1(\rho)| |\eta + i\gamma - \zeta^2(\rho)|}. \quad (73)$$

Now observe that

$$\lim_{\rho \rightarrow \infty} \eta + i\gamma - \zeta^j(\rho) = \eta + i(\mu_k - 1)R$$

uniformly for η bounded and $j = 1, 2$. Hence it follows that

$$|J_2(\rho)| \leq \frac{C}{\rho^{2m-2}} \int_{|\eta| \leq C} \frac{d\eta}{|\eta + i(\mu_k - 1)R|^2}. \quad (74)$$

Evaluating the integral we have

$$|J_2(\rho)| \leq \frac{C}{\rho^{2m-2}(1 - \mu_k)R}. \quad (75)$$

Combining this with (72) we have

$$|J(\rho)| \leq \frac{C}{\rho^{2m-2}(1 - \mu_k)R}. \quad (76)$$

We are now ready for the final estimate. Using (62), along with (69) and (76) we have $C > 0$, independent of k , such that

$$\begin{aligned} |I_3| &\leq \frac{C}{(1 - \mu_k)R} \int_0^\infty e^{-\rho\mu_k R} \rho^{mk-2m+2} d\rho \\ &= \frac{C(mk - 2m + 2)!}{(1 - \mu_k)R(\mu_k R)^{mk-2m+3}} \\ &\leq \frac{C(mk)!}{R^{mk} B_k}, \end{aligned} \quad (77)$$

where we define

$$B_k = (1 - \mu_k)\mu_k^{mk-2m+3}(mk)(mk - 1)\dots(mk - 2m + 3).$$

Now if we choose

$$\mu_k = 1 - \frac{1}{k}$$

it follows that

$$\lim_{k \rightarrow \infty} B_k = +\infty.$$

Now if we combine estimates (55), (56) and (77) it follows that for large k we have

$$|(D_t^k F)(0)| \geq \frac{C(mk)!}{R^{mk}}.$$

Hence F is not analytic near $t = 0$. Indeed, F is no better than Gevrey class m .

10. THE METHOD OF STATIONARY PHASE

In this section we discuss the precise version of the stationary phase formula that we need. This allows us to obtain the inequality (58), which was used to estimate I_3 in the previous section. We begin with an elementary, but very precise, result.

Lemma 14. *Let f and g be smooth, complex valued functions, defined near the closed interval $[-p, p]$. We define*

$$I(f, g, p) = \int_{-p}^p e^{-f(t)} g(t) dt.$$

Assume that there exists $a \in \mathbb{C}, \Re a > 0$ and $C_1 > 0$ such that

$$|f(t) - at^2| \leq C_1 |t|^3, |t| \leq p$$

and

$$C_1 p \leq \Re a / 2.$$

Let $C_2 > 0$ satisfy

$$|g(t) - g(0)| \leq C_2 |t|, |t| \leq p.$$

Then there exists a universal constant $C > 0$ such that

$$\begin{aligned} & |I(f, g, p) - g(0) \sqrt{\frac{\pi}{a}}| \leq \\ & \frac{CC_2}{\Re a} + \frac{|g(0)|CC_1}{(\Re a)^2} + |g(0)| \sqrt{\frac{2\pi}{\Re a}} e^{-\Re a p^2/2}. \end{aligned}$$

Proof. We write $I(f, g, p) = I_1 + I_2 + I_3 - I_4$, where

$$I_1 = \int_{-p}^p e^{-f(t)} (g(t) - g(0)) dt$$

$$I_2 = g(0) \int_{-p}^p (e^{-f(t)} - e^{-at^2}) dt$$

$$I_3 = g(0) \int_{-\infty}^{\infty} e^{-at^2} dt$$

$$I_4 = g(0) \int_{|t| \geq p} e^{-at^2} dt.$$

First observe that when $a > 0$ we have $I_3 = g(0) \sqrt{\frac{\pi}{a}}$. When $\Re a > 0$, this formula persists, by analytic continuation.

Note that if $|t| \geq p$, it follows that $\Re(at^2) \geq (\Re a)p^2/2 + (\Re a)t^2/2$. Hence it follows that

$$|I_4| \leq |g(0)| e^{-\Re a p^2/2} \int_{-\infty}^{+\infty} e^{-\Re a t^2/2} dt = |g(0)| e^{-\Re a p^2/2} \sqrt{\frac{2\pi}{\Re a}}.$$

Our hypothesis implies that if $|t| \leq p$, then we have $\Re f(t) \geq (\Re a)t^2/2$. Hence

$$|I_1| \leq C_2 \int_{-p}^p e^{-\Re a t^2/2} |t| dt \leq \frac{2C_2}{\Re a} \int_{-\infty}^{\infty} e^{-s^2} |s| ds.$$

It remains to estimate I_2 . First observe that

$$e^{-f(t)} - e^{-at^2} = \int_0^1 \frac{d}{d\lambda} (e^{-(\lambda f(t) + (1-\lambda)at^2)}) d\lambda$$

$$= (at^2 - f(t))e^{-at^2} \int_0^1 e^{-\lambda(f(t)-at^2)} d\lambda.$$

Now, using the hypothesis, we obtain

$$\begin{aligned} |I_2| &\leq |g(0)|C_1 \int_{-p}^p \int_0^1 |t|^3 e^{-\Re at^2 + C_1 |t|^3} d\lambda dt \\ &\leq |g(0)|C_1 \int_{-p}^p |t|^3 e^{-\Re at^2/2} dt \leq |g(0)|C_1 \left(\frac{2}{\Re a}\right)^2 \int_{-\infty}^{\infty} e^{-s^2} |s|^3 ds. \end{aligned}$$

□

We wish to estimate $G_j(\rho, \zeta)$, $j = 1, 2$ and the quantity $(G_1^{-1} - G_2^{-1})(\rho, \zeta)$. We assume that $\varphi(s) = u(s)s^m$, $u(0) = 1$, $1/2 \leq u(s) \leq 3/2$ for $|s| \leq 2\delta$, $s \in \mathbb{R}$. We also assume that u is holomorphic for $s \in \mathbb{C}$, $|s| \leq 2\delta$. Throughout, $\zeta = \eta + i\gamma$ will be a complex variable, $\Re \zeta = \eta$, $\Im \zeta = \gamma$. We assume that there exists $M > 0$ (large) and $\epsilon' > 0$ (small) such that $|\eta| \leq \frac{\rho^{m-1}}{M}$ and $|\gamma| \leq \epsilon'|\eta|$. We define Φ as follows:

$$\Phi(t, \rho, \zeta) = 2(\zeta t + t^m u(t/\rho)), |t| \leq 2\rho\delta.$$

We define $t_*(\rho, \zeta)$ to be the solution to the equation

$$\frac{\partial \Phi}{\partial t}(t_*, \rho, \zeta) = 0.$$

Note that $t_*(\rho, \zeta) = \left(-\frac{\zeta}{m}\right)^{\frac{1}{m-1}}(1 + O(\delta))$. We always take the root closest to the real axis, so that $t_*(\rho, \eta)$ is real. Also note that

$$\left(-\frac{\zeta}{m}\right)^{\frac{1}{m-1}} = \left(-\frac{\eta}{m}\right)^{\frac{1}{m-1}}(1 + O(\epsilon')).$$

Lemma 15. *There exists $M_0 > 0$ and $\epsilon_0 > 0$ such that if $M > M_0$ and $0 < \epsilon < \epsilon_0$, and $|\eta| \leq \frac{\rho^{m-1}}{M}$ and $|\gamma| \leq \epsilon|\eta|$, then*

$$\begin{aligned} G_1(\rho, \zeta) &= e^{-\Phi(t_*, \rho, \zeta)} \left(\left(1 + O\left(\frac{1}{M^2}\right)\right) \sqrt{\frac{2\pi}{\Phi_{tt}(t_*, \rho, \zeta)}} + O\left(\frac{1}{\Phi_{tt}(t_*, \rho, \zeta)}\right) \right) \\ G_2(\rho, \zeta) &= e^{-\Phi(t_*, \rho, \zeta)} \left(\sqrt{\frac{2\pi}{\Phi_{tt}(t_*, \rho, \zeta)}} + O\left(\frac{1}{\Phi_{tt}(t_*, \rho, \zeta)}\right) \right) \\ (G_1 - G_2)(\rho, \zeta) &= \frac{\zeta O(t_*^{m-1})}{\rho^{2m-2}} e^{-\Phi(t_*, \rho, \zeta)} \left(\sqrt{\frac{2\pi}{\Phi_{tt}(t_*, \rho, \zeta)}} + O\left(\frac{1}{\Phi_{tt}(t_*, \rho, \zeta)}\right) \right). \end{aligned}$$

Furthermore we have

$$-\Phi(t_*, \rho, \zeta) = 2(m-1)\left(\frac{\eta}{m}\right)^{\frac{m}{m-1}}(1 + O(\epsilon))(1 + O(\delta))$$

and

$$\sqrt{\frac{2\pi}{\Phi_{tt}(t_*, \rho, \zeta)}} = \sqrt{\frac{\pi}{m(m-1)}} \left(\frac{\eta}{m}\right)^{\frac{2-m}{2(m-1)}} (1 + O(\epsilon))(1 + O(\delta)).$$

Proof. Let $q(t, \rho, \zeta)$ be holomorphic for $|t| \leq 2\rho\delta$, $|\gamma| \leq \epsilon'|\eta|$ for each $\rho > 0$. Assume that there exists $C_2 > 0$ such that

$$|q(t, \rho, \zeta) - q(t_*(\rho, \zeta), \rho, \zeta)| \leq C_2|t - t_*(\rho, \zeta)|.$$

Define

$$I = \int_{-\rho\delta}^{\rho\delta} e^{\Phi(t_*(\rho, \zeta), \rho, \zeta) - \Phi(t, \rho, \zeta)} q(t, \rho, \zeta) dt$$

and

$$I_1 = \int_{|t - t_*(\rho, \eta)| \leq \epsilon|t_*(\rho, \eta)|} e^{\Phi(t_*(\rho, \zeta), \rho, \zeta) - \Phi(t, \rho, \zeta)} q(t, \rho, \zeta) dt.$$

Let $I_2 = I - I_1$. Note that if $M > 0$ is large enough, we have $|t_*(\rho, \zeta)| \leq \rho\delta/2$. Hence, the major contribution will come from I_1 .

We begin by estimating I_2 . Note that $\Phi(t, \rho, \eta)$ has an absolute minimum at $t_*(\rho, \eta)$. Hence, if $|t - t_*(\rho, \eta)| \geq \epsilon|t_*(\rho, \eta)|$, we have, without loss of generality,

$$\Phi(t, \rho, \eta) \geq \Phi(t_*(\rho, \eta) + \epsilon|t_*(\rho, \eta)|, \rho, \eta).$$

It follows that if ϵ', ϵ are small enough, there exists $C > 0$ such that we have

$$\Re(\Phi(t_*(\rho, \zeta), \rho, \zeta) - \Phi(t, \rho, \zeta)) \leq -\epsilon^2 C \left|\frac{\eta}{m}\right|^{\frac{m}{m-1}}.$$

Hence there exists $C' > 0$ such that

$$|I_2| \leq C' e^{-\epsilon^2 C \left|\frac{\eta}{m}\right|^{\frac{m}{m-1}}}.$$

Hence the main contribution comes from I_1 .

To estimate I_1 , we make the shift of contour $t \rightarrow t + t_*(\rho, \zeta)$, where $|t| \leq \epsilon|t_*(\rho, \eta)|$. We have

$$I_1 = \int_{|t| \leq \epsilon|t_*(\rho, \eta)|} e^{\Phi(t_*(\rho, \zeta), \rho, \zeta) - \Phi(t + t_*(\rho, \zeta), \rho, \zeta)} q(t + t_*(\rho, \zeta), \rho, \zeta) dt + R,$$

where R is an integral over the paths $t = t_s^\pm$, $0 \leq s \leq 1$, defined by

$$t_s^\pm = s(t_*(\rho, \zeta) \pm \epsilon|t_*(\rho, \eta)|) + (1-s)(t_*(\rho, \eta) \pm \epsilon|t_*(\rho, \eta)|).$$

Observe that

$$t_s^\pm = \left(-\frac{\eta}{m}\right)^{\frac{1}{m-1}} (1 \pm \epsilon)(1 + O(\epsilon'))(1 + O(\delta)).$$

It follows that if ϵ', ϵ are small enough, there exists $C > 0$ such that we have

$$\Re(\Phi(t_*(\rho, \zeta), \rho, \zeta) - \Phi(t_s^\pm, \rho, \zeta)) \leq -\epsilon^2 C \left|\frac{\eta}{m}\right|^{\frac{m}{m-1}}.$$

Hence, there exists $C > 0, C' > 0$ such that

$$|R| \leq C' e^{-\epsilon^2 C \left|\frac{\eta}{m}\right|^{\frac{m}{m-1}}}.$$

Now it remains to study $I_1 - R$, which can be estimated using Lemma 14. \square

11. AN ALGORITHM OF PÓLYA

Lemma 16. *The roots $\pm iR$ of \mathcal{N} are simple.*

Proof. To prove that $\mathcal{N}'(iR) \neq 0$ we use a method of Pólya [25]. Pólya developed an algorithm to compute the roots of a class of entire function assuming that the Taylor coefficients are known. Let F be an entire function of finite order, with genus p , satisfying the following conditions:

- (i) F has infinitely many roots, and all the roots of F are positive;
- (ii) F is real valued on the real line;
- (iii) $F(0) = 1$.

Let $0 < \alpha_1 \leq \alpha_2 \leq \dots$ be the roots of F , and let $(-1)^k a_k$ be the Taylor coefficients of F , i.e. $F(z) = \sum_{k=0}^{\infty} (-1)^k a_k z^k$. Taking a unit root $\omega = e^{2\pi i/n}$ with $n > p$, the power series of $F(z)F(\omega z) \cdots F(\omega^{n-1}z)$ has the following form

$$F(z)F(\omega z) \cdots F(\omega^{n-1}z) = \sum_{k=0}^{\infty} (-1)^k a_{n,k} z^{nk}.$$

The coefficient $a_{n,k}$ can be computed from coefficients $\{a_k\}_{k=1}^{\infty}$. Then Pólya's algorithm provides the following estimate

$$\left(\frac{1}{a_{n,1}}\right)^{1/n} < \alpha_1 < \left(\frac{a_{n,1}}{a_{2n,1}}\right)^{1/n}. \quad (78)$$

We will use this inequality to show that the first root of \mathcal{N} is closer to the origin than the first nonzero root of \mathcal{N}' . More precisely, we apply Pólya's algorithm to the function

$$\begin{aligned} F_m(z) &= \frac{m2^{1/m-1}}{\Gamma(\frac{1}{m})} \mathcal{N}(2^{1/m-1}i\sqrt{z}) = \frac{m}{\Gamma(\frac{1}{m})} \int_0^{\infty} e^{-t^m} \cos(t\sqrt{z}) dt \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{2k+1}{m})}{\Gamma(\frac{1}{m})\Gamma(2k+1)} z^k \equiv \sum_{k=0}^{\infty} (-1)^k a_k z^k. \end{aligned}$$

It follows from Pólya [24] that F_m satisfies (i)–(iii). The genus of F_m is zero since the order of \mathcal{N} is $\frac{m}{m-1} \in (1, 2)$. So we can apply (78) with $n = 1$ to obtain

$$\alpha_1 < \frac{a_{1,1}}{a_{2,1}},$$

where

$$a_{1,1} = \frac{\Gamma(\frac{3}{m})}{2\Gamma(\frac{1}{m})}, \quad a_{2,1} = \left(\frac{\Gamma(\frac{3}{m})}{2\Gamma(\frac{1}{m})}\right)^2 - 2\frac{\Gamma(\frac{5}{m})}{\Gamma(\frac{1}{m})\Gamma(5)}.$$

Next we consider the function

$$G_m(z) = \frac{m}{\Gamma(\frac{3}{m})} \int_0^\infty \frac{\sin(t\sqrt{z})}{t\sqrt{z}} t^2 e^{-t^m} dt.$$

The function $G_m(z)$ is the derivative

$$F'_m(z) = \frac{m2^{2/m-3}}{\Gamma(\frac{1}{m})\sqrt{z}} i\mathcal{N}'(2^{1/m-1}i\sqrt{z}) = -\frac{m}{2\Gamma(\frac{1}{m})} \int_0^\infty t^2 e^{-t^m} \frac{\sin(t\sqrt{z})}{t\sqrt{z}} dt$$

with the normalization $G_m(0) = 1$. Again, it follows from Pólya [24] that G_m satisfies (i)–(iii) and that the genus of G_m is zero. Let $0 < \beta_1 \leq \beta_2 \leq \dots$ be the roots of G_m , and let $(-1)^k b_k$ be the Taylor coefficients of G_m , i.e.,

$$G_m(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\frac{2k+3}{m})}{\Gamma(\frac{3}{m})\Gamma(2k+2)} z^k \equiv \sum_{k=0}^{\infty} (-1)^k b_k z^k.$$

Applying (78) with $n = 2$ we get the lower bound

$$\left(\frac{1}{b_{2,1}}\right)^{1/2} < \beta_1$$

where

$$b_{2,1} = \left(\frac{\Gamma(\frac{5}{m})}{\Gamma(\frac{3}{m})\Gamma(4)}\right)^2 - 2\frac{\Gamma(\frac{7}{m})}{\Gamma(\frac{3}{m})\Gamma(6)}.$$

To prove that $\alpha_1 \neq \beta_1$ it is sufficient to show that

$$\frac{a_{1,1}}{a_{2,1}} < \left(\frac{1}{b_{2,1}}\right)^{1/2} \quad (79)$$

for all even integer $m \geq 4$. After a short calculation one can see that the last inequality is equivalent to

$$10\Gamma\left(\frac{1}{m}\right)\Gamma\left(\frac{3}{m}\right)\Gamma\left(\frac{5}{m}\right) < 15\Gamma\left(\frac{3}{m}\right)^3 + \Gamma\left(\frac{1}{m}\right)^2\Gamma\left(\frac{7}{m}\right). \quad (80)$$

Using the logarithmic convexity of Γ we have the estimate $\frac{1}{x}e^{-Cx} \leq \Gamma(x) \leq \frac{1}{x}$ for all $x \in (0, 1]$, with the Euler constant C . Therefore we have the upper bound $\frac{2}{3}m^3$ for the left-hand side of (80), and the lower bound $m^3(\frac{5}{9} + \frac{1}{7})e^{-9C/m}$ for the right-hand side of (80) for all $m \geq 7$. Since the inequality

$$\frac{2}{3}m^3 < m^3 \left(\frac{5}{9} + \frac{1}{7}\right) e^{-\frac{9C}{m}}$$

is valid for $m \geq 114$, we conclude that $\alpha_1(m) \neq \beta_1(m)$ provided that $m \geq 114$. The remaining cases are verified by the computer program *Mathematica*[®]. Taking $n = 2$ in (78) we have the estimate

$$\alpha_1 < \left(\frac{a_{2,1}}{a_{4,1}}\right)^{1/2}, \quad \left(\frac{1}{b_{2,1}}\right)^{1/2} < \beta_1$$

with

$$a_{4,1} = \left[\left(\frac{\Gamma(\frac{3}{m})}{2\Gamma(\frac{1}{m})} \right)^2 - 2 \frac{\Gamma(\frac{5}{m})}{\Gamma(\frac{1}{m})\Gamma(5)} \right]^2 - 2 \left(\frac{\Gamma(\frac{5}{m})}{\Gamma(\frac{1}{m})\Gamma(5)} \right)^2 + 4 \frac{\Gamma(\frac{3}{m})\Gamma(\frac{7}{m})}{\Gamma(\frac{1}{m})^2\Gamma(3)\Gamma(7)} - 4 \frac{\Gamma(\frac{9}{m})}{\Gamma(\frac{1}{m})\Gamma(9)}.$$

□

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