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A Finite Dimensional Gauge Problem in the WZNW Model ⁰

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Abstract

The left and right zero modes of the $SU(n)$ WZNW model give rise to a pair of isomorphic mutually commuting algebras \mathcal{A} and $\overline{\mathcal{A}}$. Here \mathcal{A} is the *quantum matrix algebra* [1] generated by an $n \times n$ matrix $a = (a_{\alpha}^i)$, $i, \alpha = 1, \dots, n$ (with noncommuting entries) and an abelian group consisting of products of n elements q^{p_i} satisfying $\prod_{i=1}^n q^{p_i} = 1$, $q^{p_i} a_{\alpha}^j = a_{\alpha}^j q^{p_i + \delta_i^j - \frac{1}{n}}$. For an integer $\widehat{su}(n)$ height h ($= k + n \geq n$) the complex parameter q is an even root of unity, $q^h = -1$, and \mathcal{A} admits an ideal \mathcal{I}_h generated by $\{(a_{\alpha}^i)^h, q^{2p_{ij}h} - 1, p_{ij} = p_i - p_j, \alpha, i, j = 1, \dots, n\}$ such that the factor algebra $\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h$ is finite dimensional. The structure of superselection sectors of the (diagonal) 2-dimensional (2D) WZNW model is then reduced to a finite dimensional problem of a gauge theory type. For $n = 2$ this problem is solved using a generalized BRS formalism.

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Introduction

Although the Wess–Zumino–Novikov–Witten (WZNW) model is first formulated in terms of a (multivalued) action [2], it is originally solved [3] by using axiomatic conformal field theory methods. The two dimensional (2D) Euclidean Green functions are written [4] as sums of products of analytic and antianalytic *conformal blocks*. Their operator interpretation exhibits some puzzling features: the presence of *non-integer* (“quantum”) *statistical dimensions* (that appear as positive real solutions of the *fusion rules* [5]) contrasted with the local (“Bose”) commutation relations (CR) of the corresponding 2D fields. The gradual understanding of both the factorization property and the hidden *braid group statistics* (signaled by the quantum dimensions) only begins with the development of the canonical approach to the model (for a sample of references, see [6] - [14]) and the associated splitting of the basic group valued field $g : \mathbb{S}^1 \times \mathbb{R} \rightarrow G$ into chiral parts. The resulting zero mode extended phase space displays a new type of *quantum group gauge symmetry*: on one hand, it is expressed in terms of the *quantum universal enveloping algebra* $U_q(\mathcal{G})$, a deformation of the *finite dimensional Lie algebra* \mathcal{G} of G – much like a gauge symmetry of the first kind; on the other, it requires the introduction of an *extended, indefinite metric state space*, a typical feature of a (local) gauge theory of the second kind.

Chiral fields admit an expansion into *chiral vertex operators* (CVO) which are characterized essentially by the currents’ degrees of freedom with “zero mode” coefficients that are independent of the world sheet coordinate [15, 16, 13, 14]. Such a type of quantum theory has been studied in the framework of lattice current algebras (see [8, 9, 10, 17, 18] and references therein) and has not been brought to a form yielding a satisfactory continuum limit. The direct investigation of the quantum model [13, 14, 1] has singled out a nontrivial gauge theory problem. This problem has been tackled in two steps [19, 20] in terms of a generalization of the Becchi-Rouet-Stora (BRS) [21] cohomologies.

The present paper aims at providing a concise survey of this study (including a preview of a work in progress [22]). It is organized as follows. Section 1 is devoted to a brief review of results of Gawędzki et al. [9, 10] that culminate earlier work [6, 7, 8] on the canonical approach to the classical WZNW model and on the first steps to its quantization – including the R -matrix exchange relations. Section 2 defines and studies the main object of interest to us, the quantum matrix algebra \mathcal{A} . The basic exchange relations for $a_\alpha^i \in \mathcal{A}$ involve a dynamical R -matrix $\hat{R}(p)$ of Hecke type. Section 3 introduces the Fock space (\mathcal{F}) representation of \mathcal{A} equipped with two $U_q(sl_n)$ invariant forms, a bilinear and a sesquilinear one. Both forms have a kernel $\mathcal{I}_h \mathcal{F}$ where \mathcal{I}_h is the maximal ideal of \mathcal{A} (for $q^h = -1$); the corresponding quotient of \mathcal{F}_h with respect to this kernel is finite dimensional. Section 4 identifies “the physical state space” for the 2D zero mode algebra $\mathcal{A} \otimes \bar{\mathcal{A}}$ in terms of a generalized BRS cohomology (with a BRS charge Q such that $Q^h = 0$, $h > n$). In the concluding remarks (Section 5) we indicate some open problems.

1 Canonical approach to the WZNW model

The starting point of the (first order) covariant Hamiltonian formulation of the WZNW model is the canonical 3-form [9]

$$\omega = d\alpha - \frac{k}{12\pi} \text{tr} (g^{-1}dg)^3, \quad \alpha = \frac{1}{2} \text{tr} \left(i(j^0 dx - j^1 dt) g^{-1}dg - \frac{\pi}{k} j_\nu j^\nu dt dx \right) \quad (1.1)$$

($j_\nu j^\nu = j_1^2 - j_0^2$). The coefficient to the WZ term (the closed but not exact form $\text{tr} (g^{-1}dg)^3$) is chosen in such a way that the solution of the resulting equations of motion (cf. [9, 12])

$$j_\nu = \frac{k}{2\pi i} g^{-1} \partial_\nu g, \quad \partial_\nu j^\nu = \frac{2\pi i}{k} [j_0, j_1] \quad (1.2)$$

splits into left and right movers. Indeed, Eqs.(1.2) imply

$$\partial_+(g^{-1}\partial_-g) = 0 \Leftrightarrow \partial_-((\partial_+g)g^{-1}) = 0, \quad 2\partial_\pm = \partial_1 \pm \partial_0 \quad (\partial_\nu = \frac{\partial}{\partial x^\nu}, x^0 \equiv t). \quad (1.3)$$

The general solution of Eq.(1.3) is given by

$$g_B^A(t, x) = u_\alpha^A(x+t) (\bar{u}^{-1})_B^\alpha(x-t) \quad (\text{classically } g, u, \bar{u} \in G = SU(n)) \quad (1.4)$$

provided the periodicity condition for g is weakened to a twisted periodicity condition for u and \bar{u} ,

$$g(t, x+2\pi) = g(x) \Rightarrow u(x+2\pi) = u(x)M, \quad \bar{u}(x+2\pi) = \bar{u}(x)\bar{M} \quad (1.5)$$

with *equal monodromies*, $\bar{M} = M$. The symplectic 2-form

$$\Omega^{(2)}(g, j^0) = \int_{\mathbb{S}^1 (t=\text{const})} \omega = \frac{1}{2} \text{tr} \int_{-\pi}^{\pi} dx \left((ij^0 - \frac{k}{2\pi} g^{-1} \partial_x g) (g^{-1} dg)^2 - i(dj^0) g^{-1} dg \right) \quad (1.6)$$

is expressed as a sum of two chiral 2-forms involving the monodromy,

$$\Omega^{(2)}(g, j^0) = \Omega(u, M) - \Omega(\bar{u}, M), \quad (1.7)$$

$$\Omega(u, M) = \frac{k}{4\pi} \text{tr} \left\{ \int_{-\pi}^{\pi} dx \partial (u^{-1} du) u^{-1} du - u(-\pi)^{-1} du(-\pi) dMM^{-1} + \rho(M) \right\}.$$

The 2-form $\text{tr} \rho(M)$ is only restricted by the requirement that Ω is closed:

$$d\Omega(u, M) = 0 \Leftrightarrow d \text{tr} \rho(M) = \frac{1}{3} \text{tr} (dMM^{-1})^3. \quad (1.8)$$

The different choices of ρ consistent with (1.8) correspond to different non-degenerate solutions of the classical Yang-Baxter equation (YBE). The associated classical r -matrices enter the Poisson bracket (PB) relations for the chiral dynamical variables

[9, 10, 12]. A standard choice corresponds to the Gauss decomposition $M = M_+ M_-^{-1}$ of M (M_+ and M_-^{-1} involving identical Cartan factors); then

$$\rho(M) = \text{tr} (M_+^{-1} dM_+ M_-^{-1} dM_-) . \quad (1.9)$$

(For more general choices including a monodromy dependent r -matrix see [23].)

The symplectic form $\Omega^{(2)}(g, j^0)$ becomes degenerate when extended to the space of left and right chiral variables (with a common monodromy) (u, \bar{u}, M) . This is due to the non-uniqueness of the decomposition (1.4): $g(t, x)$ does not change under constant right shifts of its chiral components ($u \rightarrow uh$, $\bar{u} \rightarrow \bar{u}h$, $M \rightarrow h^{-1}Mh$, $h \in G$). We restore non-degeneracy by further extending the phase space introducing independent monodromies M and \bar{M} for u and \bar{u} , respectively, thus completely decoupling the left and right sectors. The price is that M and \bar{M} satisfy Poisson bracket relations of opposite sign (cf. (1.7)) and monodromy invariance is only restored in a weak sense – when g is applied to physical states in the quantum theory.

Quantization is performed by requiring that it respects all symmetries of the classical chiral theory. Apart from conformal invariance and invariance under periodic left shifts the (u, M) system admits a Poisson–Lie symmetry under constant right shifts [24, 9, 10, 25, 12] which gives rise to a quantum group symmetry in the quantized theory. This requires passing from the classical to the quantum R -matrix which obeys the quantum YBE: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. We end up with quadratic exchange relations for the chiral variables:

$$P u_1(y) u_2(x) = u_1(x) u_2(y) \hat{R}(x - y), \quad \bar{u}_1(y) \bar{u}_2(x) P = \hat{R}^{-1}(x - y) \bar{u}_1(x) \bar{u}_2(y),$$

$$\hat{R}(x) = \begin{cases} \hat{R} & \text{for } x > 0 \\ P & \text{for } x = 0 \\ \hat{R}^{-1} & \text{for } x < 0 \end{cases} \quad (1.10)$$

(and associated relations for the monodromy $M = u(\pi)^{-1}u(-\pi)$). Here we are using the standard tensor product notation $u_1 u_2 = u \otimes u$, $R_{13} = (R_{\beta_1 \beta_3}^{\alpha_1 \alpha_3} \delta_{\beta_2}^{\alpha_2})$ etc. (see [26]), P stands for permutation, so that the first equation (1.10) is a shorthand for $u_\alpha^B(y) u_\beta^A(x) = u_{\alpha'}^A(x) u_{\beta'}^B(y) \hat{R}_{\alpha\beta}^{\alpha'\beta'}(x - y)$; \hat{R} is the braid operator related to the standard Drinfeld-Jimbo [27, 28] $U_q(sl_n)$ R -matrix by $\hat{R} = RP$. The quantum YBE for R implies the braid relation

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}. \quad (1.11)$$

The exchange relations for u are invariant under the left coaction of the quantum group $SL_q(n)$ [1]. A dual expression of this property is the $U_q(sl_n)$ ($\equiv U_q$) invariance of (1.10). The U_q Chevalley generators E_i, F_i, q^{H_i} can be identified with the elements of the triangular matrices M_\pm [26, 14]. The R -matrix exchange relations among (elements of) M_\pm

$$[\hat{R}, (M_\pm)_1 (M_\pm)_2] = 0 = [\hat{R}, (\bar{M}_\pm)_2 (\bar{M}_\pm)_1] \quad (1.12)$$

imply the CR for the U_q generators:

$$q^{H_i} E_j = E_j q^{H_i + c_{ij}}, \quad q^{H_i} F_j = F_j q^{H_i - c_{ij}}, \quad c_{ij} = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } |i - j| = 1 \\ 0 & \text{for } |i - j| \geq 2 \end{cases}, \quad (1.13)$$

$$[E_i, F_j] = \delta_{ij} [H_i], \quad X_i X_{i\pm 1} X_i = X_i^{[2]} X_{i\pm 1} + X_{i\pm 1} X_i^{[2]}, \quad X_i = E_i, F_i \quad (1.14)$$

where

$$X^{[m]} := \frac{1}{[m]!} X^m \quad ([m]! = [m-1]![m], \quad [m] = \frac{q^m - \bar{q}^m}{q - \bar{q}}, \quad [0]! = 1; \quad \bar{q} \equiv q^{-1}). \quad (1.15)$$

The $U_q(sl_n)$ \hat{R} -matrix can be written in the form

$$q^{\frac{1}{n}} \hat{R} = q \mathbb{I} - A, \quad A_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = q^{\epsilon_{\alpha_2 \alpha_1}} \delta_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - \delta_{\beta_2 \beta_1}^{\alpha_1 \alpha_2}, \quad q^{\epsilon_{\alpha_2 \alpha_1}} = \begin{cases} q & \text{for } \alpha_2 > \alpha_1 \\ 1 & \text{for } \alpha_2 = \alpha_1 \\ \bar{q} & \text{for } \alpha_2 < \alpha_1 \end{cases}. \quad (1.16)$$

The q -antisymmetrizer (see, e.g., [28]) A is a (non-normalized) projector,

$$A^2 = [2] A \quad ([2] = q + \bar{q}) \quad (1.17)$$

satisfying, due to the braid relation (1.11),

$$A_{12} A_{23} A_{12} - A_{12} = A_{23} A_{12} A_{23} - A_{23}. \quad (1.18)$$

Eq. (1.17) is equivalent to the *Hecke property*

$$q^{\frac{2}{n}} \hat{R}^2 = \mathbb{I} + (q - \bar{q}) q^{\frac{1}{n}} \hat{R} \quad (1.19)$$

that does not hold for a deformation of a simple Lie group different from $SL(n)$. The parameter q is expressed in terms of the height $h = k + n$ ($\geq n$) by

$$q = e^{i\frac{\pi}{h}} \quad (q^{\frac{1}{n}} = e^{i\frac{\pi}{nh}}) \quad \Rightarrow \quad q^h = -1; \quad (1.20)$$

here $k \in \mathbb{Z}_+$ is the quantized coupling constant of (1.1), (1.2) identified with the Kac-Moody level.

2 Chiral vertex operators and zero modes. The quantum matrix algebra

Let $\{v^{(i)}, i = 1, \dots, n\}$ be a symmetric "*barycentric basis*" of (linearly dependent) real traceless diagonal matrices :

$$(v^{(i)})_k^j = \left(\delta_{ij} - \frac{1}{n} \right) \delta_k^j \quad \Rightarrow \quad \sum_{i=1}^n v^{(i)} = 0. \quad (2.1)$$

The simple $sl(n)$ (co)roots $(\alpha_i^\vee =) \alpha_i$ and the corresponding (co)weights $\Lambda^{(j)}$ are expressed in terms of $v^{(i)}$ as follows:

$$\alpha_i = v^{(i)} - v^{(i+1)}, \quad \Lambda^{(j)} = \sum_{i=1}^j v^{(i)} \quad (\Rightarrow (\Lambda^{(j)} | \alpha_i) = \delta_i^j) \quad (2.2)$$

(the inner product of two matrices coinciding with the trace of their product). A *shifted dominant weight*

$$p = \Lambda + \rho, \quad \Lambda = \sum_{i=1}^{n-1} \lambda_i \Lambda^{(i)}, \quad \lambda_i \in \mathbb{Z}_+, \quad \rho = \sum_{i=1}^{n-1} \Lambda^{(i)} (= \frac{1}{2} \sum_{\alpha > 0} \alpha) \quad (2.3)$$

can be conveniently parametrized by n numbers $\{p_i\}$ ($\sum_{i=1}^n p_i = 0$) satisfying

$$p_{i+1} = \lambda_i + 1 \in \mathbb{N}, \quad i = 1, 2, \dots, n-1 \quad \text{where} \quad p_{ij} \equiv p_i - p_j. \quad (2.4)$$

(The non-negative integers $\lambda_i = p_{i+1} - 1$ count the number of columns of length i in the Young tableau that corresponds to the IR of highest weight p of $SU(n)$ – see, e.g., [29].) Dominant weights p also label highest weight representations of U_q . For integer heights $h (\geq n)$ and q satisfying (1.20) these are (unitary) irreducible if $(n-1 \leq) p_{1n} \leq h$. The *quantum dimension* of such an IR is given by [30]

$$d_q(p) = \prod_{i=1}^{n-1} \left\{ \frac{1}{[i]!} \prod_{j=i+1}^n [p_{ij}] \right\} \quad (\geq 0 \quad \text{for} \quad p_{1n} \leq h). \quad (2.5)$$

For $q \rightarrow 1$ ($h \rightarrow \infty$), $[m] \rightarrow m$ we recover the usual (integral) dimension of the IR under consideration.

Energy positivity implies that the state space of the chiral quantum WZNW theory is a direct sum of (height h) ground state modules \mathcal{H}_p of the Kac-Moody algebra $\widehat{su}(n)$ with a finite multiplicity:

$$\mathcal{H} = \bigoplus_p \mathcal{H}_p \otimes \mathcal{V}_p, \quad \dim \mathcal{V}_p < \infty. \quad (2.6)$$

Each "internal space" \mathcal{V}_p carries a representation of U_q and is an eigensubspace of all U_q Casimir invariants (whose eigenvalues are polynomials in q^{p_i}).

The analysis of the axiomatic construction of the quantum field theory generated by a chiral current algebra [3, 4, 31, 32, 33] yields the following properties of the representations of U_q in the space (2.6):

- (i) the ideal generated by $E_i^h, F_i^h, [hH_i]$ is represented trivially;
- (ii) the *integrable highest weight representations* corresponding to positive integer p_{i+1} and $n-1 \leq p_{1n} < h$ appear in pair with representations of weight \tilde{p} where

$$\tilde{p}_{12} = 2h - p_{12} \quad \text{for } n = 2; \quad \tilde{p}_{12} = h - p_{23}, \quad \tilde{p}_{23} = h - p_{12} \quad \text{for } n = 3, \text{ etc.} \quad (2.7)$$

which corresponds to the highest weight of a subspace of singular vectors in the Verma module \mathcal{H}_p .

Each \mathcal{H}_p in the direct sum (2.6) is a graded vector space,

$$\mathcal{H}_p = \bigoplus_{\nu=0}^{\infty} \mathcal{H}_p^{\nu}, \quad (L_0 - \Delta_h(p) - \nu) \mathcal{H}_p^{\nu} = 0, \quad \dim \mathcal{H}_p^{\nu} < \infty, \quad (2.8)$$

where L_0 is the chiral (Virasoro) energy operator. Here \mathcal{H}_p^0 spans an IR of $su(n)$ of (shifted) highest weight p . The *conformal dimension* (or *conformal weight*) $\Delta_h(p)$ is proportional to the ($su(n)$ -) second order Casimir operator $|p|^2 - |\rho|^2$, and $\Delta_h(\tilde{p}) - \Delta_h(p)$ is an integer; the vacuum weight is zero:

$$2h\Delta_h(p) = |p|^2 - |\rho|^2 = \frac{1}{n} \sum_{1 \leq i < j \leq n} p_{ij}^2 - \frac{n(n^2 - 1)}{12},$$

$$\Delta_h(\tilde{p}) - \Delta_h(p) = h - p_{1n} \quad \text{for } n = 2, 3; \quad \Delta_h(p^{(0)}) = 0 \quad \text{for } p_{i+1}^{(0)} = 1. \quad (2.9)$$

The eigenvalues of the braid operator \hat{R} are expressed as exponents of differences of conformal dimensions [13, 14, 1]. This yields (1.20) for q .

We shall split the $SU(n) \times SL_q(n)$ covariant field $u(x) = (u_{\alpha}^A(x))$ into factors which intertwine separately different \mathcal{H}_p and \mathcal{V}_p spaces. A CVO $u_j(x, p)$ is defined as an intertwining map between \mathcal{H}_p and $\mathcal{H}_{p+v(j)}$ (for each p in the sum (2.6)). Noting that \mathcal{H}_p is an eigenspace of $e^{2\pi i L_0}$,

$$\text{Spec } L_0 |_{\mathcal{H}_p} \subset \Delta_h(p) + \mathbb{Z}_+ \quad \Rightarrow \quad \{e^{2\pi i L_0} - e^{2\pi i \Delta_h(p)}\} \mathcal{H}_p = 0, \quad (2.10)$$

we deduce that $u_j(x, p)$ is an eigenvector of the monodromy automorphism,

$$u_j(x + 2\pi, p) = e^{-2\pi i L_0} u_j(x, p) e^{2\pi i L_0} = u_j(x, p) \mu_j(p) \quad (2.11)$$

where, in view of (2.10), we find

$$\mu_j(p) := e^{2\pi i \{\Delta_h(p) - \Delta_h(p+v(j))\}} = q^{\frac{1}{n} - 1 - 2p_j}. \quad (2.12)$$

The monodromy matrix is diagonalizable whenever its eigenvalues (2.12) are all different. The exceptional points are those p for which there exists a pair of indices $1 \leq i < j \leq n$ such that $q^{2p_{ij}} = 1$, since we have

$$\frac{\mu_j(p)}{\mu_i(p)} = q^{2p_{ij}}. \quad (2.13)$$

According to (2.5) all such "exceptional" \mathcal{V}_p have zero quantum dimension. In particular, for the "physical IR", characterized by $p_{1n} < h$, M is diagonalizable.

The *quantum matrix* $a = (a_{\alpha}^j, j, \alpha = 1, \dots, n)$ is defined to relate the $SL_q(n)$ covariant field $u(x) = (u_{\alpha}^A(x))$ with the CVO $u_j^A(x, p)$ realizing the so called vertex-IRF (interaction-round-a-face) transformation [34]

$$u(x) = a^j u_j(x, p) \quad (u_j = (u_j^A), a^j = (a_{\alpha}^j)). \quad (2.14)$$

The zero mode operators a^j are defined to intertwine the finite dimensional U_q modules \mathcal{V}_p of (2.6):

$$a^j : \mathcal{V}_p \rightarrow \mathcal{V}_{p+v^{(j)}} . \quad (2.15)$$

For $n = 2$ the operators a^2 were treated as annihilation operators [13]. In fact, in the case of irreducible \mathcal{V}_p realized (for any $n \geq 2$) in the Fock space representation of the quantum matrix algebra introduced in Section 3 below, we have the annihilation property

$$a^j \mathcal{V}_p = 0 \quad \text{if} \quad p_{j-1} = p_j + 1, \quad j > 1 \quad (2.16)$$

(thus $a^j \mathcal{V}_p$ is zero unless $p + v^{(j)}$ is again a dominant weight).

The zero modes a_α^j commute with the currents, thus leaving the $\widehat{su}(n)$ modules \mathcal{H}_p unaltered. The order of factors in (2.14) is dictated by the requirement that the commuting operators p_i , $i = 1, \dots, n$, the components of the argument p of u_j , are proportional to the unit operators on \mathcal{H}_p with eigenvalues corresponding to the label of the module. We note that

$$q^{p_i} a^j = a^j q^{p_i + \delta_{ij} - \frac{1}{n}} \Rightarrow a^j u_i(x, p) = u_i(x, p - v^{(j)}) a^j . \quad (2.17)$$

The U_q covariance properties of a_α^i can be read off their exchange relations with the Gauss components M_\pm of M [14, 22]:

$$\begin{aligned} [E_a, a_\alpha^i] &= \delta_{a\alpha-1} a_{\alpha-1}^i q^{H_a}, \quad a = 1, \dots, n-1, \\ F_a a_\alpha^i - q^{\delta_{a\alpha-1} - \delta_{a\alpha}} a_\alpha^i F_a &= \delta_{a\alpha} a_{\alpha+1}^i; \\ q^{H_a} a_\alpha^i &= a_\alpha^i q^{H_a + \delta_{a\alpha} - \delta_{a\alpha-1}}. \end{aligned} \quad (2.18)$$

Comparing (1.5), (2.11) and (2.14) we deduce that the zero mode matrix a diagonalizes the monodromy (whenever the quantum dimension (2.5) does not vanish); setting $aM = M_p a$ we find (from the above analysis of Eqs. (2.11) - (2.13)) the implication

$$d_q(p) \neq 0 \Rightarrow (M_p)_j^i = \delta_j^i \mu_j(p - v^{(j)}), \quad \mu_j(p - v^{(j)}) = q^{1 - \frac{1}{n} - 2p_j}. \quad (2.19)$$

(A careful study of the case of vanishing $d_q(p)$ and non-diagonalizable M is still lacking.)

The exchange relations (1.10) for u given by (2.14) can be translated into quadratic exchange relations for the " U_q vertex operators" a_α^i provided we assume standard braid relations for the CVO $u(x, p)$:

$$u_i^B(y, p - v^{(i)}) u_j^A(x, p - v^{(i)} - v^{(j)}) = u_k^A(x, p - v^{(k)}) u_l^B(y, p - v^{(k)} - v^{(l)}) \hat{R}(p)_{ij}^{kl} \quad (2.20)$$

(what is important here is that $\hat{R}(p)$ depends only on p). An analysis of chiral 4-point blocks [22] shows that Eq.(2.20) is indeed satisfied.

A straightforward computation then gives

$$\hat{R}(p) a_1 a_2 = a_1 a_2 \hat{R}. \quad (2.21)$$

Associativity of triple tensor products of quantum matrices together with Eq.(1.11) for \hat{R} yield, as a consistency condition of (2.21), the quantum dynamical YBE for $\hat{R}(p)$ (first studied in [35]):

$$\hat{R}_{12}(p) \hat{R}_{23}(p - v_1) \hat{R}_{12}(p) = \hat{R}_{23}(p - v_1) \hat{R}_{12}(p) \hat{R}_{23}(p - v_1) \quad (2.22)$$

where we use again the succinct notation of Faddeev et al. [26]:

$$\left(\hat{R}_{23}(p - v_1) \right)_{j_1 j_2 j_3}^{i_1 i_2 i_3} = \delta_{j_1}^{i_1} \hat{R}(p - v^{(i_1)})_{j_2 j_3}^{i_2 i_3}. \quad (2.23)$$

In deriving (2.22) from (2.21) we use (2.17). (The quantum dynamical YBE (2.22) is only *sufficient* for the consistency of the quadratic matrix algebra relation (2.21); it would be also necessary if the matrix a were invertible – i.e., if $d_q(p) \neq 0$.)

The property of the operators $\hat{R}_{i+1}(p)$ to generate a representation of the braid group is ensured by the additional requirement (reflecting the commutativity of the braid group generators B_i and B_j for $|i - j| \geq 2$)

$$\hat{R}_{12}(p + v_1 + v_2) = \hat{R}_{12}(p) \Leftrightarrow \hat{R}_{kl}^{ij}(p) a_\alpha^k a_\beta^l = a_\alpha^k a_\beta^l \hat{R}_{kl}^{ij}(p). \quad (2.24)$$

The Hecke algebra condition (1.19) follows from the analysis of braiding properties of conformal blocks [22].

We shall adopt here the following solution of Eq.(2.22) and the Hecke algebra condition (presented in a form similar to (1.16)):

$$q^{\frac{1}{n}} \hat{R}(p) = q \mathbb{I} - A(p), \quad A(p)_{kl}^{ij} = \frac{[p_{ij} - 1]}{[p_{ij}]} (\delta_k^i \delta_l^j - \delta_l^i \delta_k^j). \quad (2.25)$$

$A(p)$ satisfies similar relations as A (cf. (1.17), (1.18)); in particular,

$$[p_{ij} - 1] + [p_{ij} + 1] = [2] [p_{ij}] \Rightarrow A^2(p) = [2] A(p). \quad (2.26)$$

According to [1] the general $SL(n)$ -type dynamical R -matrix [36] can be obtained from (2.25) by either a dynamical analog of Drinfeld's twist [37] (see Lemmas 3.1 and 3.2 of [1]) or by a canonical transformation $p_i \rightarrow p_i + c_i$ where c_i are constants (numbers) such that $\sum_{i=1}^n c_i = 0$. The interpretation of the eigenvalues of p_i as (shifted) weights (of the corresponding representations of U_q) allows to dispose of the second freedom.

Inserting (2.25) into the exchange relations (2.21) allows to present the latter in the following explicit form:

$$[a_\alpha^i, a_\alpha^j] = 0, \quad a_\alpha^i a_\beta^i = q^{\epsilon_{\alpha\beta}} a_\beta^i a_\alpha^i \quad (2.27)$$

$$[p_{ij} - 1] a_\alpha^j a_\beta^i = [p_{ij}] a_\beta^i a_\alpha^j - q^{\epsilon_{\beta\alpha} p_{ij}} a_\alpha^i a_\beta^j \quad \text{for } \alpha \neq \beta \text{ and } i \neq j, \quad (2.28)$$

where $q^{\varepsilon\beta\alpha}$ is defined in (1.16). There is, finally, a relation of order n for a_α^i , derived from the following basic property of the *quantum determinant*:

$$\det(a) = \frac{1}{[n]!} \varepsilon_{i_1 \dots i_n} a_{\alpha_1}^{i_1} \dots a_{\alpha_n}^{i_n} \mathcal{E}^{\alpha_1 \dots \alpha_n} \quad (2.29)$$

where

$$\mathcal{E}^{\alpha_1 \dots \alpha_n} (= \mathcal{E}_{\alpha_1 \dots \alpha_n}) = \frac{1}{q^{\frac{n(n-1)}{4}}} (-q)^{\ell(\sigma)} \quad \text{for } \sigma = \begin{pmatrix} n, \dots, 1 \\ \alpha_1, \dots, \alpha_n \end{pmatrix}, \quad (2.30)$$

$\ell(\sigma)$ being the length of the permutation σ ; $\varepsilon_{i_1 \dots i_n}$ is the undeformed Levi-Civita tensor normalized by $\varepsilon_{n \dots 1} = 1$; the ratio $\det(a) \left(\prod_{i < j} [p_{ij}] \right)^{-1}$ belongs to the centre of the *quantum matrix algebra* $\mathcal{A} = \mathcal{A}(\hat{R}(p), \hat{R})$ – the associative algebra with generators q^{p_i} , a_α^j and relations $q^{p_i} q^{p_j} = q^{p_j} q^{p_i}$, $\prod_{i=1}^n q^{p_i} = 1$, as well as (2.17) and (2.21) (see Corollary 5.1 of Proposition 5.2 of [1]). It is, therefore, legitimate to normalize the quantum determinant setting

$$\det(a) = \prod_{i < j} [p_{ij}] \equiv \mathcal{D}(p). \quad (2.31)$$

Clearly, it is proportional (with a p -independent positive factor) to the quantum dimension (2.5).

We shall use in what follows the intertwining properties of the product $a_1 \dots a_n$ (see Proposition 5.1 of [1]):

$$\varepsilon_{i_1 \dots i_n} a_{\alpha_1}^{i_1} \dots a_{\alpha_n}^{i_n} = \mathcal{D}(p) \mathcal{E}_{\alpha_1 \dots \alpha_n}, \quad (2.32)$$

$$a_{\alpha_1}^{i_1} \dots a_{\alpha_n}^{i_n} \mathcal{E}^{\alpha_1 \dots \alpha_n} = \varepsilon^{i_1 \dots i_n}(p) \mathcal{D}(p). \quad (2.33)$$

Here $\varepsilon^{i_1 \dots i_n}(p)$ is the dynamical Levi-Civita tensor given by

$$\varepsilon^{i_1 \dots i_n}(p) = (-1)^{\ell(\sigma)} \prod_{1 \leq \mu < \nu \leq n} \frac{[p_{i_\mu i_\nu} - 1]}{[p_{i_\mu i_\nu}]}. \quad (2.34)$$

We note that the "monodromy subalgebra" of \mathcal{A} , i.e., the commutant of $\{q^{p_i}, i = 1, \dots, n\}$, is generated by U_q and $\{q^{p_i}\}$.

3 Fock space representation of \mathcal{A} and its finite dimensional quotient space

The *Fock space* $\mathcal{F} = \mathcal{F}(\mathcal{A})$ for the quantum matrix algebra \mathcal{A} is defined as a (reducible) U_q module with a 1-dimensional U_q -invariant subspace $\{\mathbb{C}|0\rangle\}$ where the

vacuum vector $|0\rangle \equiv |p^{(0)}, 0\rangle$ is cyclic for \mathcal{A} and is annihilated by all monomials in a_α^i which do not correspond to a Young tableau: $\mathcal{A}|0\rangle = \mathcal{F}$, and

$$\begin{aligned} \mathcal{P}_{m_n}(a_\alpha^n) \dots \mathcal{P}_{m_1}(a_\alpha^1)|0\rangle &= 0 \\ \text{for } \mathcal{P}_{m_i}(a_\alpha^i) &= (a_1^i)^{m_{i1}} \dots (a_n^i)^{m_{in}} \quad \text{unless } m_1 \geq m_2 \geq \dots \geq m_n \end{aligned} \quad (3.1)$$

where $m_{is} \in \mathbb{Z}_+$ and $m_i = m_{i1} + \dots + m_{in}$. The U_q structure of \mathcal{F} is given by the following statements [22].

Proposition 3.1 *The Fock space \mathcal{F} is a direct sum of finite dimensional U_q modules $\mathcal{F}(p)$ spanned by monomials of the type*

$$\mathcal{P}_{\lambda_{n-1}}(a_\alpha^{n-1}) \mathcal{P}_{\lambda_{n-2}+\lambda_{n-1}}(a_\alpha^{n-2}) \dots \mathcal{P}_{\lambda_1+\dots+\lambda_{n-1}}(a_\alpha^1)|0\rangle \in \mathcal{F}(p) \quad (3.2)$$

where $\lambda_i = p_{i+1} - 1 \in \mathbb{Z}_+$ and the factors $\mathcal{P}_m(a_\alpha^i)$ are defined in (3.1). For $p_{1n} \leq h$ the resulting U_q modules are irreducible.

Theorem 3.2 *The space \mathcal{F} admits a (symmetric) U_q invariant bilinear form $\langle \cdot, \cdot \rangle$ determined by the conditions*

$$\langle \Phi, X\Psi \rangle = \langle X'\Phi, \Psi \rangle, \quad \forall X \in \mathcal{A}, \quad \langle 0|0 \rangle = 1 \quad (3.3)$$

where the "transposition" $X \rightarrow X'$ is a linear antiinvolution on \mathcal{A} such that

$$E'_i = F_i q^{H_i-1}, \quad F'_i = q^{1-H_i} E_i, \quad (q^{H_i})' = q^{H_i}, \quad (q^{p_{ij}})' = q^{p_{ij}} \quad (3.4)$$

and

$$\begin{aligned} (a_\alpha^i)' &= \tilde{a}_i^\alpha / \mathcal{D}_i(p); \quad \tilde{a}_i^\alpha = \frac{1}{[n-1]!} \mathcal{E}^{\alpha\alpha_1 \dots \alpha_{n-1}} \varepsilon_{i i_1 \dots i_{n-1}} a_{\alpha_1}^{i_1} \dots a_{\alpha_{n-1}}^{i_{n-1}}, \\ \mathcal{D}_i(p) &= \prod_{\substack{j < \ell \\ j \neq i \neq \ell}} [p_{j\ell}]. \end{aligned} \quad (3.5)$$

For $p_{i+1} \leq h$ the U_q modules $\mathcal{F}(p)$ are unitary with respect to a sesquilinear inner product (\cdot, \cdot) such that

$$(\Phi, X\Psi) = (X^*\Phi, \Psi) \quad \text{for } X \in U_q, \quad (0|0) = 1 \quad (3.6)$$

where the hermitean conjugation $X \rightarrow X^*$ is an antilinear involutive antihomomorphism of U_q satisfying

$$E_i^* = F_i, \quad F_i^* = E_i, \quad (q^{H_i})^* = \bar{q}^{H_i}, \quad (q^{p_{ij}})^* = \bar{q}^{p_{ij}}. \quad (3.7)$$

The bilinear form $\langle \cdot, \cdot \rangle$ is majorized by the scalar product (\cdot, \cdot) :

$$|\langle \Phi, \Psi \rangle|^2 \leq (\Phi, \Phi)(\Psi, \Psi) \quad \text{for } \Phi, \Psi \in \mathcal{H}' := \oplus_{p_{i+1} \leq h} \mathcal{F}(p). \quad (3.8)$$

There exists a basis $\{\Phi_\nu\}$ (e.g., the canonical basis described below, for $n = 2, 3$) in each $\mathcal{F}(p) \subset \mathcal{H}'$ of eigenvectors of q^{H_i} such that $|\langle \Phi_\nu, \Phi_\nu \rangle| = (\Phi_\nu, \Phi_\nu)$.

We note that while the transposition is a coalgebra homomorphism, – i.e., it preserves the coproduct Δ ,

$$\Delta(X)' = \Delta(X') \quad (\text{where } \Delta(M_\pm^\alpha{}_\beta) = M_\pm^\alpha{}_\sigma \otimes M_\pm^\sigma{}_\beta), \quad (3.9)$$

the conjugation reverses the order of factors in tensor products.

The algebra \mathcal{A} admits, for q given by (1.20), a large ideal

$$\mathcal{I}_h (= \mathcal{A} \mathcal{I}_h \mathcal{A} \subset \mathcal{A}) \quad \text{generated by } (a_a^i)^h \text{ and } [hp_{ij}]. \quad (3.10)$$

This is a consequence (for $m = h$) of the general exchange relation

$$[p_{ij} - 1](a_\alpha^j)^m a_\beta^i = [p_{ij} + m - 1]a_\beta^i (a_\alpha^j)^m - q^{\epsilon_{\beta\alpha}(p_{ij}+m-1)}[m](a_\alpha^j)^{m-1} a_\alpha^i a_\beta^j \quad (3.11)$$

which follows from (2.28), (2.27). Both forms (3.3) and (3.6) are degenerate, their kernel containing $\mathcal{I}_h \mathcal{F}$:

$$\langle \Phi, \mathcal{I}_h \Psi \rangle = 0 = (\Phi, \mathcal{I}_h \Psi) \quad \forall \Phi, \Psi \in \mathcal{F}. \quad (3.12)$$

The quotient space $\mathcal{F}_h := \mathcal{F}/\mathcal{I}_h \mathcal{F}$ is finite dimensional: it is isomorphic to the subspace of \mathcal{F} spanned by vectors of the form (3.2) with $\lambda_1 + \dots + \lambda_{n-1} \leq n(h-1)$, – i.e., $p_{1n} < nh$. It carries a representation of the factor algebra \mathcal{A}_h whose p -invariant subalgebra (of elements X satisfying $q^{p_i} X \bar{q}^{p_i} = X$) is the "finite dimensional (or restricted) quantum group" U_h :

$$\mathcal{A}_h = \mathcal{A}/\mathcal{I}_h \supset U_h = U_q/\mathcal{I}_h^U, \quad \mathcal{I}_h^U = \mathcal{I}_h \cap U_q, \quad (3.13)$$

\mathcal{I}_h^U being generated by E_i^h , F_i^h and $[hH_i]$.

As an example, and in order to prepare the ground for the discussion in Section 4, we shall display the canonical basis in \mathcal{F}_h for $n = 2$. Setting in this case $p_{12} = p$ we can write

$$|p, m\rangle := (a_1^1)^m (a_2^1)^{p-1-m} |0\rangle \quad (3.14)$$

with m belonging to the intervals $0 \leq m < p$, for $0 < p \leq h$, and $p - h \leq m < h$ for $h < p < 2h$, respectively. The $U_q(sl_2)$ properties of this basis are summed up by

$$(q^{H-2m+p-1} - 1)|p, m\rangle = 0, \quad E|p, m\rangle = [p-m-1]|p, m+1\rangle, \quad F|p, m\rangle = [m]|p, m-1\rangle. \quad (3.15)$$

\mathcal{F}_h can be written as a sum of irreducible $U_q(sl_2)$ modules and its dimension can be computed as follows:

$$\begin{aligned} \mathcal{F}_h &= \left(\bigoplus_{p=1}^h \mathcal{F}(p) \right) \oplus \left(\bigoplus_{p=h+1}^{2h-1} \tilde{\mathcal{F}}(p) \right) \quad \left(\tilde{\mathcal{F}}(p) = \mathcal{F}(\tilde{p})/\mathcal{I}_h^U \mathcal{F}(\tilde{p}) \right), \quad (3.16) \\ \dim \mathcal{F}(p) &= p, \quad \dim \tilde{\mathcal{F}}(p) = 2h - p, \quad \dim \mathcal{F}_h = h^2. \end{aligned}$$

Eq.(3.5) reduces in this case to

$$(a_\alpha^i)' = \tilde{a}_i^\alpha = \mathcal{E}^{\alpha\beta} \varepsilon_{ij} a_\beta^j, \quad \text{i.e., } (a_1^1)' = \tilde{a}_1^1 = q^{\frac{1}{2}} a_2^2, \quad (a_2^1)' = \tilde{a}_1^2 = -\bar{q}^{\frac{1}{2}} a_1^2, \quad \text{etc.} \quad (3.17)$$

In order to compute the bilinear form (3.3) in the basis (3.14) one needs the relations

$$\tilde{a}_1^1 |p, m\rangle = q^{m-p+1} [m] |p-1, m-1\rangle, \quad \tilde{a}_1^2 |p, m\rangle = [p-m-1] |p-1, m\rangle. \quad (3.18)$$

We deduce that the basis (3.14) is orthogonal with respect to both inner products and verifies the last statement of Theorem 3.2:

$$\langle p, m | p', m' \rangle = \delta_{pp'} \delta_{mm'} \bar{q}^{m(p-m-1)} (p, m | p, m), \quad (p, m | p, m) = [m]! [p-m-1]!. \quad (3.19)$$

Remark 3.1 The hermitean conjugation (3.7) can be extended to the entire algebra \mathcal{A} (and to \mathcal{A}_h) setting

$$(a_1^1)^* = q^{\frac{p-H-1}{2}} \tilde{a}_1^1 (\equiv a_1 \Rightarrow a_1^1 = a_1^*), \quad (a_2^1)^* = q^{\frac{1-p-H}{2}} \tilde{a}_1^2 (\equiv a_2 \Rightarrow a_2^1 = a_2^*). \quad (3.20)$$

We then verify that products of the type XX^* ,

$$a_1^1 (a_1^1)^* (= a_1^* a_1) = \left[\frac{H+p-1}{2} \right], \quad (a_1^1)^* a_1^1 (= a_1 a_1^*) = \left[\frac{H+p+1}{2} \right], \quad (3.21)$$

$$a_2^1 (a_2^1)^* (= a_2^* a_2) = \left[\frac{p-H-1}{2} \right], \quad (a_2^1)^* a_2^1 (= a_2 a_2^*) = \left[\frac{p-H+1}{2} \right],$$

are positive semidefinite, $XX^* \geq 0$, in \mathcal{F}_h . It follows that the majorization property (3.8) extends to the entire quotient space \mathcal{F}_h .

Remark 3.2 The canonical basis $\{|p, m\rangle, 0 \leq m \leq p-1\}$ in $\mathcal{F}(p)$ for $p \leq h$ can be also expressed in terms of the lowest and highest weight vectors in $\mathcal{F}(p)$

$$\begin{bmatrix} p-1 \\ m \end{bmatrix} |p, m\rangle = E^{[m]} |p, 0\rangle = F^{[p-1-m]} |p, p-1\rangle, \quad (3.22)$$

where we are using the notation $X^{[m]}$ of (1.15). These expressions are readily generalized to the case of $U_q(\mathfrak{sl}_3)$ in terms of Lusztig's canonical basis [38] of (say, raising) operators

$$E_1^{[m]} E_2^{[\ell]} E_1^{[k]}, \quad E_2^{[k]} E_1^{[\ell]} E_2^{[m]} \quad \text{for } \ell \geq k+m \quad (3.23)$$

where the Serre relations imply

$$E_1^{[m]} E_2^{[k+m]} E_1^{[k]} = E_2^{[k]} E_1^{[k+m]} E_2^{[m]}. \quad (3.24)$$

4 The physical zero mode space: a generalized BRS construction

The zero mode part of the monodromy extended $2D$ WZNW model involves the tensor product $\mathcal{A}_h \otimes \overline{\mathcal{A}}_h$ of two isomorphic copies of the chiral factor algebra (3.13) in the finite dimensional state space

$$\mathcal{H} = \mathcal{F}_h \otimes \overline{\mathcal{F}}_h. \quad (4.1)$$

The physical space $\mathfrak{H} = \mathfrak{H}_{h,n}$, on the other hand, is known (from the axiomatic treatment of the model) to have the following properties:

- (i) it is the complexification of the real Hilbert space $\mathfrak{H}_{\mathbb{R}}$ of dimension $\binom{h-1}{n-1}$ with inner product coinciding with the bilinear form $\langle \cdot, \cdot \rangle$;
- (ii) it is monodromy invariant [13, 14],

$$a_{\alpha}^i \{ (M \bar{M}^{-1})_{\beta}^{\alpha} - \delta_{\beta}^{\alpha} \} \bar{a}_j^{\beta} \mathfrak{H} = 0; \quad (4.2)$$

- (iii) it is invariant under the diagonal (coproduct) action of $U_q(sl_2)$

$$\{ \Delta(X) - \epsilon(X) \} \mathfrak{H} = 0 \quad \text{or, explicitly,} \quad \{ (\bar{M}_{\pm}^{-1} M_{\pm})_{\beta}^{\alpha} - \delta_{\beta}^{\alpha} \} \mathfrak{H} = 0, \quad (4.3)$$

where M_{\pm} are the Gauss components of M (see Section 1) defined, in the quantum case, by $q^{n-\frac{1}{n}} M = M_+ M_-^{-1}$.

The following question arises: can we construct \mathfrak{H} as a subquotient of \mathcal{H} using a kind of a BRS formalism? A constructive answer to this question has only been given for $n = 2$; we will summarize it below. We shall proceed, following the historical development, in two steps.

First, we note that the full symmetry of the $2D$ zero mode problem is given by

$$\mathcal{U}_q = U_q(sl_2)_{\Delta} \otimes U_q(sl_2)_b \quad (4.4)$$

where the diagonal (coproduct) U_q action Δ is defined in (4.3) (that is fully deciphered in Eqs. (2.2) and (2.3) of Ref. [20]) while the Chevalley generators of the second factor are

$$b = a_{\alpha}^1 \bar{a}_2^{\alpha}, \quad b' = -a_{\alpha}^2 \bar{a}_1^{\alpha} \quad \text{and} \quad q^{\pm(p-\bar{p})} \quad (4.5)$$

satisfying

$$[b, b'] = [p - \bar{p}], \quad q^{p-\bar{p}} b = q^2 b q^{p-\bar{p}}, \quad q^{p-\bar{p}} b' = \bar{q}^2 b' q^{p-\bar{p}}. \quad (4.6)$$

Proposition 4.1 ([14, 19]) *The subspace \mathcal{H}_I of \mathcal{U}_q -invariant vectors of \mathcal{H} is $2h - 1$ dimensional and is spanned by vectors of the form*

$$|\lambda + 1\rangle_I = B'^{[\lambda]} |1, 0\rangle \otimes |1, 0\rangle, \quad B'^{[\lambda]} = \sum_{\nu=m}^{\lambda-m} q^{\nu(\lambda-\nu)} B'_1^{[\nu]} B'_2^{[\lambda-\nu]} \quad (4.7)$$

where $B'_\alpha = a_\alpha^1 \bar{a}_1^\alpha$, $\alpha = 1, 2$, $m = \max(0, \lambda - h + 1)$, $0 \leq \lambda \leq 2h - 2$. The \mathcal{U}_q -invariant subalgebra of observables of \mathcal{A}_h is generated by the pair $B' = B'_1 + B'_2$, $B = a_\alpha^2 \bar{a}_2^\alpha = B_1 + B_2$ satisfying

$$B' B = [p] [p - 1], \quad B B' = [p] [p + 1] \quad (\text{on } \mathcal{H}_I); \quad (4.8)$$

the operators $B, B', q^{\pm 2p}$ give rise to a q -deformation of the $su(1, 1)$ Lie algebra:

$$[B, B'] = [2p], \quad q^{2p} B = \bar{q}^2 B q^{2p}, \quad q^{2p} B' = q^2 B' q^{2p}. \quad (4.9)$$

B and B' act on the basis (4.7) according to

$$B |p\rangle_I = [p] |p - 1\rangle_I \quad B' |p\rangle_I = [p] |p + 1\rangle_I. \quad (4.10)$$

The restriction to \mathcal{H}_I of the bilinear form on \mathcal{H} gives

$$\langle p | p' \rangle = \delta_{pp'} [p] \quad (\Rightarrow \text{sign } \langle p | p \rangle = \text{sign } (h - p)). \quad (4.11)$$

The space \mathcal{H}_I satisfies conditions (ii) and (iii) – but not (i) – of the desiderata for the physical space \mathfrak{H} listed in the beginning. It was demonstrated in [19] that the space of physical states can be obtained from \mathcal{H}_I by applying the generalized (BRS) cohomology of [39, 40].

Theorem 4.2

- (a) The conditions $B_\alpha^h = 0$ and $B_2 B_1 = q^2 B_1 B_2$ imply $B^h \equiv (B_1 + B_2)^h = 0$ in \mathcal{H} .
(b) Each of the generalized cohomologies

$$H^{(p)}(\mathcal{H}_I, B) = \text{Ker } B^p / \text{Im } B^{h-p}, \quad p = 1, \dots, h - 1 \quad (4.12)$$

is one dimensional and is given by $\{ \mathbb{C} |p\rangle, 0 < p < h \}$.

These results leave something to desire: in the above treatment the \mathcal{U}_q symmetry of

$$\mathfrak{H} = \bigoplus_{p=1}^{h-1} H^{(p)}(\mathcal{H}_I, B) \quad (4.13)$$

was not an outcome of the (generalized) BRS construction but a precondition imposed on $\mathcal{H}_I \subset \mathcal{H}$. The first question addressed (and answered in the negative) in [20] was: is there a h -differential complex (\mathcal{H}, Q) with the same generalized cohomology as (\mathcal{H}_I, B) ?

Proposition 4.3 ([20], Section 3) *If $H^{(p)}(\mathcal{H}, Q) = H^{(p)}(\mathcal{H}_I, B)$ (for $Q^h = 0 = B^h$, $0 < p < h$), then $\dim \mathcal{H} = \pm 1 \pmod{h}$.*

This result excludes the existence of a generalized BRS charge Q in \mathcal{H} with the desired properties since $\dim \mathcal{H} = h^4$ for \mathcal{H} given by (4.1).

A solution to the resulting puzzle, given in [20] (Section 4), consists in embedding \mathcal{H} in a graded vector space

$$\mathcal{H}^\bullet = \oplus_{\nu \geq 0} \mathcal{H}^\nu \text{ with } \mathcal{H}^0 = \mathcal{H}, \quad \mathcal{H}^\nu = \mathcal{H}/\mathcal{H}_I, \quad 1 \leq \nu \leq h-1, \quad \mathcal{H}^\nu = 0, \quad \nu \geq h, \quad (4.14)$$

introducing a h -differential, $d : \mathcal{H}^\nu \rightarrow \mathcal{H}^{\nu+1}$ ($d^h = 0$) and extending B to \mathcal{H}^\bullet in such a way that the nilpotent operator $Q = d + B$ defines the same cohomology in \mathcal{H}^\bullet as B in \mathcal{H}_I (Theorem 1 of [20]):

$$Q^h \equiv (d + B)^h = 0, \quad H^{(p)}(\mathcal{H}^\bullet, Q) = H^{(p)}(\mathcal{H}_I, B). \quad (4.15)$$

A "physical geometric" interpretation of this result in terms of (generalized) Hochschild cochains was also proposed in [20] (Sections 5,6).

5 Concluding remarks

The finite dimensional gauge problem extracted from the WZNW model displays a rich structure and opens the way of applying recently developed generalizations of BRS cohomology. It supports the maverick opinion that, contrary to the common beliefs, the true understanding of solvable $2D$ current algebra models still lies in the future.

The results of Section 4 only apply to the $SU(2)$ WZNW model. Their extension to higher rank groups offers an obvious suggestion for further work. More important, in our view, is the problem of making full use of the extended space, the necessity of which is indicated in Proposition 4.3. The possibility to relate it to the recently advanced operad approach to quantum field theory [41] looks tantalizing.

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