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Vienna, Preprint ESI 2232 (2010) March 2, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

$\overline{\partial}$ AND THE DIRAC OPERATOR.

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Abstract.
In the present paper, we prove an abstract functional analytic criterion for an elliptic linear partial differential operator acting on a domain in $\mathbb{R}^n$ to have compact resolvent. This is applied to the $\overline{\partial}$-Neumann problem in weighted L^2-spaces on $\mathbb{C}^n$ to obtain necessary and sufficient conditions for existence and compactness of the $\overline{\partial}$-Neumann operator for a class of weight functions that is more general than the ones considered in the literature up to now. As another application, we give some embedding Theorems for certain weighted Sobolev spaces. Moreover, we point out the relationship between the $\overline{\partial}$-Laplacian and the Dirac operator in real dimension two and prove a non-compactness result for its resolvent.

1. Introduction.

The subject of the present paper is the $\overline{\partial}$-Neumann problem in weighted L^2-spaces on $\mathbb{C}^n$. The weighted $\overline{\partial}$-Neumann operator is the inverse of the weighted complex Laplacian, see Section 2 for the precise Definitions. For background on the $\overline{\partial}$-Neumann problem, we refer the reader to [4], [10] and [7].
The weighted $\overline{\partial}$-equation is one of the fundamental tools in complex analysis, for various applications see e.g. [18]. Weighted problems also arise naturally when studying the unweighted one: In the case of complete pseudoconvex Hartogs domains it is known for instance, that the $\overline{\partial}$-Neumann problem can be reduced to a corresponding weighted problem on the base domain, see [3], [21].
For the case of weighted spaces on $\mathbb{C}^n$ it is known that studying the $\overline{\partial}$-Neumann problem is equivalent to studying certain Schrödinger operators (or Witten Laplacians in the complex higher dimensional case) on $\mathbb{R}^{2n}$, which comes from the absence of boundary conditions, see [8], [14] and the discussion in Section 2.

The class of weight functions Φ we are working with is the following. Let $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ be the Bergman space of entire functions that are square-integrable with respect to the weight φ. Define the set

$$\Phi = \{\varphi \in PSH(\mathbb{C}^n) \cap \mathcal{C}^\infty(\mathbb{C}^n) \mid \dim \mathcal{A}^2_\varphi \geq 1\}.$$

Assuming the existence of an integrable holomorphic function is reasonable from the complex analysis point of view. By a result in [11] (see also Section 2), smoothness of the weight function is no restriction. The class Φ is more general than the ones considered before: In [8], M. Christ imposes a doubling condition on $\triangle\varphi$ as well

2000 *Mathematics Subject Classification.* Primary 32W05; Secondary 35J10, 47A10, 46E35.
Key words and phrases. $\overline{\partial}$-Neumann problem, compactness, Dirac operator, Pauli operator, compact resolvent, Sobolev spaces.
Supported by the FWF-grants P19147 and P19667.

as the condition $\int_{\mathbb{B}} \triangle\varphi > \delta$ for some fixed $\delta > 0$ and all balls with radius 1. An approximation Theorem for doubling measures (Theorem 14 in [22]) combined with Proposition 4.2 shows that in this case the Bergman space will even be infinite dimensional. Confer also Propostion 1.10 in [8]. Similar arguments show the same for the classes of weights in [14], [23] and [11].

2. Preliminaries.

Let $\mathbb{C}^n$ be the n-dimensional complex Euclidian space and for $z \in \mathbb{C}^n$ denote the coordinates by $z = (z_1, \dots, z_n)$. We will often use the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ without pointing it out in particular, denoting the coordinates in $\mathbb{R}^{2n}$ by $(x_1, y_1, \dots, x_n, y_n)$. Let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic and smooth weight function. Define the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \to \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-2\varphi} \, d\mu < \infty\},$$

where μ is the Lebesgue measure. By Lemma 2.3 in [11], assuming smoothness of the weight function is no loss of generality when considering the $\overline{\partial}$-Neumann problem. Similarly define $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$, the space of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$, the space of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. The weighted inner product in $L^2(\mathbb{C}^n, \varphi)$ is defined to be

$$\langle f, g \rangle_\varphi = \int_{\mathbb{C}^n} f \, \overline{g} \, e^{-2\varphi} \, d\mu$$

and the norm $\|f\|^2_\varphi = \langle f, f \rangle_\varphi$. Let $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ be the Bergman space of entire functions belonging to $L^2(\mathbb{C}^n, \varphi)$. The $\overline{\partial}$-operator on $(0,q)$-forms with coefficients in $\mathcal{C}^\infty_0(\mathbb{C}^n)$, i.e., the space of smooth functions with compact support, is given by

$$\overline{\partial}\left(\sum_J{}' a_J \, d\overline{z}_J\right) = \sum_{j=1}^n \sum_J{}' \frac{\partial a_J}{\partial \overline{z}_j} d\overline{z}_j \wedge d\overline{z}_J,$$

where $\sum'$ means that the sum is only taken over strictly increasing multi-indices J. We shall work with the maximal closure of this operator, which we again denote by $\overline{\partial}$. Consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n, \varphi),$$

where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

where the symbol $\Box_\varphi$ is again to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}^\infty_0$. The weighted $\overline{\partial}$-Neumann operator N_φ is – if it exists – the bounded inverse of $\Box_\varphi$.

In the case of weighted spaces over $\mathbb{C}^n$ there is a connection to spectral theory, see [15], [14] and also [8]. Let $\overline{D}_q := e^{-\varphi}\overline{\partial}_q e^\varphi$, where we added a subscript to indicate the form-level on which the operator is acting. Then $\overline{D}_q$ is a closed densely defined operator on $L^2(\mathbb{C}^n)$ and we denote its L^2-adjoint by $\overline{D}_q^*$. The $\overline{D}$-Laplacians $\Box_\varphi^{(0,0)}$ and $\Box_\varphi^{(0,1)}$ are defined to be $\Box_\varphi^{(0,0)} = \overline{D}_1^*\overline{D}_1$ and $\Box_\varphi^{(0,1)} = \overline{D}_1\overline{D}_1^* + \overline{D}_2^*\overline{D}_2$, respectively. If one denotes by

$$M_\varphi = \left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k}\right)_{jk}$$

the complex Hessian of φ and defines its action on (0,1)-forms $g = \sum_{j=1}^n g_j d\overline{z}_j$ to be

$$M_\varphi g = \sum_{j,k=1}^n \frac{\partial^2\varphi}{\partial z_j \partial \overline{z}_k} g_j d\overline{z}_k,$$

one can write the expression for $\Box_\varphi^{(0,1)}$ in the more elegant way

$$\Box_\varphi^{(0,1)} = \Box_\varphi^{(0,0)} \otimes Id + 2M_\varphi, \tag{2.1}$$

see [14] for the computation. This kind of operator is called a Witten Laplacian. Defining the magnetic Schrödinger operator $\triangle_\varphi$ acting on $L^2(\mathbb{R}^{2n})$

$$\triangle_\varphi = -\sum_{j=1}^n \left(\left(\frac{\partial}{\partial x_j} + i\frac{\partial\varphi}{\partial y_j}\right)^2 + \left(\frac{\partial}{\partial y_j} - i\frac{\partial\varphi}{\partial x_j}\right)^2\right), \tag{2.2}$$

we also notice that

$$4\Box_\varphi^{(0,0)} = \triangle_\varphi - \triangle\varphi, \tag{2.3}$$

where $\triangle$ means the ordinary "negative" Laplacian, $\triangle = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2}$. A Pauli operator is a special kind of a Schrödinger operator – for Pauli operators, the electric potential and the magnetic field coincide up to the sign. The precise Definition in real dimension two is

$$P_\pm = -\left(\frac{\partial}{\partial x} - iA_1(x,y)\right)^2 - \left(\frac{\partial}{\partial y} - iA_2(x,y)\right)^2 \pm B(x,y), \tag{2.4}$$

where

$$B(x,y) = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

By $P_\pm^{(j)}$ we mean the Pauli operator obtained by taking $A_1 = -\varphi_{y_j}$ and $A_2 = \varphi_{x_j}$. Consequently, we have $4\Box_\varphi^{(0,0)} = \sum_{j=1}^n P_-^{(j)}$ and in complex dimension one, $4\Box_\varphi^{(0,1)} = P_+$. Note that with this Definition, the magnetic potentials A_j in general depend on all variables z_k, $1 \le k \le n$.

Let $\mathfrak{L}$ be the space of bounded linear operators on $L^2_\varphi(\mathbb{C}^n)$. For a closed, densely defined operator A on L^2_φ we define the resolvent set of A to be

$$\rho(A) := \{\lambda \in \mathbb{C} : (A-\lambda)^{-1} \in \mathfrak{L}\}.$$

To be more precise, $\lambda \in \rho(A)$ if and only if $(A-\lambda) : dom(A) \to L^2_\varphi$ is bijective and its inverse is bounded. By the Closed Graph Theorem, it suffices to check that $A-\lambda$ is bijective. The complement of the resolvent set is the spectrum

$$\sigma(A) = \mathbb{C} \setminus \rho(A) \tag{2.5}$$

of A. The essential spectrum of A, $\sigma_{ess}(A)$, is the set of all $\lambda \in \sigma(A)$ such that $A-\lambda$ is not a Fredholm operator. The discrete spectrum $\sigma_d(A)$ of A is the complement

$\sigma_d(A) = \sigma(A) \setminus \sigma_{ess}(A)$. Note that σ_d is not necessarily a closed set. With these Definitions, a point in the discrete spectrum corresponds to an isolated eigenvalue with finite multiplicity. A point in the essential spectrum is either an eigenvalue of infinite multiplicity, an accumulation point of eigenvalues or a point in the interior of $\sigma(A)$, the so-called continuous spectrum. To give an example, for a compact operator A it always holds $0 \in \sigma_{ess}(A)$.
The function

$$R_A : \rho(A) \to \mathfrak{L},\ \lambda \mapsto (A - \lambda)^{-1}$$

is the resolvent of A. We say that the operator A has compact resolvent, if there is some $\lambda \in \rho(A)$ such that the resolvent $R_A(\lambda)$ is a compact operator. By the so-called first resolvent identity

$$R_A(\lambda) - R_A(\lambda') = (\lambda - \lambda')R_A(\lambda)R_A(\lambda')$$

for all $\lambda, \lambda' \in \rho(A)$, it follows that in this case the resolvent $R_A(\lambda)$ is compact for any $\lambda \in \rho(A)$. This is also equivalent to the statements $\sigma_{ess}(A) = \emptyset$ or $\sigma(A) = \sigma_d(A)$. The interested reader can find more details in any introductory book on spectral theory, see for instance [27] or [16].
It is essentially contained in [14], that N_φ is a compact operator if and only if $\Box_\varphi^{(0,1)}$ has compact resolvent, confer also [15]. Similarly it can be shown that N_φ is bounded if and only if $\Box_\varphi^{(0,1)}$ is strictly positive, i.e., there is $\varepsilon > 0$ such that $\langle \Box_\varphi^{(0,1)} u, u\rangle \geq \varepsilon \langle u, u\rangle$ for all $u \in dom(\Box_\varphi^{(0,1)})$.

3. Some functional analysis.

Let us start by taking a closer look at our operators.

Lemma 3.1. *The kernel of the Schrödinger operator* $4\Box_\varphi^{(0,0)} = \triangle_\varphi - \triangle\varphi$ *consists exactly of those* $L^2(\mathbb{C}^n)$*-functions of the form* $u = fe^{-\varphi}$, *where* f *is holomorphic.*

Proof. We have $4\Box_\varphi^{(0,0)} = 4\overline{D}_1^*\overline{D}_1$. So if $\Box_\varphi^{(0,0)} f = 0$ for some $f \in dom(\Box_\varphi^{(0,0)})$, then $\overline{D}_1^*\overline{D}_1 f = 0$, hence also $\|\overline{D}_1 f\|^2 = 0$ and the kernels of $\Box_\varphi^{(0,0)}$ and $\overline{D}_1$ coincide. But since $\overline{D}_1 = e^{-\varphi}\overline{\partial}_1 e^{\varphi}$, the statement is obviously true for $\overline{D}_1$, so the Lemma follows.

□

Lemma 3.2. *Let* $u = fe^{-\varphi} \in L^2(\mathbb{C}^n)$. *Then*

$$4\Box_\varphi^{(0,0)} u = -\sum_{j=1}^{n} \left[\left(f_{x_j x_j} + f_{y_j y_j}\right) - 2(\varphi_{x_j} - i\varphi_{y_j}) \left(f_{x_j} + i f_{y_j}\right)\right] e^{-\varphi}.$$

Proof. The proof is a straight forward computation:

$$\begin{aligned}\triangle_\varphi(fe^{-\varphi}) &= -\sum_{j=1}^n\left[\left(\frac{\partial}{\partial x_j}+i\varphi_{y_j}\right)^2+\left(\frac{\partial}{\partial y_j}-i\varphi_{x_j}\right)^2\right](fe^{-\varphi})\\ &= -\sum_{j=1}^n\left(\frac{\partial}{\partial x_j}+i\varphi_{y_j}\right)\left(f_{x_j}e^{-\varphi}+f\varphi_{x_j}e^{-\varphi}+if\varphi_{y_j}e^{-\varphi}\right)\\ &\quad -\sum_{j=1}^n\left(\frac{\partial}{\partial y_j}-i\varphi_{x_j}\right)\left(f_{y_j}e^{-\varphi}-f\varphi_{y_j}e^{-\varphi}-if\varphi_{x_j}e^{-\varphi}\right)\\ &= -\sum_{j=1}^n\left(f_{x_jx_j}+f_{y_jy_j}\right)e^{-\varphi}-2(\varphi_{x_j}-i\varphi_{y_j})\left(f_{x_j}+if_{y_j}\right)e^{-\varphi}+\triangle\varphi fe^{-\varphi}\end{aligned}$$

□

Lemma 3.3. *Let $u = fe^{-\varphi} \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Then*

$$\langle\Box_\varphi^{(0,0)}u,u\rangle=\sum_{j=1}^n\int_{\mathbb{C}^n}\left|\frac{\partial f}{\partial\overline{z}_j}\right|^2e^{-2\varphi}d\mu$$

Proof. It obviously suffices to do the calculation in one variable. By Lemma 3.2, we have

$$\begin{aligned}4\langle\Box_\varphi^{(0,0)}u,u\rangle &= \int_{\mathbb{C}^n}(-\triangle f)\overline{f}e^{-2\varphi}d\mu+8\int_{\mathbb{C}^n}\overline{f}\frac{\partial f}{\partial\overline{z}}\frac{\partial\varphi}{\partial z}e^{-2\varphi}d\mu\\ &= \int_{\mathbb{C}^n}(-\triangle f)\overline{f}e^{-2\varphi}d\mu-4\int_{\mathbb{C}^n}\overline{f}\frac{\partial f}{\partial\overline{z}}\frac{\partial}{\partial z}e^{-2\varphi}d\mu.\end{aligned}$$

Since f has compact support, we can integrate the second term by parts and the Lemma follows from $4\frac{\partial^2}{\partial z\partial\overline{z}} = \triangle$.

□

The following Lemma gives a functional analytic characterization of precompact sets in weighted Lebesgue spaces.

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^n$ and $\mathcal{A}$ be a bounded subset of $L^2(\Omega,\varphi)$. Then $\mathcal{A}$ is precompact if and only if the following two conditions hold:*

(1) for all $\varepsilon > 0$ and all $\Omega' \subset\subset \Omega$ there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_{L^2(\Omega',\varphi)} < \varepsilon$$

for each $h \in \mathbb{R}^n$ with $|h| < \delta$ and all $f \in \mathcal{A}$, where $\tau_h f(x) = f(x+h)$.

(2) for all $\varepsilon > 0$ there exists $\Omega_\varepsilon \subset\subset \Omega$ such that

$$\|f\|_{L^2(\Omega\setminus\Omega_\varepsilon,\varphi)} < \varepsilon$$

for each $f \in \mathcal{A}$.

For the proof we refer to [1], Theorem 2.32. See also [6], Corollaire IV.26.
The next Proposition is a reformulation of Lemma 3.4 and will be the basic tool in proofing our results. A variant of it fitted to the $\overline{\partial}$-Neumann problem already appeared in [13]. One should also compare the Main Theorem in [19].

Proposition 3.5. *Let T be a linear partial differental operator acting on dom(T), which is closed, densely defined and elliptic in the interior of Ω. Let T^*_φ be its adjoint in $L^2(\Omega,\varphi)$ and set $P = T^*_\varphi T$.*
Then the follwing are equivalent:

(1) *P has compact resolvent.*
(2) *The injection j_φ of dom(T) equipped with the graph norm $u \mapsto \|Tu\|_\varphi$ into $L^2(\Omega,\varphi)$ is compact.*
(3) *Let $\mathcal{L}$ be the unit ball in dom(T) with the graph norm. Then for all $\varepsilon > 0$ there is $\Omega_\varepsilon \subset\subset \Omega$ such that $\|u\|_{L^2(\Omega\setminus\Omega_\varepsilon,\varphi)} \leq \varepsilon$ for all $u \in \mathcal{L}$.*
(4) *There is a smooth function λ, such that $\lambda \to \infty$ for $z \to \partial\Omega$ and*

$$\langle Pu, u\rangle_\varphi \geq \int_\Omega \lambda |u|^2 e^{-2\varphi} d\mu$$

for all $u \in$ dom(P).

Proof. Let P^{-1} be the inverse of P and let j_φ be the injection of *dom*(T) into $L^2(\Omega,\varphi)$. We first show that $P^{-1} = j_\varphi \circ j^*_\varphi$. For all $u, v \in$ *dom*(P) it holds

$$\langle u, v\rangle_\varphi = \langle u, j_\varphi v\rangle_\varphi = \langle j^*_\varphi u, v\rangle_{Q_T},$$

and on the other hand

$$\langle u, v\rangle_\varphi = \langle PP^{-1}u, v\rangle_\varphi = \langle TP^{-1}u, Tv\rangle_\varphi = \langle P^{-1}u, v\rangle_{Q_T}.$$

Hence, $P^{-1} = j^*_\varphi$ as an operator to *dom*(T) and consequently $P^{-1} = j_\varphi \circ j^*_\varphi$ as an operator to $L^2(\Omega,\varphi)$. This proves the equivalence of (1) and (2).

Now we show (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (2). Suppose that the injection j_φ is compact. Hence $\mathcal{L}$ is precompact in $L^2(\Omega,\varphi)$, thus (2) $\Longrightarrow$ (3) by Lemma 3.4.
If (3) holds, then by linearity of T for all $\varepsilon > 0$ there is $\Omega_\varepsilon \subset\subset \Omega$, such that $\|u\|_{L^2(\Omega\setminus\Omega_\varepsilon)} \leq \varepsilon\|u\|_T$ for all $u \in$ *dom*(T). Thus for all $u \in$ *dom*(T):

$$\begin{aligned}\int_\Omega |u|^2 e^{-2\varphi} d\mu \leq &\int_{\Omega\setminus\Omega_{\frac14}} 1 \cdot |u|^2 e^{-2\varphi} d\mu + \int_{\Omega_{\frac14}\setminus\Omega_{\frac18}} 2 \cdot |u|^2 e^{-2\varphi} d\mu + \int_{\Omega_{\frac18}\setminus\Omega_{\frac1{16}}} 4 \cdot |u|^2 e^{-2\varphi} d\mu + \dots \\ \leq &\ 2\ \|u\|_T^2.\end{aligned}$$

Hence it is clear that one can find a smooth function λ tending to infinity at the boundary of Ω such that

$$\langle Pu, u\rangle_\varphi = \|u\|_T^2 \geq \int_\Omega \lambda |u|^2 e^{-2\varphi} d\mu$$

for all $u \in$ *dom*(P).
Finally suppose that (4) holds and let $\varepsilon > 0$ be given. Choose M such that $1/M \leq \varepsilon$ and $\Omega_M \subset\subset \Omega$ such that $\lambda \geq M$ on $\Omega \setminus \Omega_M$. Since by positivity of P we can without loss of generality assume $\lambda \geq 0$ on Ω, we have for all $u \in$ *dom*(P)

$$\begin{aligned}\|u\|_\varphi^2 \leq &\int_{\Omega_M} |u|^2 e^{-2\varphi} d\mu + \int_{\Omega\setminus\Omega_M} \frac{\lambda}{M} |u|^2 e^{-2\varphi} d\mu \\ \leq &\|u\|^2_{L^2(\mathbb{B}_R,\varphi)} + \varepsilon\|u\|_T^2,\end{aligned}$$

which implies compactness of the embedding since *dom*(P) is dense in *dom*(T) and $\|.\|^2_{L^2(\mathbb{B}_R,\varphi)}$ is strictly weaker than $\|.\|_T^2$ by Gårding's inequality combined with the

Rellich – Kondrachov Theorem.

□

Remark. This generalizes the Main Theorem in [19], where the same result was proven for Schrödinger operators with electric potentials that are semibounded from below.
If $\mathcal{C}_0^\infty(\Omega)$ is a core in the form domain, one can push the analogy to [19] even further by also adding the bottom of the spectrum of the Dirichlet realization to the picture. This is, P has compact resolvent if and only if the lowest eigenvalue of the Dirichlet realization of P in $\Omega \setminus K_j$ tends to infinity for $j \to \infty$, for any sequence of compact sets K_j exhausting Ω. Note that if $\Omega = \mathbb{R}^n$, then $\mathcal{C}_0^\infty$ is always a core.

Remark. The Proposition holds in particular for $T = \overline{\partial} \oplus \overline{\partial}_\varphi^*$ and $P = T_\varphi^* T = \overline{\partial}\overline{\partial}_\varphi^* + \overline{\partial}_\varphi^* \overline{\partial}$. In this case, $dom(T) = dom(\overline{\partial}) \cap dom(\overline{\partial}_\varphi^*)$.

4. On the dimension of the space $\mathcal{A}_\varphi^2(\mathbb{C}^n)$

In this Section we present a result due to I. Shigekawa ([25], Lemma 3.4) which will be useful to determine wether a weight function belongs to the class Φ. We can give a simplified version of the proof, avoiding arguments involving Kähler manifolds and even yielding a slightly sharper result by also making a density statement, see [24]. First we prepare the following Lemma.

Lemma 4.1. *Let $g(z) = \log(1 + K|z-\xi|^2)$ and $K > 0$. Then for all $w \in \mathbb{C}^n$ it holds*

$$\frac{K}{(1+K|z-\xi|^2)^2}|w|^2 \leq \sum_{j,k=1}^{n} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \leq \frac{K}{1+K|z-\xi|^2}|w|^2$$

Proof. Without loss of generality we can assume $\xi = 0$. Differentiating, we find

$$\begin{aligned} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k}(z) &= -\frac{K^2 \overline{z}_j z_k}{(1+|z|^2)^2} + \frac{K\delta_{jk}}{1+K|z|^2} \\ &= \frac{K}{(1+K|z|^2)^2}\big((1+K|z|^2)\delta_{jk} - K\overline{z}_j z_k\big) \end{aligned}$$

and hence

$$\sum_{j,k} \frac{\partial^2 g}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k = \frac{K}{(1+K|z|^2)^2}\big((1+K|z|^2)|w|^2 - K|\langle w, z\rangle|^2\big).$$

This last line makes the statement of the Lemma immediate.

□

Proposition 4.2. *Let φ be a weight function of class $\mathcal{C}^2$ on $\mathbb{C}^n$ and denote by $\lambda_1(z)$ the lowest eigenvalue of the complex Hessian of φ. Suppose that it holds*

$$\lim_{|z|\to\infty} |z|^2 \lambda_1(z) = \infty. \tag{4.1}$$

Then the weighted Bergman space $\mathcal{A}^2_\varphi(\mathbb{C}^n)$ is dense in $\mathcal{H}(\mathbb{C}^n)$, the space of entire functions, in the topology of uniform convergence on compact sets. In particular $\dim \mathcal{A}^2_\varphi = \infty$.

Proof. By assumption on $\lambda_1(z)$, there exist $R > 0$ and $s > -2$ such that $\lambda_1(z) \geq |z|^s$ for all $|z| \geq R$. So we can find $\varepsilon > 0$ such that $\lambda_1(z) - \frac{1}{|z|^{2-\varepsilon}} > 0$ on the complement of a compact set. According to Lemma 4.1

$$\sum_{j,k=1}^n \frac{\partial^2(\log(1+|z|^2))}{\partial z_j \partial \overline{z}_k} w_j \overline{w}_k \geq \frac{1}{(1+|z|^2)^2}|w|^2,$$

hence we can choose an integer N large enough for

$$\Psi(z) = \varphi(z) - \frac{1}{\varepsilon^2}|z|^\varepsilon + N\log(1+|z|^2)$$

to be a plurisubharmonic function. By Theorem 4.4.4. in [18], the space

$$\mathcal{A} = \{f : f \in \mathcal{H}(\mathbb{C}^n) \text{ and } \exists\ K \text{ s.t. } \int_{\mathbb{C}^n} |f|^2(1+|z|^2)^{-K} e^{-\Psi(z)} d\mu < \infty\}$$

is dense in $\mathcal{H}(\mathbb{C}^n)$. Hence it suffices to show $\mathcal{A} \subseteq \mathcal{A}^2_\varphi$. To this end choose any $f \in \mathcal{A}$ and note that

$$\begin{aligned}
\int_{\mathbb{C}^n} |f|^2 e^{-\varphi(z)} d\lambda &= \int_{\mathbb{C}^n} |f|^2 e^{-\Psi - \frac{1}{\varepsilon^2}|z|^\varepsilon + N\log(1+|z|^2)} d\mu \\
&= \int_{\mathbb{C}^n} |f|^2 (1+|z|^2)^{-K} e^{-\Psi} (1+|z|^2)^{K+N} e^{-|z|^\varepsilon/\varepsilon^2} d\mu \\
&\leq \sup_{z\in\mathbb{C}^n} \{(1+|z|^2)^{K+N} e^{-|z|^\varepsilon/\varepsilon^2}\} \int_{\mathbb{C}^n} |f|^2(1+|z|^2)^{-K} e^{-\Psi} d\mu \\
&< \infty,
\end{aligned}$$

since $(1+|z|^2)^{K+N} e^{-|z|^\varepsilon/\varepsilon^2}$ is a bounded function.

□

Remark. Note that we do not assume plurisubharmonicity of the weight function in Proposition 4.2.

5. Compactness.

Theorem 5.1. *Suppose that $4\square^{(0,0)}_\varphi = \triangle_\varphi - \triangle\varphi$ acting on $L^2(\mathbb{C}^n)$ has compact resolvent. Then $\mathcal{A}^2_\varphi = \{0\}$. In particular, $\square^{(0,0)}_\varphi$ has non-compact resolvent for all $\varphi \in \Phi$.*

Proof. Combining Proposition 3.5 with Lemma 3.3, compactness of the resolvent of $\square^{(0,0)}_\varphi$ implies

$$4\sum_{j=1}^n \int_{\mathbb{C}^n} \left|\frac{\partial f}{\partial \overline{z}_j}\right|^2 e^{-2\varphi} d\mu \geq \int_{\mathbb{C}^n} \lambda |f|^2 e^{-2\varphi} d\mu$$

for some function λ with $\lambda \to \infty$ as $|z| \to \infty$ and all $f \in \mathcal{C}_0^\infty(\mathbb{C}^n)$. Without loss of generality we can assume that $\lambda \geq \varepsilon > 0$, since the resolvent of $\Box_\varphi^{(0,0)}$ is in particular bounded, which is equivalent to $\langle \Box_\varphi^{(0,0)} f, f\rangle \geq \varepsilon \|f\|^2$ for all $f \in dom(\Box_\varphi^{(0,0)})$.
Let $\{\chi_R\}_{R\in\mathbb{N}}$ be a family of smooth cut-off functions which are identically one on $\mathbb{B}_R$, supported in $\mathbb{B}_{R+1}$ and have uniformly bounded first order derivatives. In fact, we can assume that $\sup |\nabla\chi_R(z)| \leq 2$. Now let $h \not\equiv 0$ be an entire function. Then $\chi_R h \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ and by assumption we have

$$4 \int\limits_{supp(\nabla\chi_R)} |h|^2 \sum_{j=1}^n \left|\frac{\partial \chi_R}{\partial \overline{z}_j}\right|^2 e^{-2\varphi} d\mu \geq \int\limits_{supp(\chi_R)} \lambda |\chi_R h|^2 e^{-2\varphi} d\mu,$$

since h is holomorphic. Using the assumption on the derivatives of χ_R, we have

$$8 \int\limits_{\mathbb{B}_{R+1}\setminus\mathbb{B}_R} |h|^2 e^{-2\varphi} d\mu \geq \int\limits_{\mathbb{B}_R} \lambda |\chi_R h|^2 e^{-2\varphi} d\mu \geq \varepsilon \int\limits_{\mathbb{B}_1} |h|^2 e^{-2\varphi} d\mu > \delta > 0,$$

yielding that

$$\|h\|_\varphi^2 = \sum_{R\in\mathbb{N}} \int\limits_{\mathbb{B}_{R+1}\setminus\mathbb{B}_R} |h|^2 e^{-2\varphi} d\mu$$

can not be finite.

□

Theorem 5.2. *Let $\varphi \in \Phi$. Then $\triangle_\varphi$ acting on $L^2(\mathbb{C}^n)$ has compact resolvent if and only if $\triangle\varphi \to \infty$.*

Proof. Since $\triangle_\varphi \geq \triangle\varphi$ in the sense of self-adjoint operators, $\triangle\varphi \to \infty$ is sufficient for compactness of the resolvent by Proposition 3.5 or general well-known facts about Schrödinger operators, see for instance [19].
Suppose conversely that $\triangle_\varphi$ has compact resolvent. By Lemma 3.3 and Proposition 3.5, there is a function λ such that $\lambda \to \infty$ for $|z| \to \infty$ and

$$\langle \triangle_\varphi u, u\rangle_\varphi = 4\sum_{j=1}^n \int_{\mathbb{C}^n} \left|\frac{\partial u}{\partial \overline{z}_j}\right|^2 e^{-2\varphi} d\mu + \int_{\mathbb{C}^n} \triangle\varphi |u|^2 e^{-2\varphi} d\mu \geq \int_{\mathbb{C}^n} \lambda |u|^2 e^{-2\varphi} d\mu$$

for all $u \in \mathcal{C}_0^\infty$. Without loss of generality we can assume λ to be the best possible of all such functions, i.e., $\lambda(z) = \sup \Lambda(z)$, where the supremum is taken over all smooth functions Λ such that $\triangle_\varphi \geq \Lambda$. Thus, λ is at least measurable and since $\triangle_\varphi \geq \triangle\varphi$, we have $\lambda \geq \triangle\varphi$ a.e. in particular.
Suppose now that there is a set E of positive measure such that $\lambda > \triangle\varphi$ a.e. on E. This set must contain an open set U and by possibly shrinking U we can assume that there is $\varepsilon > 0$ such that $\lambda - \triangle\varphi \geq \varepsilon$ a.e. on U. Then

$$4\sum_{j=1}^n \int_{\mathbb{C}^n} \left|\frac{\partial u}{\partial \overline{z}_j}\right|^2 e^{-2\varphi} d\mu \geq \int_{\mathbb{C}^n} (\lambda - \triangle\varphi)|u|^2 e^{-2\varphi} d\mu \geq \varepsilon \int_U |u|^2 e^{-2\varphi} d\mu$$

and similarly to the proof of Theorem **??**, we deduce that $\dim \mathcal{A}_\varphi^2 = 0$ which contradicts our assumption $\varphi \in \Phi$. Hence $\triangle\varphi = \lambda$ a.e. and in particular $\triangle\varphi \to \infty$ for $|z| \to \infty$.

□

Remark. Note that our arguments heavily rely on the quite special form of our operator. For general Schrödinger operators, the connection between the magnetic field and compactness of the resolvent is much more involved. Cf. the examples in [19] and see in particular Theorem 1.2 in [20].
Note that it is also crucial that there is no boundary in our situation to be able to choose the gradients of the cut-off funtions uniformly bounded.

Remark. In complex dimension one, the Theorem states that if $\varphi \in \Phi$, N_φ is compact on $L^2(\mathbb{C}, \varphi)$ if and only if $\triangle\varphi \to \infty$. This should be compared with Theorem 1.3 of [23], where it is proved that if $\triangle\varphi$ defines a doubling measure, compactness of N_φ is equivalent to

$$\rho_\varphi(z)^{-1} = \int_{\mathbb{B}(z,1)} \triangle\varphi \, d\mu \to \infty$$

for $|z| \to \infty$. In that case, φ can be regularized, see Theorem 14 in [22]. If $\triangle\varphi$ is doubling, then there is a smooth function ψ such that $|\varphi - \psi| \le C$ and $\triangle\psi \sim \frac{1}{\rho_\psi^2} \sim \frac{1}{\rho_\varphi^2}$. We have $\psi \in \Phi$ by Proposition 4.2. So in case that $\triangle\varphi$ is doubling and $\rho_\varphi^{-1} \not\sim \triangle\varphi$, then $\mathcal{A}^2_\varphi$ can not contain a non-trivial entire function.

Corollary 5.3. *Suppose that $n = 1$, $\varphi \in \Phi$ and that N_φ is a bounded operator on $L^2(\mathbb{C}, \varphi)$. Then $\dim \mathcal{A}^2_\varphi = \infty$.*

Proof. If N_φ is bounded, then there is $\varepsilon > 0$ such that $\langle \triangle_\varphi f, f\rangle \ge \varepsilon \|f\|^2$ for all $f \in dom(\triangle_\varphi)$. By the same arguments as in the proof of Theorem 5.2, this implies that $\triangle\varphi \ge \varepsilon$. Thus it remains to use Proposition 4.2.

□

Corollary 5.4. *Suppose that $\varphi \in \Phi$ and suppose that the $\overline{\partial}$-Neumann operator N_φ is compact on $L^2(\mathbb{C}^n, \varphi)$. Then $\triangle\varphi \to \infty$ for $|z| \to \infty$.*

Proof. Since φ is plurisubharmonic, all eigenvalues of the complex Hessian are positive. Thus we can estimate M_φ in (2.1) by its trace:

$$\Box_\varphi^{(0,1)} \le \left(\Box_\varphi^{(0,0)} + \frac{1}{2}\triangle\varphi \right) \otimes Id \le 2\triangle_\varphi \otimes Id.$$

If N_φ is compact, $\Box_\varphi^{(0,1)}$ has compact resolvent, hence also $\triangle_\varphi$. Thus the Corollary follows from Theorem 5.2.

□

Remark. For plurisubharmonic weight functions, $\triangle\varphi$ is of the same order as the largest eigenvalue λ_n of the complex Hessian. So Corollary 5.4 gives a necessary condition for compactness of N_φ on λ_n, which should be compared with sufficent conditions on the lowest eigenvalue λ_1, see [14], [11] and [12]. From there it is known, that the $\overline{\partial}$-Neumann operator N_φ is compact if $\lambda_1 \to \infty$ for $|z| \to \infty$. This now follows also easily from Proposition 3.5, since $\Box_\varphi = (\overline{\partial} + \overline{\partial}^*_\varphi)(\overline{\partial} + \overline{\partial}^*_\varphi)^*$ and $\Box_\varphi \ge M_\varphi \ge \lambda_1$ in the sense of selfadjoint operators.

Remark. Let $\lambda_1, \ldots, \lambda_n$ denote the eigenvalues of the complex Hessian of φ ordered increasingly. Suppose there is a smooth form $f = \sum f_k d\overline{z}_k$ such that $M_\varphi f = \lambda_j f$ and suppose that $f_k \in \mathcal{A}^2_\varphi$ are holomorphic. Then, if N_φ is compact, necessarily $\lambda_j \to \infty$ for $|z| \to \infty$. This can be proven similarly to Theorem 5.2.

6. Weighted Sobolev spaces.

As another application of Proposition 5.2, we revisit in this Section notions of weighted Sobolev spaces, which appeared in [5] and were used in [11] to show compactness results for the $\overline{\partial}$-Neumann operator. Confer also [12], Section 4.

Definition 6.1. *Denote the coordinates in $\mathbb{R}^n$ by $(x_1, \ldots, x_n)$. Let*

$$H^1_\varphi(\mathbb{R}^n) := \{f \in L^2(\mathbb{R}^n, \varphi) \ : \ \frac{\partial f}{\partial x_j} \in L^2(\mathbb{C}^n, \varphi) \text{ for } 1 \leq j \leq n\},$$

with the norm

$$\|f\|^2_{1,\varphi} = \|f\|^2_\varphi + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|^2_\varphi .$$

The main result of this Section is the following.

Theorem 6.2. *The injection $H^1_\varphi(\mathbb{R}^n) \hookrightarrow L^2(\mathbb{R}^n, \varphi)$ is never compact.*

Proof. By Proposition 5.2, this injection is compact if and only if there is a function λ such that $\lambda \to \infty$ for $|x| \to \infty$ and

$$\sum_{j=1}^n \int_{\mathbb{R}^n} \left| \frac{\partial f}{\partial x_j} \right|^2 d\mu \geq \int_{\mathbb{R}^n} \lambda |f|^2 d\mu$$

for all $f \in \mathcal{C}^\infty_0(\mathbb{R}^n)$. Similarly to the proof of Theorem 5.1, this forces $1 \notin L^2_\varphi(\mathbb{R}^n)$. But on the other hand it was shown in [2], Theorem 3.3, that $1 \in L^2_\varphi(\mathbb{R}^n)$ is a necessary condition for compactness of this embedding.

□

Definition 6.3. *For $j = 1, \ldots, n$ let*

$$X_j = \frac{\partial}{\partial x_j} - 2 \frac{\partial \varphi}{\partial x_j}$$

and define

$$H^1(\mathbb{R}^n, \varphi, \nabla\varphi) = \{f \in L^2(\mathbb{R}^n, \varphi) \ : X_j f \in L^2(\mathbb{R}^n, \varphi) \text{ for } 1 \leq j \leq n\},$$

with norm

$$\|f\|^2_{\varphi, \nabla\varphi} = \|f\|^2_\varphi + \sum_{j=1}^n \|X_j f\|^2_\varphi .$$

Let moreover $H^1_0(\mathbb{R}^n, \varphi, \nabla\varphi)$ be the closure of $\mathcal{C}^\infty_0(\mathbb{R}^n)$ under the norm defined above.

Theorem 6.4. *Suppose that the weighted $\overline{\partial}$-Neumann operator N_φ is compact on $L^2(\Omega, \varphi)$. Then $H^1_0(\Omega, \varphi, \nabla\varphi)$ is compactly embedded in $L^2(\Omega, \varphi)$.*

Proof. Let us first consider the complex one-dimensional case. Then, compactness of N_φ implies compactness of the resolvent of $\overline{DD}^*$. Thus by Proposition 3.5, there is a function λ such that $\lambda \to \infty$ for $z \to \partial\Omega$ and

$$\int_\Omega \lambda |f|^2 d\mu \leq \int_\Omega |\overline{D}^* f|^2 d\mu = \int_\Omega \left| \frac{\partial f}{\partial z} - \frac{\partial \varphi}{\partial z} f \right|^2 d\mu$$

for all $f \in \mathcal{C}_0^\infty$. Hence, by triangle inequality, also

$$\int_\Omega \lambda |f|^2 d\mu \leq \int_\Omega (|X_1 f|^2 + |X_2 f|^2) d\mu,$$

which shows compactness of the embedding $H_0^1(\Omega, \varphi, \nabla\varphi) \hookrightarrow L^2(\Omega, \varphi)$ for $\Omega \subset \mathbb{C}$. In higher dimension, compactness of N_φ on the level of $(0,1)$-forms implies by standard arguments compactness of the $\overline{\partial}$-Neumann operator on $(0,n)$-forms. There, one easily checks in complete analogy to the one-dimensional case that compactness implies

$$\int_\Omega \lambda |f|^2 d\mu \leq \int_\Omega \sum_{j=1}^n |\overline{D}_j^* f|^2 d\mu,$$

where $\overline{D}_j = e^{-\varphi} \frac{\partial}{\partial z_j} e^\varphi$. So the conclusion follows as before.

□

Remark. For any domain $\Omega \subset \mathbb{R}^n$, one can characterize compactness of the injection $W_0^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$ in terms of capacity, see Theorem 6.19 in [1]. Setting $\varphi \equiv 0$ and combining this with the Theorem, one gets necessary conditions on Ω for compactness of the unweighted $\overline{\partial}$-Neumann operator on unbounded domains.

7. Decoupled weight functions.

Proposition 7.1. *Let $n \geq 2$ and let $\varphi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic decoupled weight function, i.e., of the form*

$$\varphi(z) = \sum_{j=1}^n \varphi_j(z_j).$$

Suppose that $\varphi_j \in \Phi$ for some $j \in \{1, \ldots, n\}$. Then the $\overline{\partial}$-Neumann operator acting on $L^2(\mathbb{C}^n, \varphi)$ is not compact.

Proof. It was shown in [14], Section 6, that for decoupled weights compactness of the canonical solution operator to $\overline{\partial}$ implies compactness of the resolvents of the Pauli operators

$$P_\pm^{(l)} = -\left(\frac{\partial}{\partial x_l} - i\frac{\partial \varphi_l}{\partial y_l}\right)^2 - \left(\frac{\partial}{\partial y_l} + i\frac{\partial \varphi_l}{\partial x_l}\right)^2 \pm \triangle \varphi_l(x_l, y_l)$$

for all $1 \leq l \leq n$. The reason is that in this case $\Box_\varphi^{(0,1)}$ acts diagonally on $(0,1)$-forms, each component $\mathcal{S}_k$ of the diagonal being

$$\mathcal{S}_k = \sum_{j \neq k} P_-^{(j)} + P_+^{(k)}$$

and that the operators $P_\pm^{(l)}$ act separately in each variable, cf. also [15]. By assumption $\varphi_j \in \Phi$ and by Theorem 5.1, $P_-^{(j)}$ has non-compact resolvent, which proves the Proposition.

□

Corollary 7.2. *Suppose that $\varphi(z)$ is plurisubharmonic and of the form*

$$\varphi(z) = \varphi_1(z_1) + \varphi_2(z_2) + \varphi_3(z_3, \ldots, z_n).$$

Suppose furthermore that $\varphi_2 \in \Phi$. Then N_φ is not compact.

Proof. By the assumed form of $\varphi(z)$, an easy computation shows that

$$\square_\varphi^{(0,1)}(u_1 d\overline{z}_1) = \frac{1}{4}\left(P_+^{(1)} + P_-^{(2)} + \triangle_{\varphi_3} - \triangle\varphi_3\right) u_1 d\overline{z}_1.$$

Hence compactness of N_φ implies compactness of the resolvent of $P_-^{(2)}$ since the operators are acting in each variable separately. So the Corollary follows from Theorem 5.1.

□

8. The Dirac operator.

In mathematical physics, the Dirac equation describes the behavior of a "free" relativistic particle, see e.g. [26]. In the real two dimensional case, the Dirac operator $\mathbb{D}$ acting on $L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$ is defined by

$$\mathbb{D} = \sigma_1 \left(-i\frac{\partial}{\partial x} - A_1(x,y)\right) + \sigma_2 \left(-i\frac{\partial}{\partial y} - A_2(x,y)\right), \tag{8.1}$$

where the standard choice of the matrices σ_j is

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{8.2}$$

There is also a notion of the Dirac operator in real dimension three, see e.g. [17]. It is conjectured that in real dimension two, the Dirac operator never admits a compact resolvent, cf. [9], [17]. The main result of [17] is the following: Let

$$m_q(x) = \sum_{|\alpha|=q-1} |\partial^\alpha B(x)| \qquad \text{and} \qquad m^r(x) = 1 + \sum_{q=0}^{r} m_q(x).$$

Suppose that there exists a sequence of pairwise disjoint balls each one of radius greater than 1, such that

$$m_{r+1}(x) \leq C m^r(x) \tag{8.3}$$

holds on the union of these balls. Then the Dirac operator has non-compact resolvent.
Note that this condition is for instance satisfied, if the magnetic potentials are polynomials.
Our condition does not make assumptions on the derivatives of the magnetic field, but on its structure and growth. The reader can easily convince himself that there are function satisfying the assumptions of Theorem 8.1 but not (8.3) and vice versa.

Theorem 8.1. *Suppose that the magnetic field*

$$B(x,y) = \frac{\partial A_2}{\partial x}(x,y) - \frac{\partial A_1}{\partial y}(x,y)$$

is of the form $B = \triangle\varphi$ for some function φ. Suppose furthermore that there is an entire function, which is square-integrable with respect to the weight $e^{-2\varphi}$. Then the Dirac operator has non-compact resolvent.

Remark. Note that one can always solve Poisson's equation to find such a function φ. By Proposition 4.2, the assumption of the Theorem is for instance satisfied if $B(x,y) \geq (x^2+y^2)^{-2}$. Note also that the case $|B(x,y)| \to 0$ is covered by the result in [17].

Proof of Theorem 8.1. Suppose that $\mathbb{D}$ has compact resolvent. Then also $\mathbb{D}^2$ has, since

$$(\mathbb{D}^2-1)^{-1} = (\mathbb{D}-i)^{-1}(\mathbb{D}+i)^{-1}.$$

It is a standard fact that for the square of the Dirac operator it holds

$$\mathbb{D}^2 = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix}.$$

This implies that both $P_\pm$ have compact resolvent, which contradicts our assumption by Theorem 5.1.

□

Aknowledgement. The author is thankful to B. Helffer for useful and interesting discussions on the Dirac operator. Moreover he wants to thank F. Haslinger for discussions on the complex analytic part of this work and in particular for drawing his attention at Lemma 3.4.

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