ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

On the Bergman Representative Coordinates

.Zywomir Dinew

Vienna, Preprint ESI 2229 (2010) February 24, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

ON THE BERGMAN REPRESENTATIVE COORDINATES

ŻYWOMIR DINEW

Abstract. We study the set where the so-called Bergman representative coordinates (or Bergman functions) form an immersion. We provide an estimate of the size of a maximal geodesic ball with respect to the Bergman metric, contained in this set. By concrete examples we show that these estimates are the best possible.

0. Introduction

Bergman representative coordinates were introduced by Bergman in [2] as a tool in his program of generalizing the Riemann mapping theorem to $\mathbb{C}^n, n > 1$. Their usefulness is based (among others) on the fact that biholomorphic mappings become linear when represented in these coordinates (See e.g., [12]).

It is hard to work with these coordinates mainly because they are not defined globally even in the domain case, nevertheless some remarkable results were obtained by using them. Lu Qi-Keng [17] proved that any domain with complete Bergman metric of constant negative holomorphic sectional curvature is biholomorphic to the unit ball in $\mathbb{C}^n$.

The so-called Bergman representative coordinates, respective to a point z_0 are:

$$w_i(z) = \sum_{j=1}^{n} T^{\bar{j}i}(z_0) \frac{\partial}{\partial \bar{\zeta}_j} \log \frac{K(z,\zeta)}{K(\zeta,\zeta)}_{|\zeta=z_0},$$

where $K(z,\zeta)$ is the Bergman kernel of the domain Ω and $T^{\bar{j}i}(z_0)$ is the inverse matrix of the matrix $(T_{i\bar{j}}(z_0))_{i,j=1..n} = \left(\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,z)\right)_{i,j=1..n}\Big|_{z=z_0}$ (we refer to section 1 for all the definitions).

Looking at the definition one immediately comes upon two issues: Are the above expressions well defined? Are they indeed coordinates?

Clearly in a small neighborhood of the point z_0 the answers to both questions are affirmative since $\frac{\partial(w_1,..,w_n)}{\partial(z_1,..,z_n)}_{|z=z_0}$ is the identity matrix.

Concerning the first question, one immediately sees that the only possible obstruction which may appear is that $K(z,\zeta)$ may have zeros. This is the reason for which studying zeros of the Bergman kernel attracted so much interest. Domains for which the Bergman kernel is zero-free are known as domains satisfying the Lu Qi-Keng conjecture or just Lu Qi-Keng domains. From nowadays perspective it is known that virtually all (in a sense, see [5]) domains are not Lu Qi-Keng domains. On the other hand it is clear that for fixed z_0 (as is in our case) the zero set of

2000 *Mathematics Subject Classification.* Primary 32A36, 32H10, 32H02, 32A60; Secondary 32F45, 32Q15.

Key words and phrases. Representative coordinates, Bergman metric, geodesic ball, Lu Qi-keng conjecture, Hermitian geometry.

the Bergman kernel will be an analytic set and hence w_i are well defined almost everywhere (on an open dense subset) in Ω. This topological information is one of the main ingredients in the Lu Qi-Keng's argument [17]. On the other hand for many geometric problems just topological information on the domain of definition (since one already knows that there is no hope to define the coordinates globally in general, but a local definition is at hand) is not enough. One would like to know whether there are subdomains Ω'_{z_0} of these domains of definition, which are related not to the topology but to the geometry of Ω. In particular one would like to know "how small" these neighborhoods of z_0 in which w_i are well defined must be, can one control them in a reasonable way (e.g., with dependence on the geometry of Ω) when the point z_0 is perturbed?

Even if well defined, the representative coordinates would have been useless if they do not yield a basis of local vector-fields, i.e.,

$$\det \frac{\partial(w_1, .., w_n)}{\partial(z_1, .., z_n)} \neq 0 \tag{0.1}$$

should hold in a prescribed neighborhood of z_0. Although these conditions still fail to yield "coordinates" in the broad sense (since one does not have the injectivity of the mapping $z \to (w_1, w_2, ..., w_n)^t$), this information will do for the purposes of this article.

Quite unexpectedly it occurs that the functions w_i are well defined in a geodesic ball of radius that does not depend not only on the choice of z_0, but is also independent of Ω. Thus we have

Theorem 0.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain equipped with the Bergman metric. For any $z_0 \in \Omega$ The Bergman kernel $K(z, z_0)$ does not vanish in the geodesic ball $\{z \in \Omega : dist_\Omega(z, z_0) < \frac{\pi}{2}\}$.*

Here the geodesic distance is with respect to the Riemannian metric yielded by the (Kählerian) Bergman metric. We note that Teorem0.1 is just a matter of looking from a different viewpoint at known facts.

Concerning the problem of linear independency it comes out that (0.1) is satisfied in a geodesic ball of radius that depends only on the Ricci curvature of the Bergman metric.

Theorem 0.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain equipped with the Bergman metric. Let $c \in (-\infty, n + 1)$ be a global lower bound of the Ricci curvature of the Bergman metric. For any $z_0 \in \Omega$ the mapping*

$$z \to (w_1(z), w_2(z)..., w_n(z))^t$$

is an immersion in the geodesic ball $\{z \in \Omega : dist_\Omega(z, z_0) < \frac{\pi}{2\sqrt{n+1-c}}\}$.

The lower bound c is defined as usually as a constant for which $Ric_{i\bar{j}} - cT_{i\bar{j}}$ is a positive definite matrix.

In the theorem above we assumed that the Ricci curvature is bounded below. This is not the case in general, see [7] and [20]. Note that the Ricci curvature is the same as the sectional, holomorphic sectional and Gaussian curvature, when the dimension is 1. Moreover, the example from [7], after delicate smoothing of the boundary, where the annuli overlap, shows that even so strong assumptions as being bounded smooth and strictly pseudoconvex except at a single point (at which both smoothness and strict pseudoconvexity fail), which is a peak point for any reasonable algebra of holomorphic functions, are not enough to guarantee boundedness of the Ricci curvature.

On the other hand for $\mathcal{C}^2$ (see [10]) strictly pseudoconvex domains or for domains of finite type in $\mathbb{C}^2$ (this fact is not stated explicitly in the literature, for nontangential approach see [13]) one has a global lower bound for the Ricci curvature of the Bergman metric.

Nevertheless, for any bounded domain Ω we have the following local substitute for Theorem 0.2.

Theorem 0.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded domain equipped with the Bergman metric. Let $U \subset \Omega$ be an open set for which $\inf_{\substack{z\in U\\ X\in\mathbb{C}^n\setminus\{0\}}} \frac{Ric_{i\bar{j}}(z)X_i\bar{X}_j}{T_{i\bar{j}}(z)X_i\bar{X}_j} > c$. For any $z_0 \in \Omega$ the mapping*

$$z \to (w_1(z), w_2(z)..., w_n(z))^t$$

is an immersion in the set $U \cap \{z \in \Omega : dist_\Omega(z, z_0) < \frac{\pi}{2\sqrt{n+1-c}}\}$.

The proofs of both Theorem 0.1 and Theorem 0.2 are quite similar and consist of using the Kobayashi construction (see [11]) of an imbedding in the infinite dimensional projective space and in the second case the target is also the projective space, however one uses an imbedding due to Lu Qi-Keng (see[16]) in the infinite dimensional Grassmannian, and the Plücker imbedding afterwards.

It is also of interest whether the estimates in Theorem 0.1 and Theorem 0.2 are optimal (whether the radii of the geodesic balls are the maximal possible). From the point of view of Riemannian geometry the generic optimality of the radius in Theorem 0.1 would mean that the Kobayashi embedding, restricted to a real submanifold of Ω is totally geodesic and that the cut-locus of z_0 lies outside $\{z \in \Omega : dist_\Omega(z, z_0) < \frac{\pi}{2}\}$. Especially the first condition is very restrictive and hence one would expect that the radius estimate is not optimal. In spite of this we prove

Theorem 0.4. *For any $\varepsilon > 0$ there exists a bounded domain Ω_ε such that there exists $z_0 \in \Omega_\varepsilon$ for which $K(z, z_0)$ has a zero in the geodesic ball*

$$\{z \in \Omega_\varepsilon : dist_{\Omega_\varepsilon}(z, z_0) < \frac{\pi}{2} + \varepsilon\}.$$

Concerning Theorem 0.2 the radius is not optimal since the theorem takes into account the minimum and not the actual value of the Ricci curvature. However if the right metric is assumed then a result of optimality does also hold

Theorem 0.5. *For any $\varepsilon > 0$ there exists a bounded domain Ω_ε such that there exists $z_0 \in \Omega_\varepsilon$ for which $z \to (w_1(z), w_2(z), .., w_n(z))^t$ fails to be an immersion in the whole geodesic ball $\{z \in \Omega_\varepsilon : \tilde{dist}_{\Omega_\varepsilon}(z, z_0) < \frac{\pi}{2} + \varepsilon\}$.*

Acknowledgement. This paper was written when the author was a Junior Fellow at the Erwin Schrödinger Institute in Vienna. I would like to thank the institute for hospitality and for providing perfect working conditions. I would like also to thank professor Takeo Ohsawa for turning my attention to hyperelliptic Riemann surfaces and professor Siqi Fu for pointing out the case of domains of finite type in $\mathbb{C}^2$.

1. The case of a bounded domain in $\mathbb{C}^n$

In this section Ω will be a bounded domain in $\mathbb{C}^n$. Let $\varphi = \{\varphi_0, \varphi_1, \varphi_2 ...\}$ be an orthonormal basis of the Hilbert space $\mathcal{O} \cap L^2(\Omega)$ of square-integrable holomorphic functions on Ω. The Bergman kernel of Ω, $K(z, w) = K_\Omega(z, w)$ is defined as follows

$$K(z,w) := \sum_{i=0}^{\infty} \varphi_i(z)\overline{\varphi_i(w)}. \tag{1.1}$$

With this kernel one associates a differential $(1, 1)$-form,

$$\sqrt{-1}\sum_{i,j=1}^{n} T_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j := \sqrt{-1}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z,z) dz_i \wedge d\bar{z}_j \tag{1.2}$$

In our setting this form will be everywhere positive definite and moreover one easily sees that it is a Kähler form with global potential. The associated metric $\sum_{i,j=1}^{n} T_{i\bar{j}} dz_i d\bar{z}_j$ is called the Bergman metric and the square of the length of a vector X, measured in this metric at the point $z \in \Omega$ is

$$\beta^2(z,X) = \beta^2_\Omega(z,X) := \sum_{i,j=1}^{n} T_{i\bar{j}}(z) X_i \bar{X}_j, \tag{1.3}$$

for any vector $X \in \mathbb{C}^n$. One defines the length of a piecewise $\mathcal{C}^1$ curve

$$\gamma : [0,1] \ni t \to \gamma(t) \in \Omega,$$

as

$$\ell(\gamma) := \int_0^1 \beta(t, \gamma'(t)) dt \tag{1.4}$$

and the Bergman distance between two points $z, w \in \Omega$

$$dist_\Omega(z,w) := \inf\{\ell(\gamma) : \gamma \text{ is a piecewise } \mathcal{C}^1 \text{ curve s.t. } \gamma(0) = z, \gamma(1) = w\}. \tag{1.5}$$

The Bergman distance is indeed a distance and hence endows Ω with the structure of a metric space.

Further let $g(z) := \det(T_{i\bar{j}}(z))_{i,j=1..n}$. Recall that

$$\sqrt{-1}\sum_{i,j=1}^{n} Ric_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j := -\sqrt{-1}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log g(z) dz_i \wedge d\bar{z}_j \tag{1.6}$$

is the Ricci form of the Bergman metric and let

(1.7)
$$\sqrt{-1}\sum_{i,j=1}^{n} \tilde{T}_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j := \sqrt{-1}\sum_{i,j=1}^{n} \left((n+1)T_{i\bar{j}}(z) + \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log g(z)\right) dz_i \wedge d\bar{z}_j.$$

It follows that this form is also positive definite (see e.g., [16]) in our setting and Kähler, with Kähler potential $\log\big(K(z,z)^{n+1} g(z)\big)$. Slightly different construction was assumed in [9]. As above one defines the square of the length of a vector $\tilde{\beta}^2(z, X)$, the length of a curve $\tilde{\ell}(\gamma)$ and distance $\tilde{dist}_\Omega(z, w)$ with respect to this new Kähler metric.

1.1. **The Kobayashi embedding.** The Kobayashi embedding is the holomorphic embedding of the domain Ω into the projective space over the Hilbert space dual of the Hilbert space $L^2 \cap \mathcal{O}(\Omega)$ which is naturally identified with the infinite dimensional projective space $\mathbb{CP}^\infty$. The construction goes as follows. Fix an orthonormal basis $\varphi = \{\varphi_0, \varphi_1, \varphi_2 ...\}$, $\varphi_j \in L^2 \cap \mathcal{O}(\Omega)$. The Kobayashi embedding is the mapping $\iota_{Ko,\varphi}$ defined by

$$\Omega \ni z \to \iota_{Ko,\varphi}(z) = [(\varphi_0(z), \varphi_1(z), \varphi_2(z), ...)] \in \mathbb{CP}^\infty,$$

where the above notation is with respect to the homogeneous coordinates in $\mathbb{CP}^\infty$. One easily sees that

$$\iota_{Ko,\varphi}(z) = [\langle \circ, K(., z)\rangle_{L^2(\Omega)}]. \tag{1.8}$$

What makes this construction so important is the fact that the embedding is isometric in the sense that the pullback $\iota^*_{Ko,\varphi}\omega_{FS}$ of the standard Fubini-Study metric on $\mathbb{CP}^\infty$ is exactly the Bergman metric of Ω. This combined with the formula for the distance on the projective space gives one the following inequality (see [4] or [3] and Proposition 4.1.6 therein):

$$dist_\Omega(z, z_0) \geq \arccos \frac{|K(z, z_0)|}{\sqrt{K(z, z)K(z_0, z_0)}}. \tag{1.9}$$

It is clear that equality need not hold in (1.9).

Once one has (1.9), Theorem 0.1 follows easily, since $\arccos 0 = \frac{\pi}{2}$

1.2. **The Lu Qi-Keng embedding.** The Lu Qi-Keng embedding is in some way similar to the Kobayashi embedding however the target manifold is different.

It is defined as

$$\Omega \ni z \to \left[\begin{pmatrix} \varphi_0 \frac{\partial \varphi_1}{\partial z_1} - \varphi_1 \frac{\partial \varphi_0}{\partial z_1} & \varphi_0 \frac{\partial \varphi_2}{\partial z_1} - \varphi_2 \frac{\partial \varphi_0}{\partial z_1} & \varphi_1 \frac{\partial \varphi_2}{\partial z_1} - \varphi_2 \frac{\partial \varphi_1}{\partial z_1} & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \varphi_0 \frac{\partial \varphi_1}{\partial z_n} - \varphi_1 \frac{\partial \varphi_0}{\partial z_n} & \varphi_0 \frac{\partial \varphi_2}{\partial z_n} - \varphi_2 \frac{\partial \varphi_0}{\partial z_n} & \varphi_1 \frac{\partial \varphi_2}{\partial z_n} - \varphi_2 \frac{\partial \varphi_1}{\partial z_n} & \cdots \end{pmatrix}\right] \in \mathbb{F}(n, \infty),$$

the infinite-dimensional Grassmanian of n-dimensional complex linear subspaces. Here $[\cdot]$ is the class with respect to the following equivalency relation: Two $n \times \infty$ matrices A and B of rank n (representing n-dimensional subspaces) are equivalent if and only if there exists a nonsingular $n \times n$ matrix C, such that $A = CB$.

It is implicitly assumed that if

$$p = \left[\begin{pmatrix} p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \cdots \\ p_{n1} & p_{n2} & \cdots \end{pmatrix}\right] \in \mathbb{F}(n, \infty), \text{ then } \sum_{j=1}^{\infty} |p_{ij}|^2 < \infty, i = 1..n. \tag{1.10}$$

It is proved in [16] that the pullback of the Fubini-Study metric of the Grassmanian $(\tilde{\omega}_{FS})$ which can be seen as the metric associated to the form

$$Tr((I_n + ZZ^*)^{-1}dZ \wedge (I + Z^*Z)^{-1}dZ^*),$$

in local coordinates Z of the Grassmannian is

$$\iota^*_{Lu,\varphi}\tilde{\omega}_{FS} = (n+1)T_{i\bar{j}} - Ric_{i\bar{j}}, \tag{1.11}$$

where Ric is the Ricci tensor of the Bergman metric (and hence coincides with $\tilde{\beta}^2(., \circ)$).

One has that $\tilde{\omega}_{FS}$ is itself the pullback via the Plücker embedding of ω_{FS} in $\mathbb{CP}^\infty$,

$$\tilde{\omega}_{FS} = \iota^*_{Pl\ddot{u}}\omega_{FS}. \tag{1.12}$$

This is almost immediate generalization of the finite-dimensional case, however the author was unable to find this result in the literature and hence a proof is provided below. For the finite-dimensional Grassmannian this is done in [14], Satz 7.

Without loss of generality (by a transitivity argument) one can assume that $p \in \mathbb{F}(n,\infty)$ lies in the subset of $\mathbb{F}(n,\infty)$ for which the matrix representing p,

$$\begin{pmatrix} p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \dots \\ p_{n1} & p_{n2} & \cdots \end{pmatrix}, \text{ for } p = \left[\begin{pmatrix} p_{11} & p_{12} & \cdots \\ \vdots & \vdots & \dots \\ p_{n1} & p_{n2} & \cdots \end{pmatrix}\right],$$

has the property that exactly the first $n \times n$ minor, $\begin{pmatrix} p_{11} \cdots p_{1n} \\ \vdots \ddots \vdots \\ p_{n1} \cdots p_{nn} \end{pmatrix}$ is nonsingular. In fact every matrix representing the class p will have the required property. Let

$$Z = \begin{pmatrix} z_{11} & z_{12} & \cdots \\ \vdots & \vdots & \dots \\ z_{n1} & z_{n2} & \cdots \end{pmatrix} := \begin{pmatrix} p_{11} \cdots p_{1n} \\ \vdots \ddots \vdots \\ p_{n1} \cdots p_{nn} \end{pmatrix}^{-1} \begin{pmatrix} p_{1n+1} & p_{1n+2} & \cdots \\ \vdots & \vdots & \dots \\ p_{nn+1} & p_{nn+2} & \cdots \end{pmatrix}.$$

Then the matrix obtained by pairing the blocks (I_n, Z) represents p, moreover a representative of this type is unique. One says that Z is the local coordinate of p in the neighborhood of $\left[\begin{pmatrix} 1 & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \dots \\ 0 & \cdots & 1 & 0 & \cdots \end{pmatrix}\right]$.

The Plücker embedding sends the vector space spanned by the vectors $v_1, ..., v_n \in \mathbb{C}^\infty$ into the element $[v_1 \wedge \cdots \wedge v_n] \in P(\Lambda^n \mathbb{C}^\infty) \cong \mathbb{CP}^\infty$, where $P(\Lambda^n \mathbb{C}^\infty)$ is the projectivization of $\Lambda^n \mathbb{C}^\infty$.

In local coordinates this reads

$$\left[\begin{pmatrix} 1 & \cdots & 0 & z_{11} & z_{12} & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \dots \\ 0 & \cdots & 1 & z_{n1} & z_{n2} & \cdots \end{pmatrix}\right]_{\mathbb{F}(n,\infty)} \to [(e_1 + z_{11}e_{n+1} + z_{12}e_{n+2} + \cdots)\wedge$$

$$\wedge(e_2 + z_{21}e_{n+1} + z_{22}e_{n+2} + \cdots) \wedge \cdots \wedge (e_n + z_{n1}e_{n+1} + z_{n2}e_{n+2} + \cdots)]_{P(\Lambda^n \mathbb{C}^\infty)} =$$

$$\left[e_1 \wedge \cdots \wedge e_n + \sideset{}{'}\sum_{\substack{(j_1,j_2,\ldots,j_n)\neq \\ (-n+1,-(n-1)+1,\ldots,0)}} \det \begin{pmatrix} z_{1j_1} \cdots z_{1j_n} \\ \vdots \ddots \vdots \\ z_{nj_1} \cdots z_{nj_n} \end{pmatrix} e_{j_1+n} \wedge \cdots \wedge e_{j_n+n}\right],$$

where we assume $\begin{pmatrix} z_{1-n} \cdots z_{10} \\ \vdots \ddots \vdots \\ z_{n-n} \cdots z_{n0} \end{pmatrix} = I_n$

The isomorphism of $P(\Lambda^n \mathbb{C}^\infty)$ with $\mathbb{CP}^\infty$ is realized by enumerating lexicographically

$$\tilde{e}_s = e_{j_1(s)+n} \wedge e_{j_2(s)+n} \wedge \cdots \wedge e_{j_n(s)+n}.$$

Because $\tilde{e}_0 = e_1 \wedge ... \wedge e_n$ the local coordinate of the image of p in $\mathbb{CP}^\infty$ will be

$$\sum_{s=1}^{\infty} \det \begin{pmatrix} z_{1j_1(s)} \cdots z_{1j_n(s)} \\ \vdots \ddots \vdots \\ z_{nj_1(s)} \cdots z_{nj_n(s)} \end{pmatrix} \tilde{e}_s.$$

The Fubini-Study metric ω_{FS} is the metric associated to $\partial\bar{\partial}\log(1+WW^*)$, for $W = (w_1, w_2, ...)$, the local coordinate of $w = [(1, w_1, w_2, ...)]$, the line with direction $\tilde{e}_0 + \sum_{s=1}^{\infty} w_s \tilde{e}_s$ in $\mathbb{C}^\infty$.

The metric at the image point of (I_n, Z) is associated to

$$\partial\bar{\partial}\log\left(1 + \sum_{s=1}^{\infty} \left| \det \begin{pmatrix} z_{1j_1(s)} \cdots z_{1j_n(s)} \\ \vdots \ddots \vdots \\ z_{nj_1(s)} \cdots z_{nj_n(s)} \end{pmatrix} \right|^2 \right),$$

which by the Cauchy-Binet formula equals $\partial\bar{\partial}\log\det(I+ZZ^*)$ (there is no problem with convergence here, by the assumption (1.10)).

We use the well known expressions for the derivative of the determinant and the inverse matrix (all the notations are to be understood in the obvious sense).

$$\bar{\partial}\det A = \det A\, Tr(A^{-1}\bar{\partial}A)$$
$$\partial A^{-1} = -A^{-1}(\partial A)A^{-1}$$

$$\partial\bar{\partial}\log(\det(I+ZZ^*)) = \partial\frac{det(I+ZZ^*)Tr((I_n+ZZ^*)^{-1}\,\bar{\partial}\,(I_n+ZZ^*))}{det(I+ZZ^*)} =$$

$$\partial Tr((I_n+ZZ^*)^{-1} ZdZ^*) = Tr(-(I_n+ZZ^*)^{-1}(\partial\,(I_n+ZZ^*))\,(I_n+ZZ^*)^{-1} ZdZ^* +$$
$$+(I_n+ZZ^*)^{-1} dZdZ^*) = Tr((I_n+ZZ^*)^{-1} dZ(I - Z^*(I_n+ZZ^*)^{-1} Z)dZ^*)).$$

What remains is to show that

$$I - Z^*(I_n+ZZ^*)^{-1} Z = (I+Z^*Z)^{-1}$$

Multiplying with $I+Z^*Z$ gives one

$$(I - Z^*(I_n+ZZ^*)^{-1} Z)(I+Z^*Z) = I + Z^*Z - Z^*(I_n+ZZ^*)^{-1}(Z+ZZ^*Z) =$$
$$I + Z^*Z - Z^*(I_n+ZZ^*)^{-1}(I_n+ZZ^*)Z = I + Z^*Z - Z^*Z = I,$$

hence

$$Tr(I_n+ZZ^*)^{-1}dZ \wedge (I+Z^*Z)^{-1}dZ^* = \partial\bar{\partial}\log\det(I+ZZ^*),$$

which proves (1.12).

Now combining (1.12) and (1.11) one has

$$(n+1)T_{i\bar{j}} - Ric_{i\bar{j}} = \iota^*_{Lu,\varphi}\iota^*_{Pl\ddot{u}}\omega_{FS}. \tag{1.13}$$

And hence $\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}$ is an isometric embedding of Ω with the metric $\tilde{\beta}$ to $\mathbb{CP}^\infty$ with the Fubini-Study metric. Now using the formula for the geodesic distance in $\mathbb{CP}^\infty$ one obtains, like (1.9), our main inequality

$$\tilde{dist}(z,\zeta) \geq$$
$$\arccos \frac{|(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(z))(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(\zeta))^*|}{\sqrt{(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(z))(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(z))^*(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(\zeta))(\iota_{Pl\ddot{u}} \circ \iota_{Lu,\varphi}(\zeta))^*}} =$$

$$\arccos \frac{\left|\det\left(K(z,\zeta)^2 \frac{\partial^2}{\partial z_i \partial \bar{\zeta}_j} \log K(z,\zeta)\right)_{i,j=1..n}\right|}{\sqrt{\det(K(z,z)^2 T_{i\bar{j}}(z))_{i,j=1..n} \det(K(\zeta,\zeta)^2 T_{i\bar{j}}(\zeta))_{i,j=1..n}}}. \tag{1.14}$$

In the expression for w_i the term $T^{\bar{j}i}(z_0)$ is introduced for the sake of normalization and is irrelevant when it comes to linear independency. Hence

$$z \to (w_1(z), w_2(z) ..., w_n(z))^t$$

is an immerison exactly when

$$z \to \left(\frac{\partial}{\partial \bar{\zeta}_1} \log \frac{K(z,\zeta)}{K(\zeta,\zeta)}_{|\zeta=z_0}, \frac{\partial}{\partial \bar{\zeta}_2} \log \frac{K(z,\zeta)}{K(\zeta,\zeta)}_{|\zeta=z_0}, ..., \frac{\partial}{\partial \bar{\zeta}_n} \log \frac{K(z,\zeta)}{K(\zeta,\zeta)}_{|\zeta=z_0}\right)^t$$

is an immersion. The determinant of the Jacobian of the latter expression is

$$\det\left(\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{\zeta}_j} \log \frac{K(z,\zeta)}{K(\zeta,\zeta)}_{|\zeta=z_0}\right)_{i,j=1..n} = \det\left(\frac{\partial^2}{\partial z_j \partial \bar{\zeta}_j} \log K(z,\zeta)_{|\zeta=z_0}\right)_{i,j=1..n}. \tag{1.15}$$

Comparing (1.14) and (1.15) one notices that the zero-sets of the determinants are the same, with possible difference of the singular loci. Hence for fixed z_0 the nearest point z for which $\frac{\partial(w_1(z),..,w_n(z))}{\partial(z_1,..,z_n)} = 0$ must satisfy

$$\tilde{dist}(z, z_0) \geq \arccos 0 = \frac{\pi}{2}. \tag{1.16}$$

Theorems 0.2 and 0.3 follow.

We note that the same conclusion can be obtained by directly calculating the distance on the Grassmanian, however this is technically involved, see [1] for a sketch in the finite-dimensional case.

Remark 1.1. *Let W_{z_0} denote the set $\{z \in \Omega : K(z,z_0) = 0\}$ and $\tilde{W}_{z_0}$ denote the set $\{z \in \Omega : \det\left(K(z,z_0)^2 \frac{\partial^2}{\partial z_i \partial \bar{\zeta}_j} \log K(z,\zeta)_{|\zeta=z_0}\right)_{i,j=1..n} = 0\}$. If $n > 1$ one has that $W_{z_0} \subset \tilde{W}_{z_0}$.*

Proof. It follows by a simple calculation that

$$\det\left(K(z,z_0)^2 \frac{\partial^2}{\partial z_i \partial \bar{\zeta}_j} \log K(z,\zeta)_{|\zeta=z_0}\right)_{i,j=1..n} =$$
$$\det\left(K(z,z_0) \frac{\partial^2}{\partial z_i \partial \bar{\zeta}_j} K(z,\zeta)_{|\zeta=z_0} - \frac{\partial}{\partial z_i} K(z,z_0) \frac{\partial}{\partial \bar{\zeta}_j} K(z,\zeta)_{|\zeta=z_0}\right)_{i,j=1..n}.$$

When $z \in W_{z_0}$ this reduces to

$$\det\left(-\frac{\partial}{\partial z_i} K(z,z_0) \frac{\partial}{\partial \bar{\zeta}_j} K(z,\zeta)_{|\zeta=z_0}\right)_{i,j=1..n} = 0,$$

since the matrix is of rank 1. □

2. The manifold case

The essential difference between the manifold and the domain cases is that there do not exist coordinates in the large. Moreover in the compact case there are no nonconstant holomorphic functions. Therefore one has to modify the construction of the Bergman kernel and to employ forms of top degree instead of functions.

Let M be a n- dimensional complex manifold. The space of top degree holomorphic forms is denoted by $H^0(M, K_M)$, which can also be viewed as the space of global holomorphic sections of the canonical bundle K_M over M. One can restrict to the space $H^0_{(2)}(M, K_M) = H^0(M, K_M) \cap L^2(M, K_M)$ of square-integrable global holomorphic forms of top degree, i.e.,

$$H^0_{(2)}(M, K_M) = \{f \in H^0(M, K_M) : \sqrt{-1}^{n^2} \int_M f \wedge \bar{f} < \infty\}.$$

Now $H^0_{(2)}(M, K_M)$ can be equipped with an inner product

$$H^0_{(2)}(M, K_M) \ni f, g \to \sqrt{-1}^{n^2} \int_M f \wedge \bar{g} \in \mathbb{C}\,.$$

This inner product turns $H^0_{(2)}(M, K_M)$ into a (possibly finite dimensional) Hilbert space. Note that if M is compact $H^0_{(2)}(M, K_M) \equiv H^0(M, K_M)$ and $H^0_{(2)}(M, K_M)$ is finitely-dimensional. Let $\varphi = \{\varphi_0, \varphi_1, \varphi_2 ...\}$ be an orthonormal basis of $H^0_{(2)}(M, K_M)$. The Bergman kernel form is

$$K = \sum_{j>0} \varphi_j \wedge \bar{\varphi}_j$$

In local coordinates one can write.

$$K(z, \zeta) = K^*(z, \zeta) dz_1 \wedge dz_2 \wedge .. \wedge dz_n \wedge d\bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge .. \wedge d\bar{\zeta}_n,$$

where $K^*(z, \zeta)$ is a function which is defined only locally. The $(1, 1)$- differential form

$$\partial\bar{\partial} \log K^*(z, z) \tag{2.1}$$

is however globally defined, which can be easily seen by expressing $K(z, \zeta)$ in different local coordinates.

One says that that the manifold M has a Bergman metric if the form (2.1) is globally strictly positive. In such a case the Bergman metric is the metric associated to (2.1). One can therefore define $T_{i\bar{j}}(z)$, $\beta(z, X)$, $dist_M(z, w)$, $Ric_{i\bar{j}}$, $\tilde{\beta}(z, X)$, and $\tilde{dist}_M(z, w)$ like in the previous section with the constraints that K^* is defined only locally and X must be taken from the tangent space to M at z.

Suppose now that M carries a Bergman metric. Having the starred counterparts of functions in $\mathbb{C}^n$ one sees that the representative coordinates can also be defined at least locally. We take an open cover $\{U_i\}$ of M subordinate to local coordinate charts. Let $z_0 \in U_1$ be fixed. After probably shrinking we can arrange the sets $U_i \times U_1$ to cover $M \times U_1$ and in every $U_i \times U_1$, $K(z, \zeta)$ can be expressed by

$$K(z, \zeta) = K^*_{U_i}(z, \zeta) dz_1 \wedge .. \wedge dz_n \wedge d\bar{\zeta}_1 \wedge .. \wedge d\bar{\zeta}_n.$$

In $U_i \times U_1 \cap U_j \times U_1$ the change of coordinates $(z, \zeta) \overset{(\tilde{z}(.), id)}{\longrightarrow} (\tilde{z}(z), \zeta)$ yields

$$K^*_{U_j}(\tilde{z}(z), \zeta) \frac{\partial(\tilde{z}(z)_1, .., \tilde{z}(z)_n)}{\partial(z_1, .., z_n)} = K^*_{U_i}(z, \zeta)$$

For every $s \in \{1..n\}$ one has

$$\frac{\partial}{\partial \bar{\zeta}_s} \log K^*_{U_i}(z,\zeta) = \frac{\partial}{\partial \bar{\zeta}_s} \log \left(K^*_{U_j}(\tilde{z}(z),\zeta) \frac{\partial(\tilde{z}(z)_1,..,\tilde{z}(z)_n)}{\partial(z_1,..,z_n)} \right) = \frac{\partial}{\partial \bar{\zeta}_s} \log K^*_{U_j}(\tilde{z},\zeta).$$

Hence the expressions

$$w_l(z) = \sum_{k=1}^{n} T^{\bar{k}l}(z_0) \frac{\partial}{\partial \bar{\zeta}_k} \log \frac{K^*_{U_i}(z,\zeta)}{K^*_{U_1}(\zeta,\zeta)}_{|\zeta=z_0}, l = 1..n$$

glue up to global functions. As in the $\mathbb{C}^n$ case the only obstruction that can appear is that $K_{U_i}(z,z_0)$ may be zero for some z (this is clearly independent of the set U_i, the representation will be zero for every U_j, $z \in U_j$).

We remark that representative coordinates for the Bergman metric on manifolds were previously studied in [6], however there the Bergman kernel function instead of the Bergman kernel form was used. This substantially limited the range of assumed manifolds, for example every compact manifold was excluded from consideration.

From now on the convention will be that f^* is the local coefficient of the form $f(z) = f^*(z)dz_1 \wedge dz_2 \wedge .. \wedge dz_n$.

Unlike the situation in $\mathbb{C}^n$, M does not obviously possess Bergman metric. The necessary and sufficient conditions for M to have a Bergman metric are the following:

• For every $z \in M$ there exists $f \in H^0_{(2)}(M, K_M)$ such that $f^*(z) \neq 0$ (condition A.1 in [11]).

• For every $z \in M$ and for every nonzero vector X in the complex tangent space at z there exists $g \in H^0_{(2)}(M, K_M)$ such that $g^*(z) = 0$ and $X(g^*)(z) \neq 0$ (condition A.2 in [11]).

The later condition is clearly equivalent to

For every $z \in M$ and for a basis $X_i, i = 1..n$ of the complex tangent space at z there exist $g_i \in H^0_{(2)}(M, K_M)$ such that $g_i(z) = 0$ and $X_i(g_i^*)(z) \neq 0$.

These conditions are hard to check for an abstract complex manifold, however one immediately sees that a necessary condition for M to have a Bergman metric is that the $(n,0)$- Hodge number $h^{n,0}(M)$ (or geometric genus) satisfies $h^{n,0}(M) \geq n+1$.

It turns out (see [11]) that A.1 and A.2 are also necessary and sufficient conditions for $\iota_{Ko,\varphi}$ to be an immersion (A.1 solely is necessary and sufficient for the Kobayashi mapping to be well defined), whereas for an injection one needs another condition (A.3 in [11]):

• For every two points $z, z_0 \in M$ there exists $h \in H^0_{(2)}(M, K_M)$ such that $h(z) = 0$ and $h(z_0) \neq 0$.

In order to carry the Lu Qi-Keng construction on manifolds, one first has to check that $\iota_{Lu,\varphi}$ does not depend on local holomorphic coordinate changes (this is not completely obvious, since there are partial derivatives in the expression of $\iota_{Lu,\varphi}$).

Let

$$\varphi_j = \varphi_j^*(\tilde{z}) d\tilde{z}_1 \wedge .. \wedge d\tilde{z}_n = \varphi_j^*(\tilde{z}(z)) Jac\left(\frac{\partial \tilde{z}}{\partial z}\right) dz_1 \wedge .. \wedge dz_n$$

Now

$$\varphi_j^*(z) \frac{\partial \varphi_k^*(z)}{\partial z_s} - \varphi_k^*(z) \frac{\partial \varphi_j^*(z)}{\partial z_s} =$$

$$\tilde{\varphi}_j^*(\tilde{z}(z))Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)\frac{\partial \tilde{\varphi}_k^*(\tilde{z}(z))Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)}{\partial z_s} - \tilde{\varphi}_k^*(\tilde{z}(z))Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)\frac{\partial \tilde{\varphi}_j^*(\tilde{z}(z))Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)}{\partial z_s} =$$

$$Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)^2 \tilde{\varphi}_j^*(\tilde{z}(z))\sum_{m=1}^{n}\frac{\partial \tilde{\varphi}_k^*}{\partial \tilde{z}_m}\frac{\partial \tilde{z}_m}{\partial z_s} - Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)^2 \tilde{\varphi}_k^*(\tilde{z}(z))\sum_{m=1}^{n}\frac{\partial \tilde{\varphi}_j^*}{\partial \tilde{z}_m}\frac{\partial \tilde{z}_m}{\partial z_s}.$$

Hence

$$\left(\varphi_j^*(z)\frac{\partial \varphi_k^*(z)}{\partial z_s} - \varphi_k^*(z)\frac{\partial \varphi_j^*(z)}{\partial z_s}\right)_{\substack{s=1..n \\ j<k}} =$$

$$Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)^2 \begin{pmatrix} \frac{\partial \tilde{z}_1}{\partial z_1} & \cdots & \frac{\partial \tilde{z}_n}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{z}_1}{\partial z_n} & \cdots & \frac{\partial \tilde{z}_n}{\partial z_n} \end{pmatrix} \left(\tilde{\varphi}_j^*(\tilde{z})\frac{\partial \tilde{\varphi}_k^*(\tilde{z})}{\partial \tilde{z}_s} - \tilde{\varphi}_k^*(\tilde{z})\frac{\partial \tilde{\varphi}_j^*(\tilde{z})}{\partial \tilde{z}_s}\right)_{\substack{s=1..n \\ j<k}}$$

and the classes of these matrices coincide since $Jac\left(\frac{\partial \tilde{z}}{\partial z}\right)^2 \begin{pmatrix} \frac{\partial \tilde{z}_1}{\partial z_1} & \cdots & \frac{\partial \tilde{z}_n}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \tilde{z}_n}{\partial z_1} & \cdots & \frac{\partial \tilde{z}_n}{\partial z_n} \end{pmatrix}$ is non-singular.

Fix now local coordinates in a neighborhood of $z \in M$. A careful analysis of [16] gives that the necessary and sufficient conditions for $\iota_{Lu,\varphi}$ to be well defined are

• For every $z \in M$ there exists $f \in H^0_{(2)}(M, K_M)$ such that $f^*(z) \neq 0$ (the same as condition A.1).

• For every $z \in M$ there exist $f_1, .., f_n \in H^0_{(2)}(M, K_M)$ such that $f_i^*(z) = 0, i = 1..n$ and $\begin{pmatrix} \frac{\partial f_1^*}{\partial z_1} & \cdots & \frac{\partial f_n^*}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1^*}{\partial z_n} & \cdots & \frac{\partial f_n^*}{\partial z_n} \end{pmatrix}$ is non-singular at z (condition B.1)

The necessary and sufficient condition for $\iota_{Lu,\varphi}$ to be an immersion, in addition to A.1 and B.1, is

• For every $z \in M$ there exist $g_1, .., g_{\frac{n(n+1)}{2}} \in H^0_{(2)}(M, K_M)$ (some of them probably 0) such that $g_i^*(z) = 0, i = 1..\frac{n(n+1)}{2}$, $dg_i^* = 0, i = 1..\frac{n(n+1)}{2}$ at the point z and the $n \times \frac{n^2(n+1)}{2}$ matrix

$$\begin{pmatrix} \frac{\partial^2 g_1^*}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 g_1^*}{\partial z_n \partial z_1} & \frac{\partial^2 g_2^*}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 g_2^*}{\partial z_n \partial z_1} & \cdots & \frac{\partial^2 g^*_{\frac{n(n+1)}{2}}}{\partial z_1 \partial z_1} & \cdots & \frac{\partial^2 g^*_{\frac{n(n+1)}{2}}}{\partial z_n \partial z_1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g_1^*}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 g_1^*}{\partial z_n \partial z_n} & \frac{\partial^2 g_2^*}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 g_2^*}{\partial z_n \partial z_n} & \cdots & \frac{\partial^2 g^*_{\frac{n(n+1)}{2}}}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 g^*_{\frac{n(n+1)}{2}}}{\partial z_n \partial z_n} \end{pmatrix}$$

has rank n at z (condition B.2)

Finally the necessary and sufficient condition for $\iota_{Lu,\varphi}$ to be an injection is

• For every pair of distinct points $z, w \in M$ with fixed local coordinates in neighborhoods of z and w and for every nonsingular $n \times n$ matrix $(p_{ij})_{i,j=1..n}$ there exist $f, g \in H^0_{(2)}(M, K_M)$ such that

$$\left(f^*(z)\frac{\partial g^*}{\partial z_i}(z)-g^*(z)\frac{\partial f^*}{\partial z_i}(z)\right)^t \neq (p_{ij})_{i,j=1..n}\left(f^*(w)\frac{\partial g^*}{\partial z_j}(w)-g^*(w)\frac{\partial f^*}{\partial z_j}(w)\right)^t,$$

as vectors i.e., the vector with i-th component - the left hand side is not equal to the vector with i-th component- the expression on the right hand side. For some i the components may however be equal. (condition B.3)

Proposition 2.1. *Let M be a complex manifold of dimension $n>1$ for which the Lu Qi-Keng mapping is well defined and the Kobayashi mapping is an injection. Then the Lu Qi-Keng mapping is also an injection.*

Proof. Fix the points $z, w \in M$ and the matrix $(p_{ij})_{i,j=1..n}$. It is enough to find f and g satisfying B.3. Consider the condition B.1 at z. There must be a non-singular 2×2 minor $\begin{pmatrix} \frac{\partial f_p^*}{\partial z_r} & \frac{\partial f_q^*}{\partial z_r} \\ \frac{\partial f_p^*}{\partial z_s} & \frac{\partial f_q^*}{\partial z_s}\end{pmatrix}$, $p\neq q$, $r\neq s$ of the matrix in condition B.1. One can find (complex) constants A, B such that $A\frac{\partial f_p^*}{\partial z_r}(z)+B\frac{\partial f_q^*}{\partial z_r}(z)=0$ and $A\frac{\partial f_p^*}{\partial z_s}(z)+B\frac{\partial f_q^*}{\partial z_s}(z)=1$. Let $f=Af_p+Bf_q$. It is clear that $f^*(z)=0$ and $\frac{\partial f^*}{\partial z_r}(z)=0$, $\frac{\partial f^*}{\partial z_s}(z)=1$. For $f^*(w)$ there are two possibilities.

In case $f^*(w)\neq 0$ one can find (by condition A.2, following from B.1) $g\in H^0_{(2)}(M,K_M)$ such that $g^*(w)=0$ and $X(g^*)(w)\neq 0$ where $X=(p_{r1},p_{r2},...,p_{rn})^t$. The left hand side of the expression in condition B.3 is 0 for $i=r$ and the left hand side becomes

$$\sum_{j=1}^n p_{rj}f^*(w)\frac{\partial g^*}{\partial w_j}(w)=f^*(w)X(g^*)(w)\neq 0.$$

In case $f^*(w)=0$ one can find (by condition A.3) $g\in H^0(M,K_M)$ such that $g^*(w)=0$ and $g^*(z)\neq 0$. So

$$f^*(z)\frac{\partial g^*}{\partial z_s}(z)-g^*(z)\frac{\partial f^*}{\partial z_s}(z)=-g^*(z)\neq 0.$$

And

$$\sum_{j=1}^n p_{sj}\left(f^*(w)\frac{\partial g^*}{\partial z_j}(w)-g^*(w)\frac{\partial f^*}{\partial z_j}(w)\right)=\sum_{j=1}^n p_{sj}0=0.$$

□

The simplest example of a manifold for which the Kobayashi mapping is an immersion almost everywhere and not allowing the Lu Qi-Keng mapping is a compact Riemann surface of genus 2 (That the Kobayashi construction is an immersion outside the Weierstrass points follows by [15]. Note that there "Bergman metric" is different from our notion of Bergman metric. On the other hand every compact Riemann surface of genus 2 is necessarily hyperelliptic and hence the Kobayashi mapping is not a global immersion). The generic non-hyperelliptic compact Riemann surface of genus 3 is an example of a manifold for which the Kobayashi mapping is a global immersion, however the Lu Qi-Keng mapping fails to be an immersion exactly at the 24 Weierstrass points.

Now we see that Theorems 0.1,0.2 and 0.3 hold also for complex manifolds under the assumption that the Kobayashi (respectively Lu Qi-Keng) mapping is an immersion.

Proposition 2.2. *Let M be a compact complex manifold admitting the Bergman metric. Then $diam M \geq \frac{\pi}{2}$ where the diameter is taken with respect to the Bergman metric.*

Proof. Fix a point $z_0 \in M$ if $diam M < \frac{\pi}{2}$ then M is contained in the geodesic ball $\{z \in M : dist_M(z, z_0) < \frac{\pi}{2}\}$ and hence by Theorem 0.1 w_1 is a globally defined nonconstant holomorphic function which clearly can not exist. □

3. Examples

It was Skwarczyński (see[19]) that first observed that for some domains $K(z,\zeta)$ has zeros, namely he proved that this is the case for the circular annulus

$$P_r := \{z \in \mathbb{C} : r < |z| < 1\}, r < e^{-2}.$$

Later Rosenthal (see [18]) extended this result for all nondegenerate annuli by using different method. Although technically complicated the case of a planar annulus is still the easiest to study. What follows is essentially a more detailed study of the analysis in [19]. Recall that the Bergman kernel of P_r is

$$K(z,\zeta) = -\frac{1}{\pi z\overline{\zeta}\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}}{\left(-r^{2+2j}+z\overline{\zeta}\right)^2} + \frac{r^{2j}}{\left(1-r^{2j}z\overline{\zeta}\right)^2}\right). \tag{3.1}$$

Fix a positive $\varepsilon << 1$. From now on we restrict the range of r to the values for which all the following three inequalities hold simultaneously

$$\left|\frac{1}{\log(r^2)}\right| < \varepsilon^2, \tag{3.2}$$

$$\left|r\log(r^2)\right| < \varepsilon, \tag{3.3}$$

$$\frac{r^2}{1-r^2} < \varepsilon^2. \tag{3.4}$$

It is easy to see that all these are satisfied by all sufficiently small r.

For the special choice

$$z = \frac{1}{\sqrt{|\log(r^2)|}}, \zeta = \frac{-1}{(1+\varepsilon)\sqrt{|\log(r^2)|}},$$

(3.1) becomes

$$-\frac{1+\varepsilon}{\pi\frac{1}{\log(r^2)}\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}}{\left(-r^{2+2j}+\frac{1}{(1+\varepsilon)\log(r^2)}\right)^2} + \frac{r^{2j}}{\left(1-r^{2j}\frac{1}{(1+\varepsilon)\log(r^2)}\right)^2}\right).$$

One of course has to check that $r < |z|, |\zeta| < 1$, for sufficiently small r, to ensure that this special pair of points belongs to the annulus. This is obvious. Now consequently using the negativity of $\log(r)$, (3.3) and (3.4) one has

$$-\frac{1+\varepsilon}{\pi} + \pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}(1+\varepsilon)^2(\log(r^2))^2}{\left(1-r^{2+2j}(1+\varepsilon)\log(r^2)\right)^2} + \frac{r^{2j}}{\left(1-r^{2j}\frac{1}{(1+\varepsilon)\log(r^2)}\right)^2}\right) \leq \tag{3.5}$$

$$-\frac{1+\varepsilon}{\pi} + \pi^{-1}\sum_{j=0}^{\infty}\left(r^{2j}(1+\varepsilon)^2\varepsilon^2 + r^{2j}\right) =$$

$$-\frac{1+\varepsilon}{\pi}+\frac{1}{\pi}((1+\varepsilon)^2\varepsilon^2+1)+\frac{1}{\pi}\frac{r^2}{1-r^2}((1+\varepsilon)^2\varepsilon^2+1)\leq$$
$$\frac{-\varepsilon+((1+\varepsilon)^2(\varepsilon^2+1)+1)\varepsilon^2}{\pi}.$$

Clearly this is negative for sufficiently small ε.

Similarly for the special choice

$$z=\frac{1}{\sqrt{|\log(r^2)|}},\zeta=\frac{-1}{(1-\varepsilon)\sqrt{|\log(r^2)|}},$$

which is also good for small r, (3.1) becomes

(3.6)

$$-\frac{1-\varepsilon}{\pi\frac{1}{\log(r^2)}\log(r^2)}+\pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}}{\left(-r^{2+2j}+\frac{1}{(1-\varepsilon)\log(r^2)}\right)^2}+\frac{r^{2j}}{\left(1-r^{2j}\frac{1}{(1-\varepsilon)\log(r^2)}\right)^2}\right)=$$

$$-\frac{1-\varepsilon}{\pi}+\pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}(1-\varepsilon)^2(\log(r^2))^2}{(1-r^{2+2j}(1-\varepsilon)\log(r^2))^2}+\frac{r^{2j}}{\left(1-r^{2j}\frac{1}{(1-\varepsilon)\log(r^2)}\right)^2}\right)\geq$$

$$-\frac{1-\varepsilon}{\pi}+\frac{1}{\pi}\frac{r^0}{\left(1-r^0\frac{1}{(1-\varepsilon)\log(r^2)}\right)^2}\geq$$

$$-\frac{1-\varepsilon}{\pi}+\frac{1}{\pi}\frac{(1-\varepsilon)^2}{(1-\varepsilon+\varepsilon^2)^2},$$

by (3.2). Now expanding into Taylor series gives one $\left(\frac{1-x}{1-x+x^2}\right)^2=1-2x^2+o(x^2)$, hence our expression is approximately

$$\frac{\varepsilon-2\varepsilon^2}{\pi}>0,$$

for sufficiently small ε.

So $K(z,\zeta)$ is real for $z=\frac{1}{\sqrt{|\log(r^2)|}}$ and ζ from the interval

$$\left[\frac{-1}{(1-\varepsilon)\sqrt{|\log(r^2)|}},\frac{-1}{(1+\varepsilon)\sqrt{|\log(r^2)|}}\right]$$

and has different sign on the endpoints of this interval. Therefore it must have a zero there.

To compute the Bergman distance between z and ζ one has to find the Bergman metric first

$$\beta^2(z)=\frac{\partial^2\log K(z,z)}{\partial z\partial\overline{z}}=\frac{K(z,z)_{1\bar{1}}K(z,z)-K(z,z)_1K(z,z)_{\bar{1}}}{K(z,z)^2}. \tag{3.7}$$

Since this expression is invariant under rotations it will be enough to compute $\beta(z)$, for $z=\frac{1}{c\sqrt{|\log(r^2)|}}\in\mathbb{R}, c$ close to 1 or -1. The expressions involved in formula (3.7) are

$$K(z,z) = -\frac{1}{\pi z\overline{z}\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\left(\frac{r^{2+2j}}{(-r^{2+2j}+z\overline{z})^2} + \frac{r^{2j}}{(1-r^{2j}z\overline{z})^2}\right)$$

$$K(z,z)_1 = \frac{1}{\pi z^2\overline{z}\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\left(-\frac{2r^{2+2j}\overline{z}}{(-r^{2+2j}+z\overline{z})^3} + \frac{2r^{4j}\overline{z}}{(1-r^{2j}z\overline{z})^3}\right)$$

$$K(z,z)_{\overline{1}} = \frac{1}{\pi z\overline{z}^2\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\left(-\frac{2r^{2+2j}\overline{z}}{(-r^{2+2j}+z\overline{z})^3} + \frac{2r^{4j}\overline{z}}{(1-r^{2j}z\overline{z})^3}\right)$$

$$K(z,z)_{1\overline{1}} = -\frac{1}{\pi(z\overline{z})^2\log(r^2)} + \pi^{-1}\sum_{j=0}^{\infty}\Big(\frac{6r^{2+2j}z\overline{z}}{(-r^{2+2j}+z\overline{z})^4} - \frac{2r^{2+2j}}{(-r^{2+2j}+z\overline{z})^3} + \frac{6r^{6j}z\overline{z}}{(1-r^{2j}z\overline{z})^4} + \frac{2r^{4j}}{(1-r^{2j}z\overline{z})^3}\Big).$$

The expressions seem complicated, however almost every summand in the series above is negligible. To show this one proceeds similarly as in (3.5), (3.6).

$$\left|\frac{Ar^B\left(\frac{1}{c\sqrt{|\log(r^2)|}}\right)^D}{\left(-r^B - \frac{1}{c^2\log(r^2)}\right)^F}\right| = \left|\frac{Ar^B}{(c\sqrt{|\log(r^2)|})^{D-2F}(1+c^2\log(r^2)r^B)^F)}\right| \leq \tag{3.8}$$

$$\leq \left|\frac{Ar^B}{(c\sqrt{|\log(r^2)|})^{D-2F}(1-c^2\varepsilon)^F}\right|,$$

when $B > 0$.

The other terms are estimated by

$$\left|\frac{A'r^{B'}\left(\frac{1}{c\sqrt{|\log(r^2)|}}\right)^{D'}}{\left(1 + r^{E'}\frac{1}{c^2\log(r^2)}\right)^{F'}}\right| \leq \left|\frac{A'r^{B'}}{\left(c\sqrt{|\log(r^2)|}\right)^{D'}\left(1-\frac{\varepsilon^2}{c^2}\right)^{F'}}\right|. \tag{3.9}$$

Now because $r^B\log(r)^E$ tends to zero for positive B and arbitrary E one can employ either of the estimates (3.2),(3.3) and finally come with estimate of the sort $H|r|^G$. The only exception is clearly when B or $B' = 0$ i.e., $j = 0$, so adding up one gets a geometric power control on the series.

Now

$$K\left(\frac{1}{c\sqrt{|\log(r^2)|}}, \frac{1}{c\sqrt{|\log(r^2)|}}\right) = \frac{c^2}{\pi} + \frac{1}{\pi\left(1+\frac{1}{c^2\log(r^2)}\right)^2} + o(C) = \frac{c^2+1}{\pi} + o(C),$$

$$K_1 = K\left(\frac{1}{c\sqrt{|\log(r^2)|}}, \frac{1}{c\sqrt{|\log(r^2)|}}\right)_{\overline{1}} = -\frac{c^3\sqrt{|\log(r^2)|}}{\pi} + o(C),$$

$$K\left(\frac{1}{c\sqrt{|\log(r^2)|}}, \frac{1}{c\sqrt{|\log(r^2)|}}\right)_{1\overline{1}} = \frac{2}{\pi\left(1+\frac{1}{c^2\log(r^2)}\right)^3} - \frac{c^4\log(r^2)}{\pi} + o(C) =$$

$$= \frac{2 - c^4 \log(r^2)}{\pi} + o(C),$$

where the convention is $o(C) \equiv o(const)$.

Finally (3.7) becomes

$$\beta = \sqrt{\frac{(c^2+1+o(C))(2 - c^4\log(r^2) + o(C)) - (c^3\sqrt{|\log(r^2)|} + o(C))^2}{(c^2+1+o(C))^2}}.$$

The path which approximates the distance is as follows. First one joins ζ with the point $\frac{-1}{\sqrt{|\log(r^2)|}}$ via a linear segment. Then this point is joined with $z = \frac{1}{\sqrt{|\log(r^2)|}}$ via the half-circle

$$[0,1] \ni t \to e^{(\pi - \pi t)i}\frac{1}{\sqrt{|\log(r^2)|}}.$$

The segment will be denoted by γ_1 and the half-circle by γ_2.

The geodesics of the Bergman metric in the annulus are classified in [8] and one easily sees that our path is not a geodesic, however the integral distance over it is a close enough approximation.

The integrals that one has to assume are

$$I_1 := \int_0^1 \beta(\gamma_1(t)) \left|\frac{\partial \gamma_1}{\partial t}(t)\right| dt$$

$$I_2 := \int_0^1 \beta(\gamma_2(t)) \left|\frac{\partial \gamma_2}{\partial t}(t)\right| dt$$

Let $\zeta = \frac{-1}{s\sqrt{|\log(r^2)|}}$, $s \in [1-\varepsilon, 1+\varepsilon]$. The parametrization of γ_1 will be

$$[0,1] \ni t \to \frac{-1}{(s + t(1-s))\sqrt{|\log(r^2)|}}.$$

After straightforward computations one obtains

$$\int_0^1 \lim_{r\to 0^+} \beta(\gamma_2(t)) \left|\frac{\partial \gamma_2}{\partial t}(t)\right| dt =$$

$$\int_0^1 |s-1|\sqrt{\frac{c^4}{(c^2+1)^2c^2}}dt,$$

where $c = s + t(1-s)$. Now since $|s-1| < \varepsilon$ and since the integral is clearly finite we conclude that

$$\lim_{\varepsilon\to 0^+}\lim_{r\to 0^+} I_1 = 0.$$

$$\lim_{r\to 0^+} I_2(r) = \int_0^1 \lim_{r\to 0^+} \beta(\gamma_2(t)) \left|\frac{\partial \gamma_2}{\partial t}(t)\right| dt$$

In this case $c = 1$, β is constant on γ_2 and equals

$$\sqrt{\frac{(2+o(C))(2 - \log(r^2) + o(C)) - (\sqrt{|\log(r^2)|} + o(C))^2}{(2+o(C))^2}}.$$

Further $\frac{\partial \gamma_2}{\partial t} = -\pi i e^{(\pi - \pi t)i}\frac{1}{\sqrt{|\log(r^2)|}}$, hence

$$\beta(\gamma_2(t))\left|\frac{\partial \gamma_2}{\partial t}(t)\right| \approx \pi\sqrt{\frac{2(2-\log(r^2)) + \log(r^2)}{4}\frac{1}{|\log(r^2)|}} \to \frac{\pi}{2},$$

when $r \to 0$.

Now the Bergman distance between z and ζ when $r \to 0$ is bounded between $\frac{\pi}{2}$ by Theorem 0.1 and $I_1 + I_2$ and hence also tends to $\frac{\pi}{2}$. This proves Theorem 0.4.

Now we provide the example proving Theorem 0.5. As above we study the circular annulus P_r. To prove non-immersivity, following (1.14) and (1.15) one has to localize the zeros of $\det K^2(z,\zeta)\frac{\partial^2}{\partial z \partial \bar\zeta}\log K(z,\zeta)$, that is to say - of

$$K(z,\zeta)\frac{\partial^2}{\partial z\partial\bar\zeta}K(z,\zeta) - \frac{\partial}{\partial z}K(z,\zeta)\frac{\partial}{\partial\bar\zeta}K(z,\zeta) \tag{3.10}$$

Fix $0 < \varepsilon << 1$. Using arguments similar to 3.8,3.9 one can consider only the terms not containing an r to a positive power in the expansions of the above objects. That is to say

$$K(z,\zeta) \approx -\frac{1}{\pi}\frac{1}{z\bar\zeta\log(r^2)} + \frac{1}{\pi}\frac{1}{(1-z\bar\zeta)^2} \tag{3.11}$$

$$\frac{\partial}{\partial z}K(z,\zeta) \approx \frac{1}{\pi}\frac{1}{z^2\bar\zeta\log(r^2)} + \frac{1}{\pi}\frac{2\bar\zeta}{(1-z\bar\zeta)^3} \tag{3.12}$$

$$\frac{\partial}{\partial\bar\zeta}K(z,\zeta) \approx \frac{1}{\pi}\frac{1}{z\bar\zeta^2\log(r^2)} + \frac{1}{\pi}\frac{2z}{(1-z\bar\zeta)^3} \tag{3.13}$$

$$\frac{\partial^2}{\partial z\partial\bar\zeta}K(z,\zeta) \approx -\frac{1}{\pi}\frac{1}{z^2\bar\zeta^2\log(r^2)} + \frac{1}{\pi}\frac{6z\bar\zeta}{(1-z\bar\zeta)^4} + \frac{1}{\pi}\frac{2}{(1-z\bar\zeta)^3}, \tag{3.14}$$

of course for a suitable choice of z and ζ. In our case we fix ζ to be $\frac{1}{\sqrt[4]{|2\log(r^2)|}}$ and put $z = \frac{i}{\xi\sqrt[4]{|2\log(r^2)|}}$, where ξ is an independent of r complex variable presumably very close to 1.

Plugging these values in the expressions (3.11),(3.12),(3.13),(3.14), one sees that the dominant term in (3.11) will be $\frac{1}{\pi}\frac{1}{(1-z\bar\zeta)^2}$, i.e., $\frac{1}{\pi}\frac{1}{\left(1-\frac{i}{\xi\sqrt{|2\log(r^2)|}}\right)^2} \longrightarrow \frac{1}{\pi}$, when $r \to 0$. Both summands in both expressions (3.12) and (3.13) have the same asymptotic behavior $\sim \frac{const}{\sqrt[4]{|\log(r^2)|}} \to 0$. Finally in (3.14) the first and the last terms are dominating, summing up to

$$-\frac{2\xi^2}{\pi} + \frac{1}{\pi}\frac{2}{\left(1-\frac{i}{\xi\sqrt{|2\log(r^2)|}}\right)^3} \longrightarrow \frac{2-2\xi^2}{\pi}.$$

Back to expression (3.10) one sees that it can be written as $F(z,\zeta) + G(z,\zeta)$, where

$$F(z,\zeta) = \frac{1}{\pi}\frac{1}{(1-z\bar\zeta)^2}\left[-\frac{1}{\pi}\frac{1}{z^2\bar\zeta^2\log(r^2)} + \frac{1}{\pi}\frac{2}{(1-z\bar\zeta)^3}\right]$$

and $G(z,\zeta)$ is the sum of all the other expressions. By taking ξ on a circle of radius ε around 1 one sees that

$$\lim_{r\to 0^+}\left|F\left(\frac{i}{\xi\sqrt[4]{|2\log(r^2)|}}, \frac{1}{\sqrt[4]{|2\log(r^2)|}}\right)\right| = \left|\frac{2-2\xi^2}{\pi}\right| > 0,$$

whereas

$$\lim_{r\to 0^+}\left|G\left(\frac{i}{\xi\sqrt[4]{|2\log(r^2)|}},\frac{1}{\sqrt[4]{|2\log(r^2)|}}\right)\right|=0.$$

By Rouché's theorem

$$K(z,\zeta)\frac{\partial^2}{\partial z\partial\bar\zeta}K(z,\zeta)-\frac{\partial}{\partial z}K(z,\zeta)\frac{\partial}{\partial\bar\zeta}K(z,\zeta)\Big|_{z=\frac{i}{\xi\sqrt[4]{|2\log(r^2)|}},\zeta=\frac{1}{\sqrt[4]{|2\log(r^2)|}}},$$

as a holomorphic function of ξ, has the same number of zeros in $\{z:|z-1|<\varepsilon\}$ as

$$F(z,\zeta)\Big|_{z=\frac{i}{\xi\sqrt[4]{|2\log(r^2)|}},\zeta=\frac{1}{\sqrt[4]{|2\log(r^2)|}}},$$

provided that r is sufficiently small. Now solving

$$\frac{2}{\left(1-\frac{i}{\xi\sqrt{|2\log(r^2)|}}\right)^3}=2\xi^2$$

one can check that

$$\frac{i}{c}+\frac{(1+i\sqrt3)c^3}{\sqrt[3]{12(-9ic^8+\sqrt3\sqrt{-27c^{16}-4c^{18}})}}+\frac{(1-i\sqrt3)\sqrt[3]{-9ic^8+\sqrt3\sqrt{-27c^{16}-4c^{18}}}}{c^3 2\sqrt[3]{18}},$$

where $c=\sqrt{|2\log(r^2)|}$ is a solution of this equation, which lies in $\{z:|z-1|<\varepsilon\}$ for sufficiently small r.

Next we sketch the rest of the proof without going into the (very technical) calculational details. Once localized, the zeros of $\frac{\partial^2}{\partial z\partial\bar\zeta}\log K(z,\zeta)$ are joined by a path $\tilde\gamma_1\cup\tilde\gamma_2$, $\tilde\gamma_1$ being the linear segment joining z with the point $\frac{i}{\sqrt[4]{|2\log(r^2)|}}$. As in the proof of Theorem0.4 the integral distance over $\tilde\gamma_1$ with respect to the metric $\tilde\beta(.,\circ)$ tends to 0 when $r\to 0$ and $\varepsilon\to 0$. Now $\tilde\gamma_2$ will be the arc

$$[0,1]\ni t\longrightarrow\frac{1}{\sqrt[4]{|2\log(r^2)|}}e^{\frac{1-t}{2}\pi i}.$$

Again we use the fact that $\tilde\beta$ is constant in the first variable along this arc, by conformal invariance. It is therefore enough to compute the metric tensor of $\tilde\beta$ only at the point $\frac{1}{\sqrt[4]{|2\log(r^2)|}}=\tilde\gamma_2(1)$. Recall that

$$\tilde\beta^2(\zeta,X)=\left[2\frac{\partial^2}{\partial\zeta\partial\bar\zeta}K(\zeta,\zeta)+\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\log\frac{\partial^2}{\partial\zeta\partial\bar\zeta}K(\zeta,\zeta)\right]|X|^2$$

and hence one has to compute

$$\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\log\left[K(\zeta,\zeta)^2\frac{\partial^2}{\partial\zeta\partial\bar\zeta}K(\zeta,\zeta)\right]=$$

$$\frac{\partial^2}{\partial\zeta\partial\bar\zeta}\log\left[K(\zeta,\zeta)\frac{\partial^2}{\partial\zeta\partial\bar\zeta}K(\zeta,\zeta)-\frac{\partial}{\partial\zeta}K(\zeta,\zeta)\frac{\partial}{\partial\bar\zeta}K(\zeta,\zeta)\right]=$$

$$\frac{K(\zeta,\zeta)_1K(\zeta,\zeta)_{\bar1}K(\zeta,\zeta)_{11}K(\zeta,\zeta)_{\bar1\bar1}}{D^2}-\frac{K(\zeta,\zeta)_{11}K(\zeta,\zeta)_{\bar1\bar1}}{D}+ \tag{3.15}$$

$$\frac{K(\zeta,\zeta)K(\zeta,\zeta)_{\bar1}K(\zeta,\zeta)_{1\bar1\bar1}K(\zeta,\zeta)_{11}}{D^2}+\frac{K(\zeta,\zeta)K(\zeta,\zeta)_1K(\zeta,\zeta)_{11\bar1}K(\zeta,\zeta)_{\bar1\bar1}}{D^2}-$$

$$-\frac{K(\zeta,\zeta)^2K(\zeta,\zeta)_{1\bar{1}\bar{1}}K(\zeta,\zeta)_{11\bar{1}}}{D^2}+\frac{K(\zeta,\zeta)K(\zeta,\zeta)_{11\bar{1}\bar{1}}}{D},$$

where the denominator $D = K(\zeta,\zeta)K(\zeta,\zeta)_{1\bar{1}} - K(\zeta,\zeta)_1K(\zeta,\zeta)_{\bar{1}}$. We first obtain the asymptotic of D. Much of the analysis from (3.11)-(3.14) can be repeated to show that

$$K(\zeta,\zeta) \approx \frac{1}{\pi}$$

$$K(\zeta,\zeta)_1 = K(\zeta,\zeta)_{\bar{1}}, \text{ tends to 0 faster than } \frac{1}{\sqrt[4]{|\log(r^2)|}}$$

$$K(\zeta,\zeta)_{1\bar{1}} \approx \frac{4}{\pi}$$

The change of sign from $\approx \frac{2-2\xi^2}{\pi}$ to $\approx \frac{4}{\pi}$ is due to the absence of i in the value of $z=\zeta$. Hence $D \to \frac{4}{\pi^2}$.

Similarly it can be shown that

$$K(\zeta,\zeta)_{11} = K(\zeta,\zeta)_{\bar{1}\bar{1}} \approx \frac{4}{\pi}$$

$$K(\zeta,\zeta)_{11\bar{1}} = K(\zeta,\zeta)_{1\bar{1}\bar{1}} \approx -\frac{4\sqrt[4]{2}\sqrt[4]{|\log(r^2)|}}{\pi}$$

$$K(\zeta,\zeta)_{11\bar{1}\bar{1}} \approx \frac{8\sqrt{2}\sqrt{|\log(r^2)|}}{\pi}$$

This gives one that only the last two terms in the expression (3.15) are relevant in the asymptotic behavior of the metric tensor $\tilde{\beta}$ which is

$$\approx -\frac{\left(\frac{1}{\pi}\right)^2\frac{(-4\sqrt[4]{2})^2\sqrt{|\log(r^2)|}}{\pi^2}}{\left(\frac{4}{\pi^2}\right)^2}+\frac{\frac{1}{\pi}\frac{8\sqrt{2}\sqrt{|\log(r^2)|}}{\pi}}{\frac{4}{\pi^2}} = \sqrt{2}\sqrt{|\log(r^2)|}.$$

Now $\frac{\partial}{\partial t}\tilde{\gamma}_2(t) = \frac{1}{\sqrt[4]{|2\log(r^2)|}}e^{\frac{1-t}{2}\pi i}\left(-\frac{1}{2}\pi i\right)$. Finally the integral distance over $\tilde{\gamma}_2$ with respect to $\tilde{\beta}$ is

$$\int_0^1\sqrt{\tilde{\beta}\left(\tilde{\gamma}_2(t),\frac{\partial}{\partial t}\tilde{\gamma}_2(t)\right)} \approx \int_0^1\sqrt{\sqrt{2}\sqrt{|\log(r^2)|}\frac{1}{\sqrt{|2\log(r^2)|}}\frac{\pi^2}{4}} \to \frac{\pi}{2}, \tag{3.16}$$

when $r \to 0$. Now by (1.16), the estimate of the distance over $\tilde{\gamma}_1$ and (3.16), one has that $\tilde{dist}(z,\zeta) \to \frac{\pi}{2}$, which establishes the claim.

References

[1] S. Berceanu, *On the geometry of complex Grassmann manifold, its noncompact dual and coherent states*, Bull. Belg. Math. Soc. Simon Stevin **4** (1997), no. 2, 205–243. MR MR1443974 (98j:53058)

[2] Stefan Bergmann, *Über die Existenz von Repräsentantenbereichen in der Theorie der Abbildung durch Paare von Funktionen zweier komplexen Veränderlichen*, Math. Ann. **102** (1930), no. 1, 430–446. MR MR1512585

[3] Zbigniew Błocki, *The complex Monge-Ampère operator in pluripotential theory*, unfinished lecture notes.

[4] ———, *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. **357** (2005), no. 7, 2613–2625 (electronic). MR MR2139520 (2006d:32013)

[5] Harold P. Boas, *The Lu Qi-Keng conjecture fails generically*, Proc. Amer. Math. Soc. **124** (1996), no. 7, 2021–2027. MR MR1317032 (96i:32024)

[6] J. Davidov, *The representative domain of a complex manifold and the Lu-Qi-Keng conjecture*, C. R. Acad. Bulgare Sci. **30** (1977), no. 1, 13–16. MR MR0466638 (57 #6515)

[7] Zywomir Dinew, *An example for the holomorphic sectional curvature of the bergman metric*, 2009.
[8] Gregor Herbort, *On the geodesics of the Bergman metric*, Math. Ann. **264** (1983), no. 1, 39–51. MR MR709860 (85f:32030)
[9] Sadao Kató, *Canonical domains in several complex variables*, Pacific J. Math. **21** (1967), 279–291. MR MR0214810 (35 #5659)
[10] Kang-Tae Kim and Steven G. Krantz, *The Bergman metric invariants and their boundary behavior*, Explorations in complex and Riemannian geometry, Contemp. Math., vol. 332, Amer. Math. Soc., Providence, RI, 2003, pp. 139–151. MR MR2018337 (2004i:32019)
[11] Shoshichi Kobayashi, *Geometry of bounded domains*, Trans. Amer. Math. Soc. **92** (1959), 267–290. MR MR0112162 (22 #3017)
[12] Steven G. Krantz, *Geometric analysis and function spaces*, CBMS Regional Conference Series in Mathematics, vol. 81, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1993. MR MR1228447 (94g:32001)
[13] Steven G. Krantz and Jiye Yu, *On the Bergman invariant and curvatures of the Bergman metric*, Illinois J. Math. **40** (1996), no. 2, 226–244. MR MR1398092 (97g:32026)
[14] Kurt Leichtweiss, *Zur Riemannschen Geometrie in Grassmannschen Mannigfaltigkeiten*, Math. Z. **76** (1961), 334–366. MR MR0126808 (23 #A4102)
[15] Joseph Lewittes, *Differentials and metrics on Riemann surfaces*, Trans. Amer. Math. Soc. **139** (1969), 311–318. MR MR0237769 (38 #6050)
[16] QiKeng Lu, *Holomorphic invariant forms of a bounded domain*, Sci. China Ser. A **51** (2008), no. 11, 1945–1964. MR MR2447420
[17] Lu Qi-keng, *On Kaehler manifolds with constant curvature*, Chinese Math.–Acta **8** (1966), 283–298. MR MR0206990 (34 #6806)
[18] Paul Rosenthal, *On the zeros of the Bergman function in doubly-connected domains*, Proc. Amer. Math. Soc. **21** (1969), 33–35. MR MR0239066 (39 #425)
[19] M. Skwarczyński, *The invariant distance in the theory of pseudoconformal transformations and the Lu Qi-keng conjecture*, Proc. Amer. Math. Soc. **22** (1969), 305–310. MR MR0244512 (39 #5826)
[20] Wlodzimierz Zwonek, *Asymptotic behaviour of the sectional curvature of the bergman metric for annuli*, 2009.

Jagiellonian University, Institute of Mathematics, Łojasiewicza 6, 30-348 Kraków, Poland
E-mail address: Zywomir.Dinew@im.uj.edu.pl