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Combinatorial Bases of Modules for Affine Lie Algebra $B_2^{(1)}$

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ABSTRACT. In this paper we construct bases of standard modules $L(\Lambda)$ for affine Lie algebra of type $B_2^{(1)}$ consisting of semi-infinite monomials. The main technical ingredient is a construction of monomial bases for Feigin-Stoyanovsky's subspaces $W(\Lambda)$ of $L(\Lambda)$ by using simple currents and intertwining operators in vertex operator algebra theory. By coincidence $W(k\Lambda_0)$ for $B_2^{(1)}$ and the standard module $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same presentation $\mathcal{P}/\mathcal{I}$, so our main theorem provides a new proof of linear independence of monomial bases of $A_1^{(1)}$-modules $L(k\Lambda_0)$.

1. INTRODUCTION

B.L. Feigin and A.V. Stoyanovsky gave in [FS] a construction of bases of standard modules $L(\Lambda)$ for affine Lie algebra $\tilde{\mathfrak{g}}$ of type $A_1^{(1)}$ consisting of semi-infinite monomials. In [P1] such a construction was given for all standard modules for affine Lie algebras of type $A_n^{(1)}$, and it turned out that basis elements are closely related to $(k, n+1)$-admissible configurations — combinatorial objects introduced and studied in a series of papers [FJLMM]–[FJMMT]. On the other side, for any classical simple Lie algebra $\mathfrak{g}$ such bases are constructed in [P2] for basic modules $L(\Lambda_0)$. In a proof of linear independence a crystal base character formula [KKMMNN] is used, but it was not clear "why" this proof works and how such approach could be extended to higher level standard modules. A new understanding came from the works of G. Georgiev [G] and S. Capparelli, J. Lepowsky and A. Milas [CLM1] and [CLM2] based on a general idea of J. Lepowsky to use intertwining vertex operators to build bases of standard modules and obtain Rogers-Ramanujan-type recursions for their graded dimensions. Their way of using intertwining operators inspired simpler proof of linear independence for $A_n^{(1)}$ in [P3] and new constructions for $A_n^{(1)}$ in [T] and $D_4^{(1)}$ in [Ba]. In this paper we use Capparelli-Lepowsky-Milas' approach to extend the construction in [P2] to all standard modules $L(\Lambda)$ for affine Lie algebra $\tilde{\mathfrak{g}}$ of type $B_2^{(1)}$, and, along the way, we obtain a presentation theorem for Feigin-Stoyanovsky's subspaces and a new proof of linear independence of monomial bases of $A_1^{(1)}$-modules $L(k\Lambda_0)$ constructed in [MP1], [MP2] and [FKLMM]. The underlying structure of Feigin-Stoyanovsky's subspaces is parallel to the structure of principal subspaces studied, for example, in [G], [Cal], [CalLM] and [AKS].

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Let $\mathfrak{g}$ be a simple complex Lie algebra of type B_2, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$$

a $\mathbb{Z}$-grading of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{g}_0$. We fix a basis of $\mathfrak{g}_1$ consisting of root vectors denoted as

$$x_{\underline{2}}\,, x_0\,, x_2\,.$$

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] + \mathbb{C}c + \mathbb{C}d$ be the associated affine Lie algebra with the canonical central element c. For $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$ we write $x(n) = x \otimes t^n$. Then for integral dominant weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$$

of level $k = k_0 + k_1 + k_2$ a basis of standard module $L(\Lambda)$ can be parametrized with semi-infinite monomials

$$\prod_{j\in\mathbb{Z}} x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j}\,, \qquad c_j = b_j = a_j = 0 \quad \text{for} \quad -j \ll 0, \tag{1.1}$$

with quasi-periodic tail

$$\begin{aligned} &(\ldots,\, c_{-2n},\, b_{-2n},\, a_{-2n},\, c_{-2n-1},\, b_{-2n-1},\, a_{-2n-1},\, \ldots) \\ &= (\ldots,\, k_1,\, k_2,\, k_1,\, k_0,\, k_2,\, k_0,\, \ldots) \end{aligned}$$

for $n \gg 0$, satisfying for all $j \in \mathbb{Z}$ the so called difference conditions

$$\begin{aligned} c_{j+1} + b_{j+1} + c_j &\le k, \\ b_{j+1} + a_{j+1} + c_j &\le k, \\ a_{j+1} + c_j + b_j &\le k, \\ a_{j+1} + b_j + a_j &\le k. \end{aligned} \tag{1.2}$$

This is the Corollary 8.2 of Theorem 8.1. The main technical ingredient in a proof is a construction of monomial bases for Feigin-Stoyanovsky's subspaces defined as

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1)v_\Lambda \subset L(\Lambda),$$

where $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_1 \otimes \mathbb{C}[t, t^{-1}]$ and v_Λ is a highest weight vector in $L(\Lambda)$. By Theorem 3.1 the constructed basis for level k subspace $W(\Lambda)$ consists of finite monomials of the form (1.1) with $-j \le -1$, satisfying difference conditions (1.2) and the so called initial conditions

$$a_1 \le k_0, \quad b_1 + a_1 \le k_0 + k_2 \quad \text{and} \quad c_1 + b_1 \le k_0 + k_2.$$

Another consequence of this result is Theorem 9.1 which gives a presentation

$$W(\Lambda) \cong \mathcal{P}/\mathcal{I}_\Lambda,$$

where $\mathcal{P}$ is a polynomial algebra $\mathbb{C}[x_{\underline{2}}(j), x_0(j), x_2(j) \mid j \le -1]$ and $\mathcal{I}_\Lambda$ is the ideal generated by the set of polynomials

$$\bigcup_{n\le -k-1} U(\mathfrak{g}_0)\cdot\Big(\sum_{\substack{j_1,\ldots,j_{k+1}\le -1\\ j_1+\cdots+j_{k+1}=n}} x_2(j_1)\ldots x_2(j_{k+1})\Big)$$

$$\bigcup\{x_2(-1)^{k_0+1}\}\bigcup U(\mathfrak{g}_0)\cdot x_2(-1)^{k_0+k_2+1}.$$

¿From Theorems 9.1 and 10.1 we see that Feigin-Stoyanovsky's subspace $W(k\Lambda_0)$ for $B_2^{(1)}$ and the standard module $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same presentation $\mathcal{P}/\mathcal{I}$, so Theorem 3.1 provides a new proof of linear independence of monomial bases of $A_1^{(1)}$-modules $L(k\Lambda_0)$, originally proved by different methods in [MP1], [MP2] and

[FKLMM]. Due to this coincidence E. Feigin's fermionic formula [F] for $A_1^{(1)}$-module $L(k\Lambda_0)$ is also a character formula of Feigin-Stoyanovsky's subspace $W(k\Lambda_0)$ for $B_2^{(1)}$.

As it was already said, in our construction we use simple currents and intertwining operators for vertex operator algebra $L(\Lambda_0)$ associated with the affine Lie algebra $\tilde{\mathfrak{g}}$ at level 1. To be more precise, we use results in [DLM] and [L2] to see the existence of level 1 "simple current operators"

$$L(\Lambda_0) \xrightarrow{[\omega]} L(\Lambda_1) \xrightarrow{[\omega]} L(\Lambda_0), \qquad L(\Lambda_2) \xrightarrow{[\omega]} L(\Lambda_2)$$

which are linear bijections with the crucial property

$$x(n)[\omega] = [\omega]x(n+1) \quad \text{for all} \quad x(n) \in \tilde{\mathfrak{g}}_1. \tag{1.3}$$

¿From [L1] we have fusion rules

$$\dim \begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_2) & & L(\Lambda_0) \end{pmatrix} = 1 \qquad \text{and} \qquad \begin{pmatrix} & L(\Lambda_1) & \\ L(\Lambda_2) & & L(\Lambda_2) \end{pmatrix} = 1$$

from which we deduce that there are coefficients $[\omega_2]$ and $[\omega_{\underline{2}}]$ of intertwining operators

$$L(\Lambda_0) \xrightarrow{[\omega_2]} L(\Lambda_2) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_1), \quad v_{\Lambda_0} \xrightarrow{[\omega_2]} v_{\Lambda_2} \xrightarrow{[\omega_{\underline{2}}]} v_{\Lambda_1}, \quad [\omega_{\underline{2}}]w_{\underline{2}} = 0,$$
$$L(\Lambda_0) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_2) \xrightarrow{[\omega_2]} L(\Lambda_1), \quad v_{\Lambda_0} \xrightarrow{[\omega_{\underline{2}}]} w_{\underline{2}} \xrightarrow{[\omega_2]} v_{\Lambda_1}, \quad [\omega_2]v_{\Lambda_2} = 0$$

which commute with the action of $\tilde{\mathfrak{g}}_1$. We consider higher level standard modules as submodules of tensor products of level 1 modules

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_1)^{\otimes k_1} \otimes L(\Lambda_2)^{\otimes k_2}.$$

Behind all combinatorial properties of our construction seems to be relation (5.3) for $[\omega]v_\Lambda$, written in terms of tensor products of level 1 highest weight vectors as

$$\begin{aligned} &[\omega]\left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\right) \\ &= \left([\omega]v_{\Lambda_0}\right)^{\otimes k_0} \otimes \left([\omega]v_{\Lambda_1}\right)^{\otimes k_1} \otimes \left([\omega]v_{\Lambda_2}\right)^{\otimes k_2} \\ &= C\, x_{\underline{2}}(-1)^{k_1} x_0(-1)^{k_2} x_2(-1)^{k_1} \left(v_{\Lambda_1}^{\otimes k_0} \otimes v_{\Lambda_0}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\right). \end{aligned} \tag{1.4}$$

In particular, it is this relation that for level 1 modules makes the use of crystal base character formula [KKMMNN] in [P2] possible.

Very roughly speaking, we prove linear independence by induction on degree of basis elements in two steps: for monomial vectors $x(\pi)v_\Lambda$, which appear with nontrivial coefficients $c_\pi \neq 0$ in a linear combination $\sum c_\pi x(\pi)v_\Lambda = 0$, we first use intertwining operators $x(\pi)v_\Lambda \to x(\pi)v_{\Lambda'}$ to be able to apply formula (1.4) to vectors $x(\pi)v_{\Lambda'}$ and get a combination of monomial vectors of the form $\sum c_\pi x(\pi')[\omega]v_{\Lambda''}$. Then, as a second step, we commute $[\omega]$ to the left and, by using (1.3) and induction hypothesis, we get that c_π equals zero—a contradiction. Of course, the actual argument is a bit more complicated and, as in [Ba], we have to use two basis elements of 4-dimensional spinor $\mathfrak{g}$-module on the top of $L(\Lambda_2)$ and the corresponding coefficients $[\omega_2]$ and $[\omega_{\underline{2}}]$ of intertwining operators.

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2. Affine Lie algebra of type $B_2^{(1)}$

Let $\mathfrak{g}$ be a complex simple Lie algebra of type B_2 and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The corresponding root system R may be realized in $\mathbb{R}^2$ with the canonical basis $\varepsilon_1, \varepsilon_2$ as

$$R = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 + \varepsilon_2)\} \cup \{\pm\varepsilon_1, \pm\varepsilon_2\}.$$

We fix simple roots $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2$ and denote by $\omega_1 = \varepsilon_1$ and $\omega_2 = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)$ the corresponding fundamental weights. Note that $\theta = \varepsilon_1 + \varepsilon_2$ is the maximal root. Set

$$\Gamma = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1, \varepsilon_1 + \varepsilon_2\}.$$

Denote by $\langle \cdot, \cdot \rangle$ the normalized Killing form such that $\langle \theta, \theta \rangle = 2$, where we identify $\mathfrak{h} \cong \mathfrak{h}^*$ via $\langle \cdot, \cdot \rangle$. We fix

$$\omega = \omega_1 = \varepsilon_1.$$

Then we have $\alpha(\omega) = \langle \alpha, \omega \rangle$ and

$$\Gamma = \{\alpha \in R \mid \alpha(\omega) = 1\}.$$

Obviously we have a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ for

$$\mathfrak{g}_0 = \mathfrak{h} + \sum_{\alpha(\omega)=0} \mathfrak{g}_\alpha = \mathfrak{h} + \mathbb{C}x_{\varepsilon_2} + \mathbb{C}x_{-\varepsilon_2}, \qquad \mathfrak{g}_{\pm 1} = \sum_{\alpha \in \pm\Gamma} \mathfrak{g}_\alpha.$$

Clearly $\mathfrak{g}_1$ is an irreducible 3-dimensional $\mathfrak{g}_0$-module. We shall briefly write

$$\underline{2} = \varepsilon_1 - \varepsilon_2, \quad 0 = \varepsilon_1, \quad 2 = \varepsilon_1 + \varepsilon_2$$

so that $\Gamma = \{\underline{2}, 0, 2\}$, a notation as in [P2] and [Ba]. For each root α fix a root vector x_α. For $\alpha = \underline{2}, 0, 2$ we shall write respectively x_α as

$$x_{\underline{2}}, x_0, x_2.$$

These vectors form a basis of $\mathfrak{g}_0$-module $\mathfrak{g}_1$.

Denote by $\tilde{\mathfrak{g}}$ the affine Lie algebra of type $B_2^{(1)}$ associated to $\mathfrak{g}$,

$$\tilde{\mathfrak{g}} = \coprod_{n \in \mathbb{Z}} \mathfrak{g} \otimes t^n + \mathbb{C}c + \mathbb{C}d$$

with the canonical central element c and the degree element d such that $[d, x \otimes t^n] = n\, x \otimes t^n$. Set

$$\tilde{\mathfrak{g}}_{<0} = \coprod_{n<0} \mathfrak{g} \otimes t^n, \qquad \tilde{\mathfrak{g}}_{\leq 0} = \coprod_{n \leq 0} \mathfrak{g} \otimes t^n + \mathbb{C}c + \mathbb{C}d.$$

Let α_0, α_1 and α_2 be simple roots of $\tilde{\mathfrak{g}}$ with the root subspaces $\mathfrak{g}_{-\theta} \otimes t^1$, $\mathfrak{g}_{\alpha_1} \otimes t^0$ and $\mathfrak{g}_{\alpha_2} \otimes t^0$ respectively, and let Λ_0, Λ_1 and Λ_2 be the corresponding fundamental weights of $\tilde{\mathfrak{g}}$ (cf. [K]). We write

$$x(n) = x \otimes t^n$$

for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$ and denote by $x(z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1}$ a formal Laurent series in formal variable z. For

$$\tilde{\mathfrak{g}}_0 = \coprod_{n \in \mathbb{Z}} \mathfrak{g}_0 \otimes t^n + \mathbb{C}c + \mathbb{C}d, \qquad \tilde{\mathfrak{g}}_{\pm 1} = \coprod_{n \in \mathbb{Z}} \mathfrak{g}_{\pm 1} \otimes t^n$$

we have $\mathbb{Z}$-grading $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_{-1} + \tilde{\mathfrak{g}}_0 + \tilde{\mathfrak{g}}_1$. In particular, $\tilde{\mathfrak{g}}_1$ is a commutative Lie subalgebra of $\tilde{\mathfrak{g}}$ with a basis

$$\widetilde{\Gamma} = \{x_{\underline{2}}(n), x_0(n), x_2(n) \mid n \in \mathbb{Z}\} = \{x_\gamma(n) \mid \gamma \in \Gamma, n \in \mathbb{Z}\}.$$

On $\widetilde{\Gamma}$ we use linear order

$$\cdots \prec x_2(n-1) \prec x_{\underline{2}}(n) \prec x_0(n) \prec x_2(n) \prec x_{\underline{2}}(n+1) \prec \dots .$$

3. Feigin-Stoyanovsky's subspaces $W(\Lambda)$

Denote by $L(\Lambda)$ a standard $\tilde{\mathfrak{g}}$-module with a dominant integral highest weight

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2,$$

$k_0, k_1, k_2 \in \mathbb{Z}_+$. Throughout the paper we denote by $k = \Lambda(c)$ the level of $\tilde{\mathfrak{g}}$-module $L(\Lambda)$,

$$k = k_0 + k_1 + k_2$$

(cf. [K]). For each fundamental $\tilde{\mathfrak{g}}$-module $L(\Lambda_i)$ fix a highest weight vector v_{Λ_i}. By complete reducibility of tensor products of standard modules, for level $k > 1$ we have

$$L(\Lambda) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_1)^{\otimes k_1} \otimes L(\Lambda_2)^{\otimes k_2}$$

with a highest weight vector

$$v_\Lambda = v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}.$$

Later on we shall also realize $L(\Lambda)$ in a symmetric algebra

$$L(\Lambda) \subset S^k(L(\Lambda_0) \oplus L(\Lambda_1) \oplus L(\Lambda_2)), \qquad v_\Lambda = v_{\Lambda_0}^{k_0} v_{\Lambda_1}^{k_1} v_{\Lambda_2}^{k_2}.$$

We set $d\,v_\Lambda = 0$. Then $L(\Lambda)$ is $\mathbb{Z}$-graded by the degree operator d,

$$L(\Lambda) = L(\Lambda)_0 + L(\Lambda)_{-1} + L(\Lambda)_{-2} + \dots ,$$

and we say that $\mathfrak{g}$-module $L(\Lambda)_0 = U(\mathfrak{g})v_\Lambda$ is the "top" of $L(\Lambda)$. The top of $L(\Lambda_0)$ is trivial $\mathfrak{g}$-module $\mathbb{C}v_{\Lambda_0}$, the top of $L(\Lambda_1)$ is 5-dimensional vector representation $L(\omega_1)$ and the top of $L(\Lambda_2)$ is 4-dimensional spinor $\mathfrak{g}$-module $L(\omega_2)$.

For each integral dominant Λ we have a Feigin-Stoyanovsky's subspace

$$W(\Lambda) = U(\tilde{\mathfrak{g}}_1)v_\Lambda \subset L(\Lambda).$$

Denote by $\pi\colon \{x_\gamma(-j) \mid \gamma \in \Gamma, j \geq 1\} \to \mathbb{Z}_+$ a "colored partition" for which a finite number of "parts" $x_\gamma(-j)$ (of degree j and color γ) appear $\pi(x_\gamma(-j))$ times, and denote by

$$x(\pi) = \prod x_\gamma(-j)^{\pi(x_\gamma(-j))} \in U(\tilde{\mathfrak{g}}_1) = S(\tilde{\mathfrak{g}}_1)$$

the corresponding monomials. We can identify π with a sequence

$$a_1, b_1, c_1, a_2, b_2, c_2, \dots$$

with finitely many non-zero terms $a_j = \pi(x_2(-j))$, $b_j = \pi(x_0(-j))$, $c_j = \pi(x_{\underline{2}}(-j))$ and

$$x(\pi) = \ \dots\ x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j} \dots\ x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1} .$$

For monomial $x(\pi)$ we say that $x(\pi)v_\Lambda \in W(\Lambda)$ is a monomial vector. The main result of this paper is the following:

Theorem 3.1. *The set of monomial vectors $x(\pi)v_\Lambda$ satisfying difference conditions*

$$\begin{aligned} c_{j+1} + b_{j+1} + c_j &\leq k, \\ b_{j+1} + a_{j+1} + c_j &\leq k, \\ a_{j+1} + c_j + b_j &\leq k, \\ a_{j+1} + b_j + a_j &\leq k \end{aligned} \tag{3.1}$$

for all $j \geq 1$*, and initial conditions*

$$a_1 \leq k_0, \quad b_1 + a_1 \leq k_0 + k_2, \quad c_1 + b_1 \leq k_0 + k_2, \tag{3.2}$$

is a basis of level k *Feigin-Stoyanovsky's subspace* $W(\Lambda)$*.*

4. Difference conditions and initial conditions

By Poincaré-Birkhoff-Witt theorem we have a spanning set of monomial vectors $x(\pi)v_\Lambda$ in Feigin-Stoyanovsky's level k subspace $W(\Lambda)$. To reduce this spanning set to a basis described in Theorem 3.1 we use vertex operator algebra relations

$$x_\theta(z)^{k+1} = \sum_{n\in\mathbb{Z}} \left(\sum_{j_1+\cdots+j_{k+1}=n} x_\theta(j_1)\dots x_\theta(j_{k+1}) \right) z^{-n-k-1} = 0 \quad \text{on} \quad L(\Lambda)$$

and it's consequences $U(\mathfrak{g}_0) \cdot x_\theta(z)^{k+1} = 0$, where $\cdot$ denotes the adjoint action of $\mathfrak{g}_0$ on $\tilde{\mathfrak{g}}_1$. This is a $(2k+1)$-dimensional $\mathfrak{g}_0$-module of relations which hold on $W(\Lambda)$, and coefficients of such relations are (infinite) sums

$$\sum c_\kappa x(\kappa) = 0 \quad \text{on} \quad W(\Lambda).$$

By choosing a proper order on the set of monomials, for each sum we can determine the smallest term $x(\kappa)$, the so called leading term, which can be replaced on $W(\Lambda)$ by a sum of higher (bigger) terms. The list of leading terms is

$$\begin{aligned}
x_{\underline{2}}(-j-1)^{c_{j+1}} x_0(-j-1)^{b_{j+1}} x_{\underline{2}}(-j)^{c_j}, &\quad c_{j+1} + b_{j+1} + c_j = k+1,\\
x_0(-j-1)^{b_{j+1}} x_2(-j-1)^{a_{j+1}} x_{\underline{2}}(-j)^{c_j}, &\quad b_{j+1} + a_{j+1} + c_j = k+1,\\
x_2(-j-1)^{a_{j+1}} x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j}, &\quad a_{j+1} + c_j + b_j = k+1,\\
x_2(-j-1)^{a_{j+1}} x_0(-j)^{b_j} x_2(-j)^{a_j}, &\quad a_{j+1} + b_j + a_j = k+1
\end{aligned} \tag{4.1}$$

for all $j \geq 1$. So by induction we see that $W(\Lambda)$ is spanned by monomial vectors $x(\pi)v_\Lambda$ which don't have factors of the form (4.1), i.e., by monomial vectors which satisfy difference conditions (3.1) (for details of this argument see [LP], [MP2], [P2] or [FKLMM]).

Lemma 4.1. $x_2(-1)v_{\Lambda_1} = x_0(-1)v_{\Lambda_1} = x_{\underline{2}}(-1)v_{\Lambda_1} = 0.$

Proof. For $\alpha \in R$ denote by $\mathfrak{sl}_2(\alpha) \subset \mathfrak{g}$ a Lie subalgebra generated with x_α and $x_{-\alpha}$ and by

$$\widetilde{\mathfrak{sl}}_2(\alpha) = \coprod_{n\in\mathbb{Z}} \mathfrak{sl}_2(\alpha) \otimes t^n + \mathbb{C}c + \mathbb{C}d \subset \tilde{\mathfrak{g}}$$

denote the corresponding affine Lie algebra of type $A_1^{(1)}$. Note that for level one $\tilde{\mathfrak{g}}$-module V the restriction to $\widetilde{\mathfrak{sl}}_2(\alpha)$ is level one representation if α is a long root, and it is level two representation if α is a short root. Also note that $U(\widetilde{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ is a standard $A_1^{(1)}$-module and that its $\mathfrak{sl}_2(\alpha)$-submodule on the top is a submodule of 5-dimensional vector representation for B_2.

In the case $\alpha = \varepsilon_1 - \varepsilon_2 = \underline{2}$ we have level one representation on $U(\widetilde{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ with 2-dimensional $\mathfrak{sl}_2(\alpha)$-module on the top, so it must be the standard $A_1^{(1)}$-module $L(\Lambda_1)$. Hence $x_{\varepsilon_1-\varepsilon_2}(-1)v_{\Lambda_1} = 0$. Similarly $x_\alpha(-1)v_{\Lambda_1} = 0$ for $\alpha = \varepsilon_1 + \varepsilon_2$. On the other hand in the case $\alpha = \varepsilon_1$ we have level two representation on $U(\widetilde{\mathfrak{sl}}_2(\alpha))v_{\Lambda_1}$ with 3-dimensional $\mathfrak{sl}_2(\alpha)$-module on the top, so it must be the standard $A_1^{(1)}$-module $L(2\Lambda_1)$. Hence again $x_\alpha(-1)v_{\Lambda_1} = 0$. □

Lemma 4.2. *We have*

1. $x_2(-1)x_2(-1)v_{\Lambda_0} = x_{\underline{2}}(-1)x_{\underline{2}}(-1)v_{\Lambda_0} = 0$,
2. $x_0(-1)x_2(-1)v_{\Lambda_0} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_0} = 0$,
3. $x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = Cx_0(-1)x_0(-1)v_{\Lambda_0}$ *for some* $C \neq 0$,
4. $x_0(-1)^3v_{\Lambda_0} = 0$.

Proof. Note that $x_\gamma(j)v_{\Lambda_0} = 0$ for all $j \geq 0$, so the relation $x_\gamma(z)^2 = 0$ on $L(\Lambda_0)$ for a long root γ implies

$$x_\gamma(-1)^2v_{\Lambda_0} = (x_\gamma(-1)x_\gamma(-1) + 2x_\gamma(-2)x_\gamma(0) + 2x_\gamma(-3)x_\gamma(1) + \dots)\, v_{\Lambda_0} = 0$$

and (1) follows. Since $x_{\varepsilon_2}(0)v_{\Lambda_0} = x_{-\varepsilon_2}(0)v_{\Lambda_0} = 0$, the action of $x_{\varepsilon_2}(0)$ or $x_{-\varepsilon_2}(0)$ on (1) gives relations (2) and (3). Since for level one $\tilde{\mathfrak{g}}$-module V the restriction to $\widetilde{\mathfrak{sl}}_2(\alpha)$ is level two representation if α is a short root, on $L(\Lambda_0)$ we have a relation $x_0(z)^3 = 0$ and (4) follows. □

We fix vectors $w_2 = v_{\Lambda_2}$ and $w_{\underline{2}}$ with weights

$$\omega_2 = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) \quad \text{and} \quad \omega_{\underline{2}} = \tfrac{1}{2}(\varepsilon_1 - \varepsilon_2)$$

in the 4-dimensional spinor $\mathfrak{g}$-module on the top of $L(\Lambda_2)$. By using arguments as above we obtain the following:

Lemma 4.3. *We have*

1. $x_2(-1)v_{\Lambda_2} = 0$ *and* $x_{\underline{2}}(-1)w_{\underline{2}} = 0$,
2. $x_{\underline{2}}(-1)x_{\underline{2}}(-1)v_{\Lambda_2} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_2} = x_0(-1)x_0(-1)v_{\Lambda_2} = 0$.

Lemma 4.4. *The set of monomial vectors $x(\pi)v_\Lambda$ satisfying difference conditions (3.1) and initial conditions (3.2) span $W(\Lambda)$.*

Proof. We have already mentioned how the relation $x_\theta(z)^{k+1} = 0$ on level k standard module $L(\Lambda)$ leads to a spanning set of monomial vectors satisfying difference conditions (3.1). By following an idea from [T] we reduce the problem of initial conditions for level k Feigin-Stoyanovsky's subspace to a problem of difference conditions for level $k' < k$ Feigin-Stoyanovsky's subspace: we shall consider $\tilde{\mathfrak{g}}$-submodules in tensor products

$$L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2} \quad \text{and} \quad L(\Lambda_1)^{\otimes k_1} \otimes \left(L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2}\right)$$

of levels $k' = k_0 + k_2$ and $k = k_0 + k_1 + k_2$ generated by highest weight vectors

$$v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \quad \text{and} \quad v_{\Lambda_1}^{\otimes k_1} \otimes \left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right).$$

Assume that for

$$x(\pi) = x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{a_1}$$

the monomial vector $x(\pi)v_\Lambda$ does not satisfy initial conditions (3.2)

$$a_1 \leq k_0, \quad b_1 + a_1 \leq k_0 + k_2, \quad c_1 + b_1 \leq k_0 + k_2.$$

By the above lemmas

$$x_2(-1)v_{\Lambda_1} = 0, \quad x_2(-1)v_{\Lambda_2} = 0, \quad x_2(-1)^2v_{\Lambda_0} = 0,$$

so in the case when $a_1 > k_0$ we have that the vector

$$x_2(-1)^{a_1}\left(v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right) = x_2(-1)^{a_1-k_0}\left(v_{\Lambda_1}^{\otimes k_1} \otimes (x_2(-1)v_{\Lambda_0})^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right)$$

equals zero and we may omit it from our spanning set of monomial vectors.

Now assume that the monomial vector $x(\pi)v_\Lambda$ does not satisfy initial conditions because

$$k'' = b_1 + a_1 > k_0 + k_2.$$

Then we have a relation

$$x_2(z)^{b_1+a_1} = 0 \quad \text{on} \quad L(k_0\Lambda_0 + k_2\Lambda_2) \subset L(\Lambda_0)^{\otimes k_0} \otimes L(\Lambda_2)^{\otimes k_2},$$

and by the adjoint action of $(x_{-\varepsilon_2})^{b_1}$ we get

$$x_0(z)^{b_1} x_2(z)^{a_1} + \cdots + c_{u,v,t}\, x_{\underline{2}}(z)^u x_0(z)^v x_2(z)^t + \cdots = 0$$

with $a_1 < t$. The coefficient of z^0 gives us a relation

$$R = x_0(-1)^{b_1} x_2(-1)^{a_1} + \cdots + c'_{u,v,t}\, x_{\underline{2}}(-1)^u x_0(-1)^v x_2(-1)^t + \cdots = 0$$

on $L(k_0\Lambda_0 + k_2\Lambda_2)$. The coefficient R is an infinite sum with the leading term

$$x_0(-1)^{b_1} x_2(-1)^{a_1}. \tag{4.2}$$

In R we have monomials of the form $x_{\gamma_1}(j_1) \dots x_{\gamma_{k''}}(j_{k''})$ with $j_1 + \cdots + j_{k''} = -k''$, so either $j_1 = \cdots = j_{k''} = -1$ or we have $j_s \geq 0$ for some s. Hence Lemma 4.1 and

$$x_\gamma(j) v_{\Lambda_i} = 0 \quad \text{for all} \quad \gamma \in \Gamma,\ j \geq 0 \quad \text{and} \quad i = 0, 1, 2$$

imply

$$R v_\Lambda = R\left(v_{\Lambda_1}^{\otimes k_1} \otimes \left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right)\right) = v_{\Lambda_1}^{\otimes k_1} \otimes R\left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2}\right) = 0.$$

Since the monomial (4.2) is the leading term of the relation $Rv_\Lambda = 0$, we can express

$$x_0(-1)^{b_1} x_2(-1)^{a_1} v_\Lambda$$

as a combination of higher monomial vectors and we may omit it from the spanning set. In a similar way we argue in the case when $c_1 + b_1 > k_0 + k_2$. ☐

5. Simple current operators

Recall that we have fixed a cominimal weight $\omega = \omega_1 = \varepsilon_1 \in \mathfrak{h}$. We shall use simple current operators $[\omega]$ on level 1 modules, i.e. linear bijections

$$L(\Lambda_0) \xrightarrow{[\omega]} L(\Lambda_1) \xrightarrow{[\omega]} L(\Lambda_0), \qquad L(\Lambda_2) \xrightarrow{[\omega]} L(\Lambda_2)$$

such that

$$x_\alpha(z)[\omega] = [\omega] z^{\alpha(\omega)} x_\alpha(z) \quad \text{for all} \quad \alpha \in R$$

or, written by components,

$$x_\alpha(n)[\omega] = [\omega] x_\alpha(n + \alpha(\omega)) \quad \text{for all} \quad \alpha \in R,\ n \in \mathbb{Z}. \tag{5.1}$$

It is easy to see that, up to a scalar multiple, linear bijection $[\omega]$ between two irreducible modules is uniquely determined by (5.1). Such a map

$$[\omega] \colon L(\Lambda) \to L(\Lambda')$$

can be interpreted in terms of simple currents as the identity map

$$\mathrm{id} : L(\Lambda) \to L(\Lambda)$$

for which the target vector space $L(\Lambda)$ is endowed the structure of $L(\Lambda')$ with vertex operators

$$Y_{L(\Lambda')}(\cdot, z) = Y_{L(\Lambda)}(\Delta(\omega, z)\cdot, z) \quad \text{with} \quad \Delta(\omega, z) = z^\omega \exp -\left(\sum_{n>0} \omega(n)(-z)^{-n}/n\right)$$

(cf. [DLM] and [L2]).

We fix $v_{\Lambda_0} = \mathbf{1}$ in the vertex operator algebra $L(\Lambda_0)$. Then we have

Lemma 5.1. *With properly normalized* v_{Λ_1} *and* x_2

(1) $[\omega]v_{\Lambda_0} = v_{\Lambda_1}$,
(2) $[\omega]v_{\Lambda_1} = x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}$.

Proof. (1) For level k standard $\tilde{\mathfrak{g}}$-module $L(\Lambda)$ the new module structure $Y_{L(\Lambda')}(\cdot, z) = Y_{L(\Lambda)}(\Delta(\omega, z)\cdot, z)$ gives

$$h(0)[\omega]v_\Lambda = [\omega]\big(h(0) + \langle\omega, h\rangle k\big)v_\Lambda \quad \text{for} \quad h \in \mathfrak{h}.$$

In particular, $[\omega]v_{\Lambda_0}$ is a weight vector with level 1 weight $\Lambda_1 = \Lambda_0 + \langle\omega, \cdot\rangle$. Relation (5.1) gives

$$\begin{aligned} &x_{-\theta}(1)[\omega]v_{\Lambda_0} = [\omega]x_{-\theta}(1 - \theta(\omega))v_{\Lambda_0} = [\omega]x_{-\theta}(0)v_{\Lambda_0} = 0 \quad \text{and} \\ &x_{\alpha_i}(0)[\omega]v_{\Lambda_0} = [\omega]x_{\alpha_i}(0 + \alpha_i(\omega))v_{\Lambda_0} = [\omega]x_{\alpha_i}(\delta_{i1})v_{\Lambda_0} = 0 \quad \text{for} \quad i = 1, 2. \end{aligned}$$

Hence $[\omega]v_{\Lambda_0}$ is a highest weight vector and $L(\Lambda_0') = L(\Lambda_1)$.

(2) Like in (1) we first see that $[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0}$ is a weight vector with weight Λ_1. By using (5.1) and Lemma 4.2 we obtain

$$\begin{aligned} x_{-\theta}(1)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{-\theta}(2)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_1}(0)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{\alpha_1}(-1)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_2}(0)[\omega]^{-1}x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} &= [\omega]^{-1}x_{\alpha_2}(0)x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0} = 0. \end{aligned}$$

Hence (2) holds and $L(\Lambda_1') = L(\Lambda_0)$. ☐

Lemma 5.2. *With properly normalized* $[\omega]$, $w_{\underline{2}}$ *and* $x_0, x_{\underline{2}}$

(1) $[\omega]v_{\Lambda_2} = x_0(-1)v_{\Lambda_2} = x_2(-1)w_{\underline{2}}$,
(2) $[\omega]w_{\underline{2}} = x_0(-1)w_{\underline{2}} = x_{\underline{2}}(-1)v_{\Lambda_2}$.

Proof. (1) As in the proof of previous lemma we see that $[\omega]^{-1}x_0(-1)v_{\Lambda_2}$ is a weight vector with weight Λ_2. By using (5.1) and Lemma 4.3 we obtain

$$\begin{aligned} x_{-\theta}(1)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{-\theta}(2)x_0(-1)v_{\Lambda_0} = 0, \\ x_{\alpha_1}(0)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{\alpha_1}(-1)x_0(-1)v_{\Lambda_2} = 0, \\ x_{\alpha_2}(0)[\omega]^{-1}x_0(-1)v_{\Lambda_2} &= [\omega]^{-1}x_{\alpha_2}(0)x_0(-1)v_{\Lambda_2} = 0. \end{aligned}$$

Hence, with a proper normalization, $[\omega]^{-1}x_0(-1)v_{\Lambda_2} = v_{\Lambda_2}$. The second equality follows from Lemma 4.3 because

$$0 = x_{-\alpha_2}(0)0 = x_{-\alpha_2}(0)x_2(-1)v_{\Lambda_2} = C'x_0(-1)v_{\Lambda_2} + C''x_2(-1)w_{\underline{2}}$$

for some $C', C'' \neq 0$.

(2) The first equality follows from (1) by using the fact that $w_{\underline{2}}$ is proportional to $x_{-\alpha_2}(0)v_{\Lambda_2}$ and the fact that $x_{-\alpha_2}(0)$ commutes with $[\omega]$. The second equality follows from Lemma 4.3. ☐

We define a linear bijection $[\omega]$ on the tensor product of k fundamental modules as

$$[\omega] \otimes \cdots \otimes [\omega] \colon \bigotimes_{s=1}^{k} L(\Lambda_{i_s}) \to \bigotimes_{s=1}^{k} L(\Lambda_{i'_s}).$$

It is clear that relation (5.1) holds for $[\omega] = [\omega] \otimes \cdots \otimes [\omega]$. In particular,

$$x_\gamma(n)[\omega] = [\omega] x_\gamma(n+1) \quad \text{for} \quad \gamma \in \Gamma. \tag{5.2}$$

If we set $\mu^+(x_\gamma(n+1)) = \mu(x_\gamma(n))$, then for monomials relation (5.2) reads as

Lemma 5.3. $x(\mu)[\omega] = [\omega] x(\mu^+)$.

Remark 5.4. *For $x(\mu) = \prod x_\gamma(n)^{m_\gamma(n)}$ we have $x(\mu^+) = \prod x_\gamma(n+1)^{m_\gamma(n)}$, so we may say that $x(\mu^+)$ is obtained from a monomial $x(\mu)$ by "shifting degrees of factors" $x_\gamma(n) \to x_\gamma(n+1)$. Later on we shall also use a notation $\mu^p(x_\gamma(n+p)) = \mu(x_\gamma(n))$ for any for $p \in \mathbb{Z}$, and we shall write μ^{+p} when we want to emphasize the shift of degrees of factors.*

¿From Lemmas 5.1, 5.2, 4.1, 4.2 and 4.3 we have

$$\begin{aligned} &[\omega]\left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\right) \\ &= \left([\omega] v_{\Lambda_0}\right)^{\otimes k_0} \otimes \left([\omega] v_{\Lambda_1}\right)^{\otimes k_1} \otimes \left([\omega] v_{\Lambda_2}\right)^{\otimes k_2} \\ &= v_{\Lambda_1}^{\otimes k_0} \otimes \left(x_{\underline{2}}(-1) x_2(-1) v_{\Lambda_0}\right)^{\otimes k_1} \otimes \left(x_0(-1) v_{\Lambda_2}\right)^{\otimes k_2} \\ &= C\, x_{\underline{2}}(-1)^{k_1} x_0(-1)^{k_2} x_2(-1)^{k_1} \left(v_{\Lambda_1}^{\otimes k_0} \otimes v_{\Lambda_0}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2}\right). \end{aligned} \tag{5.3}$$

For

$$\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2 \quad \text{set} \quad \Lambda^* = k_1\Lambda_0 + k_0\Lambda_1 + k_2\Lambda_2. \tag{5.4}$$

Then (5.3) and Lemma 5.3 imply

Proposition 5.5. $[\omega] \colon L(\Lambda) \to L(\Lambda^*)$ *and* $[\omega] \colon W(\Lambda) \to W(\Lambda^*)$.

This proposition and a construction in [DLM] and [L2] show that

$$[\omega] = [\omega] \otimes \cdots \otimes [\omega]$$

is a simple current operator for level k standard modules. Later on we shall need the following:

Lemma 5.6. *For elements h of the Cartan subalgebra $\mathfrak{h}$, and the Virasoro algebra element $L(0)$, on level k standard modules we have*

(1) $[\omega]^{-n} h\, [\omega]^n = h + \langle \omega, h\rangle k$ *for all* $n \in \mathbb{Z}$, *and*
(2) $[\omega]^{-n} L(0)\, [\omega]^n = L(0) + n\,\omega + \frac{n^2}{2}\langle \omega, \omega\rangle k$ *for all* $n \in \mathbb{Z}$.

Proof. As it was already said, we can view $[\omega]$ as the identity map on $L(\Lambda) \to L(\Lambda)$, where the target space is given a new module structure $L(\Lambda')$ with a vertex operator

$$Y_{L(\Lambda')}(\cdot, z) = Y_{L(\Lambda)}(\Delta(\omega, z)\cdot, z), \qquad \Delta(\omega, z) = z^\omega \exp -\left(\sum_{n>0} \omega(n)(-z)^{-n}/n\right).$$

Then $L(0)[\omega]$ is a coefficient of z^{-2} in the vertex operator

$$Y_{L(\Lambda')}(L(-2)\mathbf{1}, z) = Y_{L(\Lambda)}\big(\Delta(\omega, z)\, L(-2)\mathbf{1}, z\big),$$

and only three terms in

$$\Delta(\omega, z) = 1 - \big(\omega(1)(-z)^{-1} + \omega(2)(-z)^{-2}/2 + \dots\big) + \tfrac{1}{2!}\big(\omega(1)(-z)^{-1} + \dots\big)^2 + \dots$$

give a contribution to this coefficient

$$L(0)[\omega] = [\omega]\big(L(0) + \omega + \tfrac{1}{2}\langle \omega, \omega\rangle k\big).$$

Now (2) follows by induction. Relation (1) is proved in a similar way. □

6. Coefficients of level 1 intertwining operators

Let V be a vertex operator algebra and let W_1, W_2 and W_3 be three V-modules. Then an intertwining operator $\mathcal{Y}$ of type $\binom{W_3}{W_1\,W_2}$ is a formal series

$$\mathcal{Y}(w,z)=\sum_{n\in\mathbb{Q}} w_n z^{-n-1},\quad w\in W_1,$$

with coefficients

$$w_n\in\operatorname{Hom}(W_2,W_3)\quad\text{for}\quad n\in\mathbb{Q}$$

such that "all the defining properties of a module action that make sense hold" (see [FHL]). In particular, for $v\in V$ we have a commutator formula

$$v_jw_n-w_nv_j=\sum_{i\geq 0}\binom{j}{i}(v_iw)_{n+j-i}$$

which we shall use. The vector space of all intertwining operators of type $\binom{W_3}{W_1\,W_2}$ is denoted by $I\binom{W_3}{W_1\,W_2}$ and its dimension is called a fusion rule. We have

$$I\binom{W_3}{W_1\,W_2}\cong I\binom{W_3}{W_2\,W_1}\cong I\binom{W_2'}{W_1\,W_3'},$$

where for V-module M we denote by M' the contragredient module (see [FHL]). If W_1 is irreducible V-module, W_2 a simple current module and W_3 a tensor product of W_1 and W_2, then by Lemma 2.3 in [L1] the fusion $\dim I\binom{W_3}{W_1\,W_2}$ is one.

Lemma 6.1. *Let $\tilde{\mathfrak{g}}$ be an affine Lie algebra and $L(k\Lambda_0)$ a vacuum level k standard $\tilde{\mathfrak{g}}$-module. Let V_1, V_2 and V_3 be irreducible modules for vertex operator algebra $V=L(k\Lambda_0)$. Let $\mathcal{Y}\neq 0$ be an intertwining operator of type $\binom{V_3}{V_1\,V_2}$, let W be the top of V_1 and $v\neq 0$ a vector on the top of V_2. Then there is $m\in\mathbb{Q}$ such that the top of V_3 is a $\mathfrak{g}$-module*

$$U(\mathfrak{g})\{w_mv\mid w\in W\}.$$

Proof. By Proposition 11.9 in [DL] we have $\mathcal{Y}(w,z)v\neq 0$ for $w\neq 0$ and, from the definition of intertwining operators, $w_nv=0$ for all n large enough. Let

$$m=\max\{n\in\mathbb{Q}\mid w_nv\neq 0 \text{ for some } w\in W\}.$$

Then we have a nonzero subspace

$$\{w_mv\mid w\in W\}\subset V_3.$$

For $x_j=x(j)$ in $\tilde{\mathfrak{g}}$ we have a commutator formula

$$x_jw_m-w_mx_j=\sum_{i\geq 0}\binom{j}{i}(x_iw)_{m+j-i}$$

which for $j>0$ implies

$$x_j(w_mv)=w_mx_jv+\sum_{i\geq 0}\binom{j}{i}(x_iw)_{m+j-i}v=(x_0w)_{m+j}v=0$$

because v and w are vectors on the top of modules and m is maximal such that w_nv can be nonzero. Since

$$U(\tilde{\mathfrak{g}}_{\leq 0})\{w_mv\mid w\in W\}\subset V_3$$

is an $\tilde{\mathfrak{g}}$-invariant subspace of irreducible $\tilde{\mathfrak{g}}$-module V_3, the space $\{w_m v \mid w \in W\}$ must be a subspace of the top of V_3 and the lemma follows. ☐

Proposition 6.2. *With a proper normalization of intertwining operators of types*

$$\begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_2) & & L(\Lambda_0) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & L(\Lambda_1) & \\ L(\Lambda_2) & & L(\Lambda_2) \end{pmatrix}$$

there are coefficients $[\omega_2]$ *and* $[\omega_{\underline{2}}]$,

$$L(\Lambda_0) \xrightarrow{[\omega_2]} L(\Lambda_2) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_1), \quad v_{\Lambda_0} \xrightarrow{[\omega_2]} v_{\Lambda_2} \xrightarrow{[\omega_{\underline{2}}]} v_{\Lambda_1}, \quad [\omega_{\underline{2}}]w_{\underline{2}} = 0,$$

$$L(\Lambda_0) \xrightarrow{[\omega_{\underline{2}}]} L(\Lambda_2) \xrightarrow{[\omega_2]} L(\Lambda_1), \quad v_{\Lambda_0} \xrightarrow{[\omega_{\underline{2}}]} w_{\underline{2}} \xrightarrow{[\omega_2]} v_{\Lambda_1}, \quad [\omega_2]v_{\Lambda_2} = 0,$$

which commute with the action of $\tilde{\mathfrak{g}}_1$.

Proof. Since $L(\Lambda_2)$ is $L(\Lambda_0)$-module we have

$$\begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_2) & & L(\Lambda_0) \end{pmatrix} \cong \begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_0) & & L(\Lambda_2) \end{pmatrix}$$

and the space of intertwining operators of this type is 1-dimensional. Since $L(\Lambda_1)$ is a simple current module such that the tensor product of $L(\Lambda_1)$ and $L(\Lambda_2)$ is $L(\Lambda_2)$ (see [L1], [L2] or [DLM]), and since both $L(\Lambda_1)$ and $L(\Lambda_2)$ are self-dual, we have

$$\begin{pmatrix} & L(\Lambda_1) & \\ L(\Lambda_2) & & L(\Lambda_2) \end{pmatrix} \cong \begin{pmatrix} & L(\Lambda_2) & \\ L(\Lambda_2) & & L(\Lambda_1) \end{pmatrix}$$

and the space of intertwining operators of this type is 1-dimensional.

Let $\mathcal{Y} \neq 0$ be an intertwining operator of type $\left(\begin{smallmatrix} & L(\Lambda_2) & \\ L(\Lambda_2) & & L(\Lambda_0) \end{smallmatrix}\right)$ and $v = v_{\Lambda_0}$ on the top of $L(\Lambda_0)$. By Lemma 6.1 there is a vector w on the top of $L(\Lambda_2)$ and an integer m such that $w_m v$ is proportional to v_{Λ_2}. It is clear that w is proportional to v_{Λ_2} and we denote by $[\omega_2] = w_m$ the corresponding coefficient of the formal series $\mathcal{Y}(v_{\Lambda_2}, z)$. Obviously for proper normalization of w we have

$$[\omega_2]v_{\Lambda_0} = v_{\Lambda_2}.$$

On the other hand, if we take $w = w_{\underline{2}}$ and the corresponding coefficient $[\omega_{\underline{2}}] = w_m$ of the formal series $\mathcal{Y}(w_{\underline{2}}, z)$, with proper normalization we have

$$[\omega_{\underline{2}}]v_{\Lambda_0} = w_{\underline{2}}.$$

Now let $\mathcal{Y} \neq 0$ be an intertwining operator of type $\left(\begin{smallmatrix} & L(\Lambda_1) & \\ L(\Lambda_2) & & L(\Lambda_2) \end{smallmatrix}\right)$ and $v = v_{\Lambda_2}$ on the top of $L(\Lambda_2)$. By Lemma 6.1 there is a vector w on the top of $L(\Lambda_2)$ and an integer m such that vector $w_m v$ generates the irreducible 5-dimensional $\mathfrak{g}$-module on the top of $L(\Lambda_1)$. Since the top of $L(\Lambda_2)$ is 4-dimensional spinor $\mathfrak{g}$-module, $\mathfrak{h}$-weight vectors of the form $w_m v$ can have weights

$$\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2), \quad -\tfrac{1}{2}(\varepsilon_1 - \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2), \quad -\tfrac{1}{2}(\varepsilon_1 + \varepsilon_2) + \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2).$$

In the first case w is proportional to $w_{\underline{2}}$ and $w_m v = C v_{\Lambda_1}$ for some scalar $C \neq 0$. Vectors in the second and third case can be transformed to the vector $w'_m v = C v_{\Lambda_1}$ in the first case by acting with Lie algebra $\mathfrak{g}$ elements $x_{\varepsilon_1 - \varepsilon_2}$ and x_{ε_1} respectively. So if we take $w = w_{\underline{2}}$ and the corresponding coefficient $[\omega_{\underline{2}}] = w_m$ of the formal series $\mathcal{Y}(w_{\underline{2}}, z)$ with proper normalization, we have

$$[\omega_{\underline{2}}]v_{\Lambda_2} = v_1.$$

Inspection of $\mathfrak{h}$-weights in 5-dimensional $\mathfrak{g}$-module on the top of $L(\Lambda_1)$ shows that $[\omega_{\underline{2}}]w_{\underline{2}} = 0$. In a similar way we see that for $w = v_{\Lambda_2}$ and the properly normalized corresponding coefficient $[\omega_2] = w_m$ of the formal series $\mathcal{Y}(v_{\Lambda_2}, z)$ we have

$$[\omega_2]w_{\underline{2}} = v_{\Lambda_1} \quad \text{and} \quad [\omega_2]v_{\Lambda_2} = 0.$$

In each of the above cases $[\omega_2]$ and $[\omega_{\underline{2}}]$ are coefficients of $\mathcal{Y}(w, z)$ with w such that

$$x(i)w = 0 \qquad \text{for all} \quad x \in \mathfrak{g}_1,\ i \geq 0.$$

Hence the commutation relations for intertwining operators imply

$$x(j)w_m - w_m x(j) = \sum_{i\geq 0} \binom{j}{i} (x(i)w)_{m+j-i} = 0 \quad \text{for all} \quad x(j) \in \tilde{\mathfrak{g}}_1.$$

$\square$

7. Proof of linear independence

By Lemma 4.4 the set of monomial vectors

$$x(\pi)v_\Lambda = \ \dots\ x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j} \dots\ x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1} v_\Lambda$$

satisfying difference conditions (3.1) and initial conditions (3.2) spans $W(\Lambda)$. We prove linear independence of this set by induction on degree

$$-n = \sum_{\gamma\in\Gamma,\, j\geq 1} -j \cdot \pi(x_\gamma(-j)) = -\left(1a_1 + 1b_1 + 1c_1 + \dots + ja_j + jb_j + jc_j + \dots\right)$$

of monomials $x(\pi)$, considering in a proof all level k modules simultaneously. In the proof we shall briefly write DC for difference conditions (3.1) and IC for initial conditions (3.2).

Step 1. The idea of proof is illustrated most clearly in a proof of linear independence of vectors

$$x(\pi)v_{k\Lambda_1}$$

of degree $-n$. As induction hypothesis we assume that vectors $x(\mu)v_{k\Lambda_0}$ of degree $> -n$ are linearly independent. Assume that

$$\sum c_\pi x(\pi)v_{k\Lambda_1} = 0. \tag{7.1}$$

By Lemma 5.1 we have $v_{\Lambda_1} = [\omega]v_{\Lambda_0}$ and hence $v_{k\Lambda_1} = [\omega]v_{k\Lambda_0}$. By Lemma 5.3

$$\sum c_\pi x(\pi)v_{k\Lambda_1} = \sum c_\pi x(\pi)[\omega]v_{k\Lambda_0} = [\omega]\sum c_\pi x(\pi^+)v_{k\Lambda_0}$$

and injectivity of $[\omega]$ implies

$$\sum c_\pi x(\pi^+)v_{k\Lambda_0} = 0. \tag{7.2}$$

Monomials $x(\pi)$ in (7.1) satisfy difference conditions, so obviously "shifted by degree" monomials $x(\pi^+)$ in (7.2) satisfy difference conditions as well. Monomials $x(\pi)$ in (7.1) satisfy initial conditions for $k\Lambda_1$, i.e., contain no part of the form $x_\alpha(-1)$. But then monomials $x(\pi^+)$ in (7.2) contain parts of the form $x_\alpha(-j)$, $j \geq 1$, and hence satisfy initial conditions for $k\Lambda_0$. Since monomials $x(\pi^+)$ in (7.2) have degrees $> -n$, the induction hypothesis implies that all $c_\pi = 0$. Hence we proved linear independence of monomial basis vectors for $W(k\Lambda_1)$ of degree $-n$.

Step 2. For $\mathcal{A} = (c_1, b_1, a_1)$ write

$$x(-1)^{\mathcal{A}} = x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1}.$$

Later on it will be convenient to write a monomial $x(\mu)$ as a product

$$\dots\, x_{\underline{2}}(-j)^{c_j} x_0(-j)^{b_j} x_2(-j)^{a_j} \dots x_{\underline{2}}(-1)^{c_1} x_0(-1)^{b_1} x_2(-1)^{a_1} = x(\mu_2) x(-1)^{\mathcal{A}_\mu}.$$

We define a partial order on the set of level k integral dominant weights:

$$\Lambda' = k_0'\Lambda_0 + k_1'\Lambda_1 + k_2'\Lambda_2 \leq \Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$$

if and only if

$$k_0' \leq k_0,$$
$$k_0' + k_2' \leq k_0 + k_2.$$

Clearly $k\Lambda_1$ is the smallest element and $k\Lambda_0$ is the largest element in the set of level k integral dominant weights.

Now we proceed with a proof of linear independence. We assume that vectors $x(\mu)v_{\Lambda'}$ of degree $\geq -n$ satisfying DC and IC are linearly independent for some set of $\Lambda' \geq k\Lambda_1$. Let Λ be a minimal level k integral weight for which we need to prove linear independence of monomial vectors of degree $\geq -n$ satisfying DC and IC. Let

$$\sum c_\pi x(\pi) v_\Lambda = 0. \tag{7.3}$$

Assume that $c_\mu \neq 0$ for some $x(\mu) = x(\mu_2)x(-1)^{\mathcal{A}_\mu}$ for

$$\mathcal{A}_\mu = (c_1, b_1, a_1), \qquad a_1 < k_0,$$

and we assume that a_1 is the smallest such. Since $[\omega_2]\colon L(\Lambda_0) \to L(\Lambda_2)$, we have the operator

$$1^{\otimes a_1} \otimes [\omega_2]^{\otimes(k_0 - a_1)} \otimes 1^{\otimes(k_2+k_1)}\colon L(\Lambda) \to L(\Lambda'),$$
$$v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \otimes v_{\Lambda_1}^{\otimes k_1} \mapsto v_{\Lambda_0}^{\otimes a_1} \otimes v_{\Lambda_2}^{\otimes(k_2+k_0-a_1)} \otimes v_{\Lambda_1}^{\otimes k_1}$$

which commutes with the action of $\tilde{\mathfrak{g}}_1$. Note that $\Lambda > \Lambda'$ so that we may use the induction hypothesis for corresponding monomial vectors. If we apply this operator on the sum (7.3) we get

$$\sum c_\pi x(\pi) v_{\Lambda'} = 0. \tag{7.4}$$

By Lemmas 4.1, 4.3, 4.2 we have $x_2(-1)v_{\Lambda_2} = 0$, $x_2(-1)v_{\Lambda_1} = 0$, $x_2(-1)^2 v_{\Lambda_0} = 0$, so for any monomial $x(\pi) = x(\pi')x_2(-1)^a$ with $a > a_1$ we have

$$x(\pi)v_{\Lambda'} = x(\pi')x_2(-1)^a \left(v_{\Lambda_0}^{\otimes a_1} \otimes v_{\Lambda_2}^{\otimes(k_2+k_0-a_1)} \otimes v_{\Lambda_1}^{\otimes k_1} \right) = 0.$$

On the other hand, vectors like $x(\mu)v_{\Lambda'}$ besides DC satisfy IC as well, i.e.,

$$a_1 \leq k_0' = a_1, \quad b_1 + a_1 \leq k_0' + k_2' = k_0 + k_2, \quad c_1 + b_1 \leq k_0' + k_2' = k_0 + k_2,$$

so by induction hypothesis the coefficient c_μ in linear combination (7.4) must be zero — a contradiction.

So in (7.3) we need to consider only monomials with $a_1 = k_0$, i.e., monomials of the form

$$x(\pi) = x(\pi')x_2(-1)^{k_0}.$$

Assume that $c_\mu \neq 0$ for some $x(\mu) = x(\mu_2)x(-1)^{\mathcal{A}_\mu}$ for

$$\mathcal{A}_\mu = (c_1, b_1, a_1), \qquad a_1 = k_0, \quad b_1 + a_1 < k_0 + k_2, \quad c_1 + b_1 < k_0 + k_2.$$

Since $[\omega_2]\colon L(\Lambda_2) \to L(\Lambda_1)$, we have the operator

$$1^{\otimes(k_0+k_2-1)} \otimes [\omega_2] \otimes 1^{\otimes k_1} \colon L(\Lambda) \to L(\Lambda'),$$
$$v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes k_2} \otimes v_{\Lambda_1}^{\otimes k_1} \mapsto v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes (k_2-1)} \otimes v_{\Lambda_1}^{\otimes (k_1+1)}$$

which commutes with the action of $\tilde{\mathfrak{g}}_1$. Note that $\Lambda > \Lambda'$ so that we may use the induction hypothesis for corresponding monomial vectors. If we apply this operator on the sum (7.3) we get

$$\sum c_\pi x(\pi) v_{\Lambda'} = 0. \tag{7.5}$$

By Lemmas 4.1, 4.3, 4.2 we have $x_2(-1)v_{\Lambda_2} = 0$, $x_2(-1)v_{\Lambda_1} = 0$, $x_2(-1)^2 v_{\Lambda_0} = 0$, so for any monomial $x(\pi) = x(\pi')x_2(-1)^{k_0}$ we have

$$\begin{aligned} x(\pi)v_{\Lambda'} &= x(\pi')x_2(-1)^{k_0}\left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes (k_2-1)} \otimes v_{\Lambda_1}^{\otimes (k_1+1)}\right) \\ &= C\, x(\pi')\left((x_2(-1)v_{\Lambda_0})^{\otimes k_0} \otimes v_{\Lambda_2}^{\otimes (k_2-1)} \otimes v_{\Lambda_1}^{\otimes (k_1+1)}\right) \end{aligned}$$

for some $C \neq 0$. If for such $x(\pi) = x(\pi_2)x(-1)^{\mathcal{A}_\pi}$ we have

$$b_1 + a_1 = k_0 + k_2 \quad \text{or} \quad c_1 + b_1 = k_0 + k_2,$$

then by Lemma 4.3 we have $x_{\underline{2}}(-1)^2 v_{\Lambda_2} = x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_2} = x_0(-1)^2 v_{\Lambda_2} = 0$ and by Lemma 4.2 we have $x_{\underline{2}}(-1)x_0(-1)v_{\Lambda_0} = 0$, so in either case at least one of $x_0(-1)$ or $x_{\underline{2}}(-1)$ must act on one copy of v_{Λ_1}. Hence by Lemma 4.1 for such $x(\pi)$ it must be

$$x(\pi)v_{\Lambda'} = 0.$$

So in (7.5) we have only vectors like $x(\mu)v_{\Lambda'}$ which besides DC satisfy IC as well, and by induction hypothesis the coefficient c_μ in linear combination (7.5) must be zero — a contradiction.

Remark 7.1. *For the rest of the proof it will be convenient to realize $L(\Lambda)$ of level k in a k^{th} component of a symmetric algebra*

$$L(\Lambda) \subset S^k(V), \qquad V = L(\Lambda_0) \oplus L(\Lambda_1) \oplus L(\Lambda_2).$$

Operator $[\omega] = S([\omega])$ acts as "a group element" on $S(V)$. On the other hand, operators A and B on V in Lemmas 7.2 and 7.3 below act as derivations on $S(V)$.

Step 3. By the previous step in the linear combination (7.3) we need to consider only monomials $x(\pi) = x(\pi_2)x(-1)^{\mathcal{A}_\pi}$ with $a_1 = k_0$ for $\mathcal{A}_\pi = (c_1, b_1, a_1)$ and

$$b_1 + a_1 = k_0 + k_2 \quad \text{or} \quad c_1 + b_1 = k_0 + k_2.$$

Assume first we have a monomial vector $x(\pi)v_\Lambda$ such that $a_1 = k_0$ and $b_1 + a_1 = k_0 + k_2$ and $c_1 + b_1 \leq k_0 + k_2$. This implies that

$$a_1 = k_0, \quad b_1 = k_2, \quad c_1 \leq k_0. \tag{7.6}$$

As above, Lemmas 4.1, 4.3 and 4.2 imply that

$$\begin{aligned} x(-1)^{\mathcal{A}_\pi}v_\Lambda &= x_{\underline{2}}(-1)^{c_1}x_0(-1)^{k_2}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right) \\ &= C\, x_{\underline{2}}(-1)^{c_1}\left(v_{\Lambda_1}^{k_1}(x_0(-1)v_{\Lambda_2})^{k_2}(x_2(-1)v_{\Lambda_0})^{k_0}\right) \\ &= C'\, v_{\Lambda_1}^{k_1}(x_0(-1)v_{\Lambda_2})^{k_2}(x_2(-1)v_{\Lambda_0})^{k_0-c_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1}. \end{aligned}$$

Let $A\colon V\to V$ be a linear operator

$$A|_{L(\Lambda_0)}=[\omega_{\underline{2}}]\colon L(\Lambda_0)\to L(\Lambda_2),\qquad A|_{L(\Lambda_1)\oplus L(\Lambda_2)}=0,$$

and let A act as a derivation on $S(V)$. By Proposition 6.2 derivation A commutes with the action of $\tilde{\mathfrak{g}}_1$ on the symmetric algebra $S(V)$. Note that $Av_{\Lambda_0}=[\omega_{\underline{2}}]v_{\Lambda_0}=w_{\underline{2}}$ by Proposition 6.2 and $x_{\underline{2}}(-1)w_{\underline{2}}=0$ by Lemma 4.3, so $A(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})=0$. Hence by Lemmas 5.1 and 5.2 we have

$$\begin{aligned}
&A^{k_0-c_1}x(-1)^{\mathcal{A}_\pi}v_\Lambda\\
&=C''\,v_{\Lambda_1}^{k_1}(x_0(-1)v_{\Lambda_2})^{k_2}(x_2(-1)w_{\underline{2}})^{k_0-c_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1}\\
&=C''\,([\omega]v_{\Lambda_0})^{k_1}([\omega]v_{\Lambda_2})^{k_2}([\omega]v_{\Lambda_2})^{k_0-c_1}([\omega]v_{\Lambda_1})^{c_1}\\
&=C''\,[\omega]\left(v_{\Lambda_0}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_2}^{k_0-c_1}v_{\Lambda_1}^{c_1}\right)\\
&=C''\,[\omega]\left(v_{\Lambda_0}^{k_1}v_{\Lambda_2}^{k_2+k_0-c_1}v_{\Lambda_1}^{c_1}\right).
\end{aligned}$$

Since A commutes with the action of $\tilde{\mathfrak{g}}_1$ we have

$$\begin{aligned}
A^{k_0-c_1}x(\pi)v_\Lambda&=x(\pi_2)Ax(-1)^{\mathcal{A}_\pi}v_\Lambda\\
&=C''\,x(\pi_2)[\omega]\left(v_{\Lambda_0}^{k_1}v_{\Lambda_2}^{k_2+k_0-c_1}v_{\Lambda_1}^{c_1}\right)\\
&=C''\,[\omega]x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}v_{\Lambda_2}^{k_2+k_0-c_1}v_{\Lambda_1}^{c_1}\right)\\
&=C''\,[\omega]x(\pi_2^+)v_{\Lambda'}
\end{aligned}$$

for some $C''\neq 0$. It is clear that "truncated and shifted by degree" monomial $x(\pi_2^+)$ satisfies DC, and IC for $x(\pi_2^+)v_{\Lambda'}$ reads

$$a_2\le k_1=k-b_1-a_1,\quad b_2+a_2\le k_1+k_2+k_0-c_1,\quad c_2+b_2\le k_1+k_2+k_0-c_1=k-c_1.$$

But these are just three difference condition relations which hold for $x(\pi)v_\Lambda$:

$$a_2+b_1+a_1\le k,\quad b_2+a_2+c_1\le k,\quad c_2+b_2+c_1\le k.$$

Hence we have proved the following:

Lemma 7.2. *In the case when (7.6) holds monomial vector*

$$x(\pi_2^+)v_{\Lambda'}=(C''\,[\omega])^{-1}A^{k_0-c_1}x(\pi)v_\Lambda$$

satisfies difference conditions (3.1) and initial conditions (3.2).

Assume now that we have a monomial vector $x(\pi)v_\Lambda$ such that $a_1=k_0$ and $b_1+a_1\le k_0+k_2$ and $c_1+b_1=k_0+k_2$. This implies that

$$a_1=k_0,\quad b_1\le k_2,\quad c_1+b_1=k_0+k_2. \tag{7.7}$$

Like before, Lemmas 4.1, 4.3 and 4.2 imply that

$$\begin{aligned}
&x(-1)^{\mathcal{A}_\pi}v_\Lambda\\
&=x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right)\\
&=C\,x_{\underline{2}}(-1)^{c_1}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2-b_1}(x_0(-1)v_{\Lambda_2})^{b_1}(x_2(-1)v_{\Lambda_0})^{k_0}\right)\\
&=C'\,v_{\Lambda_1}^{k_1}(x_{\underline{2}}(-1)v_{\Lambda_2})^{k_2-b_1}(x_0(-1)v_{\Lambda_2})^{b_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{k_0}.
\end{aligned}$$

By Lemmas 5.1 and 5.2 we further have

$$\begin{aligned}&x(-1)^{\mathcal{A}_\pi} v_\Lambda\\ &= C'\,([\omega]v_{\Lambda_0})^{k_1}([\omega]w_{\underline{2}})^{k_2-b_1}([\omega]v_{\Lambda_2})^{b_1}([\omega]v_{\Lambda_1})^{k_0}\\ &= C'\,[\omega]\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right).\end{aligned}$$

Hence we have

$$\begin{aligned}x(\pi)v_\Lambda &= x(\pi_2)x(-1)^{\mathcal{A}_\pi}v_\Lambda\\ &= C'\,x(\pi_2)[\omega]\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right)\\ &= C'\,[\omega]x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right).\end{aligned}$$

Let $B\colon V\to V$ be a linear operator

$$B|_{L(\Lambda_2)}=[\omega_2]\colon L(\Lambda_2)\to L(\Lambda_1),\qquad B|_{L(\Lambda_0)\oplus L(\Lambda_1)}=0,$$

and let B act as a derivation on $S(V)$. By Proposition 6.2 derivation B commutes with the action of $\tilde{\mathfrak{g}}_1$ on the symmetric algebra $S(V)$ and $Bw_{\underline{2}}=[\omega_2]w_{\underline{2}}=v_{\Lambda_1}$ and $Bv_{\Lambda_2}=[\omega_2]v_{\Lambda_2}=0$. Hence

$$B^{k_2-b_1}\colon v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\mapsto C''\,v_{\Lambda'}=C''\,v_{\Lambda_0}^{k_1}v_{\Lambda_1}^{k_0+k_2-b_1}v_{\Lambda_2}^{b_1}.$$

Lemma 7.3. *In the case when (7.7) holds monomial vector*

$$x(\pi_2^+)v_{\Lambda'}=B^{k_2-b_1}(C''C'\,[\omega])^{-1}x(\pi)v_\Lambda$$

satisfies difference conditions (3.1) and initial conditions (3.2).

Proof. It is clear that "truncated and shifted by degree" monomial $x(\pi_2^+)$ satisfies DC, and IC for $x(\pi_2^+)v_{\Lambda'}$ reads

$a_2\le k_1=k-c_1-b_1,\quad b_2+a_2\le k_1+b_1,\quad c_2+b_2\le k_1+b_1=k_1+k_0+k_2-c_1=k-c_1.$

But these are just three difference condition relations which hold for $x(\pi)v_\Lambda$:

$$a_2+c_1+b_1\le k,\quad b_2+a_2+c_1\le k,\quad c_2+b_2+c_1\le k.$$

□

Now we proceed with the proof of linear independence. As already noted, in the linear combination (7.3) we need to consider only

$$\begin{aligned}0 &= \sum_{\substack{a_1=k_0\\ b_1+a_1=k_0+k_2>c_1+b_1\\ \text{or}\\ a_1=k_0\\ b_1+a_1\le k_0+k_2=c_1+b_1}} C_{\dots c_1b_1a_1}\dots x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right)\\ &= \sum_{c_1<k_0} C_{\dots c_1k_2k_0}\dots x_{\underline{2}}(-1)^{c_1}x_0(-1)^{k_2}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right)\\ &+ \sum_{\substack{b_1\le k_2\\ c_1+b_1=k_0+k_2}} C_{\dots c_1b_1k_0}\dots x_{\underline{2}}(-1)^{c_1}x_0(-1)^{b_1}x_2(-1)^{k_0}\left(v_{\Lambda_1}^{k_1}v_{\Lambda_2}^{k_2}v_{\Lambda_0}^{k_0}\right).\end{aligned}$$

Note that in the first sum we have vectors of the form

$$x(\pi_2)\left(v_{\Lambda_1}^{k_1}(x_0(-1)v_{\Lambda_2})^{k_2}(x_2(-1)v_{\Lambda_0})^{k_0-c_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1}\right)\tag{7.8}$$

for $k_0 - c_1 = 1, \dots, k_0$, and that in the second sum we have vectors of the form

$$x(\pi_2)\left(v_{\Lambda_1}^{k_1}(x_{\underline{2}}(-1)v_{\Lambda_2})^{k_2-b_1}(x_0(-1)v_{\Lambda_2})^{b_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{k_0}\right) \tag{7.9}$$

for $k_2 - b_1 = 0, \dots, k_2$. In particular in (7.8) we see a factor

$$(x_2(-1)v_{\Lambda_0})^{k_0-c_1}(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{c_1} \quad \text{for} \quad c_1 = 0, \dots, k_0 - 1$$

and in (7.9) we see a factor $(x_{\underline{2}}(-1)x_2(-1)v_{\Lambda_0})^{k_0}$. Hence the operator A^{k_0} will annihilate all these terms except ones with $c_1 = 0$ and the action on linear combination (7.3) gives

$$0 = A^{k_0}\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\mathcal{A}_\pi=(0,k_2,k_0)} c_\pi C''\, x(\pi_2^+)v_{\Lambda'}.$$

Now Lemma 7.2 and the induction hypothesis imply that $c_\pi = 0$ whenever $A_\pi = (0, k_2, k_0)$. In turn this implies that in the first sum of (7.3) it is enough to consider vectors (7.8) for $c_1 = 1, \dots, k_0 - 1$. Then we apply operator A^{k_0-1} which will annihilate all these terms except ones with $c_1 = 1$ and the action on linear combination (7.3) gives

$$0 = A^{k_0}\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\mathcal{A}_\pi=(1,k_2,k_0)} c_\pi C''\, x(\pi_2^+)v_{\Lambda'}.$$

Now Lemma 7.2 and the induction hypothesis imply that $c_\pi = 0$ whenever $A_\pi = (1, k_2, k_0)$. By proceeding in this way we see that all the coefficients $c_\pi = C_{\dots c_1 b_1 a_1}$ for $c_1 < k_0$ in the first sum are equal to zero.

So we are left with the second sum

$$\sum c_\pi x(\pi)v_\Lambda = [\omega]\sum_{\substack{b_1\le k_2\\ c_1+b_1=k_0+k_2}} c_\pi C'\, x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right) = 0. \tag{7.10}$$

This implies

$$\sum_{\substack{b_1\le k_2\\ c_1+b_1=k_0+k_2}} c_\pi C'\, x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1}v_{\Lambda_1}^{k_0}\right) = 0. \tag{7.11}$$

In (7.11) we see factors

$$w_{\underline{2}}^{k_2-b_1}v_{\Lambda_2}^{b_1} \qquad \text{for} \quad b_1 = 0, \dots, k_2.$$

The operator B^{k_0} will annihilate all these terms except ones with $b_1 = 0$ and the action on linear combination (7.11) gives

$$\sum_{\substack{b_1=0\\ c_1+b_1=k_0+k_2}} c_\pi C'C''\, x(\pi_2^+)\left(v_{\Lambda_0}^{k_1}v_{\Lambda_1}^{k_0+k_2-b_1}v_{\Lambda_2}^{b_1}\right) = 0.$$

Now Lemma 7.3 and the induction hypothesis imply that $c_\pi = 0$ whenever $A_\pi = (k_0 + k_2, 0, k_0)$. In turn this implies that in (7.11) it is enough to consider vectors for $b_1 = 1, \dots, k_0$. So next we apply B^{k_0-1} and conclude that $c_\pi = 0$ whenever $A_\pi = (k_0 + k_2 - 1, 1, k_0)$. By proceeding in this way we see that all the coefficients c_π in the second sum (7.10) are equal to zero and our proof of linear independence is complete.

8. Bases consisting of semi-infinite monomials

In (5.4) we have set $\Lambda^* = k_1\Lambda_0 + k_0\Lambda_1 + k_2\Lambda_2 \quad \text{for} \quad \Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$. Note that $\Lambda^{**} = \Lambda$. The relation (5.3) applied twice together with Lemma 5.3 gives

$$\begin{aligned}
&[\omega]^2 v_\Lambda = [\omega]^2 \left(v_{\Lambda_0}^{\otimes k_0} \otimes v_{\Lambda_1}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2} \right) \\
&= C' \, [\omega] x_{\underline{2}}(-1)^{k_1} x_0(-1)^{k_2} x_2(-1)^{k_1} \left(v_{\Lambda_1}^{\otimes k_0} \otimes v_{\Lambda_0}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2} \right) \\
&= C' \, x_{\underline{2}}(-2)^{k_1} x_0(-2)^{k_2} x_2(-2)^{k_1} [\omega] \left(v_{\Lambda_1}^{\otimes k_0} \otimes v_{\Lambda_0}^{\otimes k_1} \otimes v_{\Lambda_2}^{\otimes k_2} \right) \\
&= C \, x_{\underline{2}}(-2)^{k_1} x_0(-2)^{k_2} x_2(-2)^{k_1} x_{\underline{2}}(-1)^{k_0} x_0(-1)^{k_2} x_2(-1)^{k_0} v_\Lambda
\end{aligned} \tag{8.1}$$

for some $C = C_\Lambda \neq 0$. If we set

$$x(\kappa_\Lambda) = x_{\underline{2}}(-2)^{k_1} x_0(-2)^{k_2} x_2(-2)^{k_1} x_{\underline{2}}(-1)^{k_0} x_0(-1)^{k_2} x_2(-1)^{k_0},$$

then (8.1) reads

$$[\omega]^2 v_\Lambda = C_\Lambda x(\kappa_\Lambda) v_\Lambda. \tag{8.2}$$

This relation and Lemma 5.3 imply

$$[\omega]^2 \colon L(\Lambda) \to L(\Lambda) \quad \text{and} \quad [\omega]^2 \colon W(\Lambda) \to W(\Lambda).$$

Theorem 8.1. *Let $L(\Lambda)_\mu$ be a weight subspace of a level k standard $B_2^{(1)}$-module $L(\Lambda)$. Then there exists an integer m_0 such that for any fixed $m \leq m_0$ the set of vectors*

$$[\omega]^{2m} x(\pi) v_\Lambda \in L(\Lambda)_\mu$$

such that monomial vectors $x(\pi)v_\Lambda \in W(\Lambda)$ satisfy difference conditions (3.1) and initial conditions (3.2) is a basis of $L(\Lambda)_\mu$. Moreover, for two choices of $m_1, m_2 \leq m_0$ the corresponding two bases are connected by a multiple of identity matrix.

Proof. As in [P2] we define a "Weyl group translation" operator

$$e_\alpha = \exp x_{-\alpha}(1) \exp x_\alpha(-1) \exp x_{-\alpha}(1) \exp x_\alpha(0) \exp x_{-\alpha}(0) \exp x_\alpha(0)$$

for properly normalized root vectors x_α and $x_{-\alpha}$. Then on a standard module $L(\Lambda)$ we have

$$e_\alpha x_\gamma(j) e_\alpha^{-1} = (-1)^{\gamma(\alpha^\vee)} x_\gamma(j - \gamma(\alpha^\vee))$$

for all roots γ. Let

$$e = e_{\varepsilon_1 - \varepsilon_2} e_{\varepsilon_1} e_{\varepsilon_1 + \varepsilon_2}. \tag{8.3}$$

Then (8.3) and (5.1) imply

$$e^2 x_{\pm\gamma}(j) e^{-2} = x_\gamma(j \mp 6) \quad \text{and} \quad [\omega]^6 x_{\pm\gamma}(j) [\omega]^{-6} = x_\gamma(j \mp 6) \tag{8.4}$$

for all $\gamma \in \Gamma$. As in the proofs of Lemmas 5.1 and 5.2 we see that $\mathbb{C}v_\Lambda$ is invariant for $e^2[\omega]^{-6}$, so (8.4) implies

$$e^2 = C \, [\omega]^6 \quad \text{for some} \quad C \neq 0.$$

By the proof of Proposition 5.2 in [P2] vectors of the form

$$e^m x(\pi) v_\Lambda \in L(\Lambda)_\mu$$

span $L(\Lambda)_\mu$ for a given small enough m, so by Theorem 3.1 monomial vectors satisfying DC and IC will form a basis. Note that by Lemma 5.3 and (8.2)

$$[\omega]^{2m} x(\pi) v_\Lambda = [\omega]^{2m-2} x(\pi^{-2}) [\omega]^2 v_\Lambda = C_\Lambda [\omega]^{2(m-1)} x(\pi^{-2}) x(\kappa_\Lambda) v_\Lambda$$

and the monomial vector $x(\pi)v_\Lambda$ satisfies DC and IC if and only if the monomial vector $x(\pi^{-2})x(\kappa_\Lambda)v_\Lambda$ satisfies DC and IC. We can iterate this process

$$[\omega]^{2m}x(\pi)v_\Lambda = \dots = C_\Lambda^2[\omega]^{2(m-2)}x(\pi^{-4})x(\kappa_\Lambda^{-2})x(\kappa_\Lambda)v_\Lambda = \dots .$$

Hence bases obtained for different m_1 and m_2 are connected with a multiple of identity matrix. $\square$

In the level $k=1$ case linear independence in this theorem is proved in [P2] for the basic representation $L(\Lambda_0)$ by writing basis elements as semi-infinite monomials and then "counting" them by using crystal base character formula [KKMMNN]. Such semi-infinite monomials interpretation is possible for all standard $B_2^{(1)}$-modules, like in [FS] for $A_1^{(1)}$: for fixed Λ and $m\in\mathbb{Z}$ set

$$v_{-m} = [\omega]^{-2m}v_\Lambda$$

For big enough $m\geq 0$ basis elements of the given $L(\Lambda)_\mu$ can be rewritten as

$$[\omega]^{-2m}x(\pi)v_\Lambda = x(\pi^{+2m})[\omega]^{-2m}v_\Lambda = x(\pi^{+2m})v_{-m}. \tag{8.5}$$

By iterating (8.2) and Lemma 5.3 we get

$$v_{-m} = C_\Lambda x(\kappa_\Lambda^{+2(m+1)})v_{-m-1} = C_\Lambda^2 x(\kappa_\Lambda^{+2(m+1)})x(\kappa_\Lambda^{+2(m+2)})v_{-m-2} = \dots ,$$

so basis vector (8.5) is proportional to

$$x(\pi^{+2m})x(\kappa_\Lambda^{+2(m+1)})v_{-m-1} \quad\text{and}\quad x(\pi^{+2m})x(\kappa_\Lambda^{+2(m+1)})x(\kappa_\Lambda^{+2(m+2)})v_{-m-2}.$$

We could say that "on the limit" $v_{-\infty}=\lim_{p\to\infty}v_{-m-p}$ our basis vector (8.5) is "proportional" to a semi-infinite monomial

$$x(\pi^{+2m})x(\kappa_\Lambda^{+2(m+1)})x(\kappa_\Lambda^{+2(m+2)})x(\kappa_\Lambda^{+2(m+3)})\dots v_{-\infty}.$$

In particular, v_{-m} is represented by a semi-infinite quasi-periodic monomial

$$x_{\underline{2}}(2m)^{k_1}x_0(2m)^{k_2}x_2(2m)^{k_1}x_{\underline{2}}(2m+1)^{k_0}x_0(2m+1)^{k_2}x_2(2m+1)^{k_0}$$
$$\cdot\, x_{\underline{2}}(2m+2)^{k_1}x_0(2m+2)^{k_2}x_2(2m+2)^{k_1}x_{\underline{2}}(2m+3)^{k_0}x_0(2m+3)^{k_2}x_2(2m+3)^{k_0}\dots$$

and we can determine its $\mathfrak{h}$-weight and degree by using Lemma 5.6. Hence we have:

Corollary 8.2. *We can parametrize a basis of level k standard $B_2^{(1)}$-module $L(\Lambda)$,*

$$\Lambda = \Lambda_0k_0+\Lambda_1k_1+\Lambda_2k_2\,, \qquad k=k_0+k_1+k_2\,,$$

with semi-infinite monomials

$$\prod_{j\in\mathbb{Z}} x_{\underline{2}}(-j)^{c_j}x_0(-j)^{b_j}x_2(-j)^{a_j}\,, \qquad c_j=b_j=a_j=0 \quad \textit{for} \quad -j\ll 0,$$

with quasi-periodic tail

$$\begin{aligned}&(\dots,\, c_{-2n},\, b_{-2n},\, a_{-2n},\, c_{-2n-1},\, b_{-2n-1},\, a_{-2n-1},\, \dots)\\ &=(\dots,\, k_1,\, k_2,\, k_1,\, k_0,\, k_2,\, k_0,\, \dots)\end{aligned}$$

for $n\gg 0$, satisfying for all $j\in\mathbb{Z}$ difference conditions

$$\begin{aligned}c_{j+1}+b_{j+1}+c_j&\leq k,\\ b_{j+1}+a_{j+1}+c_j&\leq k,\\ a_{j+1}+c_j+b_j&\leq k,\\ a_{j+1}+b_j+a_j&\leq k.\end{aligned}$$

Note that for semi-infinite monomials the initial conditions are built in the form of quasi-periodic tail and the difference conditions.

9. Presentation of $W(\Lambda)$

Theorem 9.1. *Let* $\Lambda = k_0\Lambda_0 + k_1\Lambda_1 + k_2\Lambda_2$ *and* $k = k_0 + k_1 + k_2$. *Let*

$$\mathcal{P} = \mathbb{C}[x_{\underline{2}}(j), x_0(j), x_2(j) \mid j \leq -1]$$

and let $\mathcal{I}_\Lambda$ *be the ideal in the polynomial algebra* $\mathcal{P}$ *generated by the set of polynomials*

$$\bigcup_{n \leq -k-1} U(\mathfrak{g}_0) \cdot \Big(\sum_{\substack{j_1,\dots,j_{k+1} \leq -1 \\ j_1+\dots+j_{k+1}=n}} x_2(j_1) \dots x_2(j_{k+1}) \Big)$$
$$\bigcup \{x_2(-1)^{k_0+1}\} \bigcup U(\mathfrak{g}_0) \cdot x_2(-1)^{k_0+k_2+1}\,,$$

where $\cdot$ *denotes the adjoint action of* $\mathfrak{g}_0$ *on* $\mathcal{P}$. *Then as vector spaces*

$$W(\Lambda) \cong \mathcal{P}/\mathcal{I}_\Lambda.$$

Proof. Since $\mathcal{P} \subset S(\tilde{\mathfrak{g}}_1) = U(\tilde{\mathfrak{g}}_1)$, we have a linear map

$$f \colon \mathcal{P} \to W(\Lambda), \quad f \colon x(\pi) \mapsto x(\pi)v_\Lambda.$$

Since $x(j)v_\Lambda = 0$ for $x \in \mathfrak{g}_1$ and $j \geq 0$, relations $U(\mathfrak{g}_0) \cdot x_\theta(z)^{k+1} = 0$ on $L(\Lambda)$ imply

$$\bigcup_{n \leq -k-1} U(\mathfrak{g}_0) \cdot \Big(\sum_{\substack{j_1,\dots,j_{k+1} \leq -1 \\ j_1+\dots+j_{k+1}=n}} x_2(j_1) \dots x_2(j_{k+1}) \Big) \subset \ker f.$$

¿From the proof of Lemma 4.4 we see that

$$\{x_2(-1)^{k_0+1}\} \bigcup U(\mathfrak{g}_0) \cdot x_2(-1)^{k_0+k_2+1} \subset \ker f.$$

Hence we have a surjective linear map

$$g \colon \mathcal{P}/\mathcal{I}_\Lambda \to W(\Lambda).$$

On the quotient $\mathcal{P}/\mathcal{I}_\Lambda$ we have relations

$$U(\mathfrak{g}_0) \cdot \Big(\sum_{\substack{j_1,\dots,j_{k+1} \leq -1 \\ j_1+\dots+j_{k+1}=n}} x_2(j_1) \dots x_2(j_{k+1}) \Big) = 0 \quad \text{for all} \quad n \leq -k-1$$

and

$$x_2(-1)^{k_0+1} = 0 \quad \text{and} \quad U(\mathfrak{g}_0) \cdot x_2(-1)^{k_0+k_2+1} = 0.$$

As in the proof of Lemma 4.4 we see that monomials $x(\pi) \in \mathcal{P}$ satisfying DC and IC span the quotient $\mathcal{P}/\mathcal{I}_\Lambda$. Since g maps this spanning set to a basis of $W(\Lambda)$, monomials $x(\pi) \in \mathcal{P}$ satisfying DC and IC are a basis of $\mathcal{P}/\mathcal{I}_\Lambda$ and g is an isomorphism. □

10. LINEAR INDEPENDENCE OF MONOMIAL BASES OF STANDARD $A_1^{(1)}$-MODULES

Let now $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the standard basis e, h, f. Then we have monomial bases of standard $\tilde{\mathfrak{g}}$-modules constructed in [MP1], [MP2] and [FKLMM]:

For integral dominant $\Lambda = k_0\Lambda_0 + k_1\Lambda_1$ *of level* $k = k_0 + k_1$ *the set of finite monomial vectors*

$$x(\pi)v_\Lambda = \ \dots f(-j)^{c_j} h(-j)^{b_j} e(-j)^{a_j} \dots f(-1)^{c_1} h(-1)^{b_1} e(-1)^{a_1} f(0)^{c_0} v_\Lambda \tag{10.1}$$

satisfying difference conditions

$$\begin{aligned} c_{j+1} + b_{j+1} + c_j &\le k, \\ b_{j+1} + a_{j+1} + c_j &\le k, \\ a_{j+1} + c_j + b_j &\le k, \\ a_{j+1} + b_j + a_j &\le k \end{aligned}$$

for all $j \ge 0$, *and initial conditions* $a_1 \le k_0$ *and* $c_0 \le k_1$, *is a basis of standard* $\tilde{\mathfrak{g}}$*-module* $L(\Lambda)$.

The proof of spanning uses the relation $e(z)^{k+1} = 0$ in a usual way, and there are two different proofs of linear independence in the works cited above. Here we give another proof of linear independence of bases (10.1) for vacuum standard modules $L(k\Lambda_0)$ based on coincidence of presentation given by Theorem 9.1 and the following:

Theorem 10.1. *Let* k *be a positive integer. Let*

$$\mathcal{P} = \mathbb{C}[f(j), h(j), e(j) \mid j \le -1]$$

and let $\mathcal{I}_{k\Lambda_0}$ *be the ideal in the polynomial algebra* $\mathcal{P}$ *generated by polynomials*

$$\bigcup_{n \le -k-1} U(\mathfrak{g}) \cdot \Big(\sum_{\substack{j_1, \dots, j_{k+1} \le -1 \\ j_1 + \dots + j_{k+1} = n}} e(j_1) \dots e(j_{k+1}) \Big)$$

(here $\cdot$ *denotes the adjoint action of* $\mathfrak{g}$ *on* $\mathcal{P}$*). Then as* $\mathbb{Z}$*-graded vector spaces and* $\mathfrak{g}$*-modules*

$$L(k\Lambda_0) \cong \mathcal{P}/\mathcal{I}_{k\Lambda_0}.$$

Proof. As in [MP2], let

$$N(k\Lambda_0) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}_{\ge 0})} \mathbb{C} v_{k\Lambda_0}$$

be a vacuum level k generalized Verma $\tilde{\mathfrak{g}}$-module and view it as a vertex operator algebra. The maximal submodule $N^1(k\Lambda_0)$ is generated by the singular vector $e(-1)^{k+1} v_{k\Lambda_0}$. Let $\bar{R}$ be a vector space spanned by all coefficients r_n of vertex operators

$$Y(r, z) = \sum_{n \in \mathbb{Z}} r_n z^{-n-1}, \qquad r \in R = U(\mathfrak{g}) \cdot e(-1)^{k+1} v_{k\Lambda_0}.$$

Then $\bar{R}$ is a loop $\tilde{\mathfrak{g}}$-module and $N^1(k\Lambda_0) = \bar{R} N(k\Lambda_0)$ (see Chapter 3 in [MP2]). Hence we have

$$\begin{aligned} &L(k\Lambda_0) \cong N(k\Lambda_0)/\bar{R}N(k\Lambda_0), \\ &N(k\Lambda_0) \cong U(\tilde{\mathfrak{g}}_{<0}), \\ &N^1(k\Lambda_0) \cong U(\tilde{\mathfrak{g}}_{<0}) \bigoplus_{n \le -k-1} U(\mathfrak{g}) \cdot \Big(\sum_{\substack{j_1, \dots, j_{k+1} \le -1 \\ j_1 + \dots + j_{k+1} = n}} e(j_1) \dots e(j_{k+1}) \Big), \end{aligned}$$

and the theorem follows by passing from the filtered enveloping algebra $U(\tilde{\mathfrak{g}}_{<0})$ to the symmetric algebra $\mathcal{P} = S(\tilde{\mathfrak{g}}_{<0})$ (cf. Ch. III §2 n°4 Proposition 2 in [Bo]). □

¿From Theorems 9.1 and 10.1 we se that $W(k\Lambda_0)$ for $B_2^{(1)}$ and $L(k\Lambda_0)$ for $A_1^{(1)}$ have the same presentation $\mathcal{P}/\mathcal{I}$, so Theorem 3.1 implies linear independence of bases (10.1) for standard $A_1^{(1)}$-modules $L(k\Lambda_0)$.

Due to this coincidence E. Feigin's fermionic formula [F] for $A_1^{(1)}$-module $L(k\Lambda_0)$ is also a character formula of Feigin-Stoyanovsky's subspace $W(k\Lambda_0)$ for $B_2^{(1)}$.

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