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Multipliers on Pseudoconvex Domains with Real Analytic Boundaries

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Multipliers on pseudoconvex domains with real analytic boundaries[1]

Dedicated to the memory of Professor Aldo Andreotti on the 30th anniversary of his death

Joseph J. Kohn

Abstract

This paper is concerned with (weakly) pseudoconvex real analytic hypersurfaces in $\mathbb{C}^n$. We are motivated by the study of local boundary regularity of the $\bar{\partial}$-Neumann problem. Subelliptic estimates in a neighborhood of a point P in the boundary (which imply regularity) are controlled by ideals of germs of real analytic functions $I^1(P), \ldots, I^{n-1}(P)$. These ideals have the property that a subelliptic estimate holds for (p,q)-forms in a neighborhood of P if and only if $1 \in I^q(P)$. The geometrical meaning of this is that $1 \in I^q(P)$ if and only if there is a neighborhood of P such that there does not exist a q-dimensional complex analytic manifold contained in the intersection of this neighborhood. Here we present a method to construct these manifolds explicitly. That is, if $1 \notin I^q(P)$ then in every neighborhood of P we give an explicit construction of such a manifold. This result is part of a program to give a more precise understanding of regularity in terms of various norms. The techniques should also be useful in the study of other systems of partial differential equations.

Introduction

Complex analysis on a domain $\Omega \subset \mathbb{C}^n$, from the point of view of partial differential equations, is basically the study of the Cauchy-Riemann equations on Ω. In particular the study of the inhomogeneous Cauchy-Riemann equations $\bar{\partial}\varphi = \alpha$, where φ is orthogonal to the nullspace of $\bar{\partial}$, leads to the $\bar{\partial}$-Neumann problem. There are a number of excellent sources which include a detailed exposition of the $\bar{\partial}$-Neumann problem, for example [CS] and [S].

In case Ω is bounded, pseudoconvex, and has a smooth boundary an important question is local regularity on the boundary. This can be formulated as follows. Suppose that $P \in b\Omega$, where $b\Omega$ denotes the boundary of Ω, and that U is a neighborhood of P such that the restriction of a (p,q)-form α to $U \cap \bar{\Omega}$ is in $C^\infty(U \cap \bar{\Omega})$. The question is: when is $\varphi \in C^\infty(U \cap \bar{\Omega})$? This

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problem is open, however it has been solved for a large class of domains. In particular, it is completely solved in the case when $U \cap b\Omega$ is real analytic (see [K1]). In general, local regularity for (p,q)-forms does not hold if there exists a complex analytic q-dimensional variety $W \subset U \cap b\Omega$. The result in the case when $U \cap b\Omega$ is real analytic is that local regularity holds for (p,q)-forms if and only if there does not exist any complex analytic q-dimensional variety $W \subset U \cap b\Omega$. To prove this result we need subelliptic estimates which we describe below.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with a smooth boundary. By a smooth boundary we mean that the boundary of Ω, denoted by $b\Omega$, has a neighborhood U on which there is a function $r \in C^\infty(U)$ with $dr \neq 0$ and $r = 0$ on $b\Omega$. If $P \in b\Omega$ we denote by $T_P^{1,0}(b\Omega)$ the space;

$$T_P^{1,0}(b\Omega) = \{L \in T^{1,0}(\mathbb{C}^n) \mid L(r)|_P = 0\}.$$

The Levi form at P is the quadratic form on $T_P^{1,0}(b\Omega)$, denoted by $\mathcal{L}_P$, defined by:

$$\mathcal{L}_P(L, \bar{L}') =< (\partial\bar{\partial} r)_P, L \wedge \bar{L}' >, \tag{1}$$

where $L, \bar{L}' \in T_P^{1,0}(b\Omega)$. Using Cartan's identity the Levi form can also be given by

$$\mathcal{L}_P(L, L') =< [L, \bar{L}'], (\partial r)_P >, \tag{2}$$

where $[L, \bar{L}'] = L\bar{L}' - \bar{L}'L$. We choose r so that $r < 0$ in $U \cap \Omega$. The domain $b\Omega$ is pseudoconvex if $\mathcal{L}_P$ is positive semidefinite for each $P \in b\Omega$. We denote by $\mathcal{A}^{p,q}$ the space (p,q)-forms on $\bar{\Omega}$ which are in $C^\infty(\bar{\Omega})$. We denote by $\bar{\partial} : C^\infty(\bar{\Omega}) \to \mathcal{A}^{0,1}$ the operator defined by $< \bar{\partial} f, \bar{L} >= \bar{L}(f)$. The induced operator on forms is also denoted by $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$. The L_2-adjoint of $\bar{\partial}$ denoted by $\bar{\partial}^*$. The domain of $\bar{\partial}^*$ intersected with $C^\infty(\bar{\Omega})$ is denoted by $\mathcal{D}^{p,q}$, so that $\bar{\partial}^* : \mathcal{D}^{p,q} \to \mathcal{A}^{p,q-1}$, where

$$\mathcal{D}^{p,q} = \{\varphi \in \mathcal{A}^{p,q} \mid \varphi \llcorner \bar{\partial} r|_{b\Omega} = 0\}, \tag{3}$$

where $\llcorner$ denotes the interior product. On $\mathcal{D}^{p,q}$ we define the "energy form" Q by

$$Q(\varphi, \psi) = (\bar{\partial}\varphi, \bar{\partial}\psi) + (\bar{\partial}^*\varphi, \bar{\partial}^*\psi). \tag{4}$$

If $P \in b\Omega$ we say that a subelliptic estimate for (p,q)-forms holds at P if there exists a neighborhood U of P and positive constants ε and C such that

$$\|\varphi\|_\varepsilon^2 \leq CQ(\varphi, \varphi), \tag{5}$$

for all $\varphi \in \mathcal{D}^{p,q}$.

The estimate (5) for $\varepsilon = \frac{1}{2}$ was first proved, for strongly pseudoconvex domains (i.e. when the Levi form is positive definite), by C. B. Morrey (see [M]). In that case (5) is used (see [K1]) to prove existence and local regularity for the $\bar{\partial}$-Neumann problem on strongly pseudoconvex domains. L. Nirenberg and I (see [KN]) proved that if (5) holds at $P \in b\Omega$ for some $\varepsilon > 0$ then the $\bar{\partial}$-Neumann problem is locally regular in a neighborhood of P. The local regularity of the $\bar{\partial}$-Neumann problem has many applications (see [CS] and [S]), including the study of the Bergmann kernel, holomorphic mappings, etc.

The key to understanding (5) at $P \in b\Omega$ is the analysis of the maximum order of contact of germs of complex analytic varieties at P to $b\Omega$. In dimension 2 it suffices to consider only non-singular varieties, necessary and sufficient conditions for (5) are given in [K2] and [G]. In higher dimensions singular varieties enter the picture and the problem is much more difficult. J. P. D'Angelo discovered (see [D'A]) that if the maximum order of contact (called the D'Angelo type) at P is finite then it is finite at all points in a neighborhood of P. Using this result D. Catlin (see [C2]) solved the problem completely by proving the remarkable result that (5) holds at P if and only if the D'Angelo type at P is finite.

This problem has been studied in [K3] using ideals of subelliptic multipliers. These are defined as follows.

Definition 1 *Suppose that $\Omega \in \mathbb{C}^n$ is a bounded pseudoconvex domain with a C^∞ boundary and that $P \in b\Omega$ then a subelliptic multiplier for (p,q)-forms on Ω at P is a germ of a C^∞ function f at P such that there exists a neighborhood U of P and positive constants ε and C such that*

$$\|f\varphi\|_\varepsilon^2 \leq CQ(\varphi,\varphi), \tag{6}$$

for all $\varphi \in \mathcal{D}^{p,q} \cap C^\infty(U \cap \bar{\Omega})$. We denote by $\mathcal{I}^{p,q}(P,\Omega)$ the set of all f with the above property.

Remark: $\mathcal{I}^{p,q}(P,\Omega)$ is independent of q, that is $\mathcal{I}^{p,q}(P,\Omega) = \mathcal{I}^{0,q}(P,\Omega)$ for all p. We will write $\mathcal{I}^q(P,\Omega)$ instead of $\mathcal{I}^{p,q}(P,\Omega)$.

Obviously (5) holds for some neighborhood of $P \in b\Omega$ if and only if $1 \in \mathcal{I}^{p,q}(P,\Omega)$. The proof in [K3] is based on a construction, recalled below, of an ascending sequence of ideals contained in $\mathcal{I}^q(P,\Omega)$. In case the defining function r is real analytic in a neighborhood of P it is shown that if 1 is not in any ideal in this sequence then there exists a manifold of holomorphic dimension q in $b\Omega$. Then, using a result of Diederich and Fornaess (see [DF]), it is shown that the existence of such manifolds in $b\Omega$ is equivalent to the existence of complex analytic manifolds in $b\Omega$. Different proofs of the result of Diederich and Fornaess were given by Bedford and Fornaess (see [BF]) and by Siu (see [Si]). Here we present an explicit construction of these manifolds.

The main theorem

In this section we will formulate the main theorem. First, we will construct the sequence of ideals mentioned above.

In [K3] it is shown that the set $\mathcal{I}^q(P,\Omega)$ has the following properties:

1. $\mathcal{I}^q(P,\Omega)$ is an ideal over C^∞.

2. $\mathcal{I}^q(P,\Omega) = \sqrt[\mathbb{R}]{\mathcal{I}^q(P,\Omega)}$, where for any ideal S of germs of functions $\sqrt[\mathbb{R}]{S}$, the real radical of S, is defined to be the ideal consisting of all g such that there exists an $m \in \mathbb{Z}^+$ and an $f \in S$ with $|g|^m \leq |f|$.

3. r and the coefficients of $\partial r \wedge \bar\partial r \wedge (\partial\bar\partial r)^{n-q}$ are in $\mathcal{I}^q(P,\Omega)$.

4. If $f_1,\dots,f_j$ are in $\mathcal{I}^q(P,\Omega) = \sqrt[\mathbb{R}]{\mathcal{I}^q_P(P,\Omega)}$ then the coefficients of $\partial f_1 \wedge \cdots \wedge \partial f_j \wedge \partial r \wedge \bar\partial r \wedge (\partial\bar\partial r)^{n-q-j}$ are in $\mathcal{I}^q(P,\Omega)$.

If S is a set of forms we will denote by $coeff\, S$ the ideal generated by the coefficients of the forms in S with respect to a holomorphic system of coordinates. Note that this ideal is independent of the coordinates used.

Definition: If $f_1,\dots,f_j$ are are germs of C^∞ functions at $P \in b\Omega$ we denote by $A^q_k(P,f_1,\dots,f_j)$ the set of germs of C^∞ functions at P defined by:

$$A^q(P,f_1,\dots,f_j) = coeff\{\partial f_1 \wedge \dots \partial f_j \wedge \partial r \wedge \bar\partial r \wedge (\partial\bar\partial r)^{n-q-j}\}.$$

Set

$$I^q_1(P) = \sqrt[\mathbb{R}]{(r, coeff\{\partial r \wedge \bar\partial r \wedge (\partial\bar\partial r)^{n-q}\})}$$

and, inductively for $k > 1$

$$I^q_k(P) = \sqrt[\mathbb{R}]{(I^q_{k-1}(P),\ \bigcup\{A^q(P,f_1,\dots,f_j)\})},$$

where the union is taken over all j-tuples of elements in $I^q_{k-1}(P)$, with $1 \leq j \leq n-q$. Then $I^q_k(P) \subset I^q_{k+1}(P)$ and we define the ideal

$$I^q(P) = \bigcup I^q_k(P)$$

Then clearly $I^q(P) \subset \mathcal{I}^q(P,\Omega)$ hence (5) holds if $1 \in I^q(P)$.

The following definition is used to formulate the main result.

Definition: Let $V \subset b\Omega$ be a real analytic variety. Then $b\Omega$ is $\underline{V\text{-convex}}$ if whenever $P \in V$ is a regular point and $L \in T^{1,0}_P(b\Omega)$ is transversal to $T^{1,0}_P(V)$ then $\mathcal{L}(L,\bar{L}) > 0$.

The main result of this paper is the following.

Main Theorem: Let $\Omega \subset \mathbb{C}^n$ be a domain with a smooth boundary $b\Omega$ and $P_0 \in b\Omega$ and assume that $b\Omega$ is $\mathcal{V}(I^q(P_0))$-convex, where $\mathcal{V}\big(I^q(P_0)\big)$ denotes the variety of $I^q(P_0)$. Then $1 \notin I^q(P_0)$ if and only if in every neighborhood U of P_0 there exists a q-dimesional complex analytic variety $W \subset U \cap \mathcal{V}\big(I^q(P_0)\big)$, so that, in particular, $W \subset U \cap b\Omega$.

Remarks:

1. In the general case when $b\Omega$ is $\mathcal{V}(I^q(P_0))$-convexity does not hold the theorem is still true but the argument is much more complicated and will be presented in a more general context in a future paper.

2. In the important case when $dim\mathcal{V}(I^q(P_0)) = n-1$, there are no transversal $(1,0)$ vectors therefore $b\Omega$ is $\mathcal{V}(I^q(P_0))$-convex.

First we will prove that the existence of such a W implies that $1 \notin I^q(P_0)$, that is:

Necessity: if in every neighborhood U of P_0 there exists a q-dimesional complex analytic variety $W \subset U \cap b\Omega$ then $1 \notin I^q(P_0)$. Since the condition $1 \in I^q(P_0)$ is open it will suffice to show that if $P \in W$ is a regular point of W then $1 \notin I^q(P)$. Let $\{z_1, \ldots, z_n\}$ be holomorphic coordinates with origin at P_0. Let U be a neighborhood of P so that all points in $U \cap W$ are regular points of W. Let L be a $(1,0)$ vectorfield tangent to W. Then $L = \sum \zeta_i \frac{\partial}{\partial z_i}$ with $L(r) = 0$ when $r = 0$. Since $\mathbb{C}T(W) = T^{1,0}(W) \oplus T^{0,1}(W)$ we obtain, applying (2), $\mathcal{L}(L,\bar{L}) =< [L,\bar{L}], \partial r >= 0$ on W. Since $\mathcal{L}$ is semi-definite on $b\Omega$ the ζ satisfy the following system of equations on $U \cap W$

$$\sum_{i=1}^{n} r_{z_i}\zeta_i = 0$$

and

$$\sum_{i=1}^{n} r_{z_i\bar{z}_j}\zeta_i = 0,$$

for $j = 1, \dots, n$. Since W is q-dimensional these equations have q linearly independent solutions on W and hence, by Cramer's rule, the coefficients of $\partial r \wedge \bar{\partial} r \wedge (\partial\bar{\partial} r)^{n-q}$ vanish on W. Therefore the elements $I_1^q(P)$ vanish on W so that if $f \in I_1^q(P)$ then $L(f) = 0$ on W. Thus the ζ satisfy the additional equations

$$L(f) = \sum_{i=1}^{n} f_{z_i}\zeta_i = 0,$$

on W for each $f \in I_1^q(P)$. Hence, adjoining these equations to the above system, we conclude that all elements of $I_2^q(P)$ vanish on W. Proceeding inductively we see that for each k all elements of $I_k^q(P)$ vanish on W and therefore $1 \notin I^q(P)$ completing the proof.

Remark: Note that if there is a q-dimensional manifold $W \subset b\Omega$ then the above implies that for each $P \in W$ we have $1 \notin I^q(P)$ and hence $W \subset \mathcal{V}(I^q(P_0))$.

From now on we will deal only with germs of real analytic functions (more precisely complex value functions whose real and imaginary parts are real analytic), thus in each of the ideals defined above we will consider only the germs of real analytic functions. We will abuse notation and denote by $I_k^q(P)$ and $I^q(P)$ the ideals consisting only of real analytic functions contained in the corresponding ideals defined above.

Definition: If I is an ideal of germs of real analytic functions at P then the Zariski tangent space at P is defined by

$$\mathcal{Z}_P^{1,0}(I) = \{L \in T_P^{1,0} \mid L(f) = 0 \text{ if } f \in I\}.$$

If $V \subset b\Omega$ is a germ of a real analytic variety at P_0 and if $P \in V$ then the Zariski tangent space to V at P is defined by

$$\mathcal{Z}_P^{1,0}(V) = \mathcal{Z}_P^{1,0}(\mathcal{I}(V)).$$

We denote by $V_k^q(P)$ and by $V^q(P)$ the varieties $\mathcal{V}(I_k^q(P))$ and $\mathcal{V}(I^q(P))$, respectively. Then from the above definitions we obtain the following lemma (see [K3]).

Lemma 1 *If $P \in V_k^q(P_0)$ then $P \in V_{k+1}^q(P_0)$ if and only if*

$$dim(\mathcal{Z}_P^{1,0}(I_k^q(P_0)) \cap \mathcal{N}_P) \geq q,$$

where $\mathcal{N}_P$ is the null space of the Levi form defined by

$$\mathcal{N}_P = \{L \in T_P^{1,0}(b\Omega) \mid \mathcal{L}_P(L, \bar{L}) = 0\}.$$

Then we have:

$$dim(\mathcal{Z}_P^{1,0}(I^q(P_0)) \cap \mathcal{N}_P) \geq q.$$

The following lemma is easily proved with a slight modification of the proof of necessity given above.

Lemma 2 *If $V \subset b\Omega$ is a germ of real analytic variety at P_0 and if there exists a sequence $\{P_\nu\}$ of regular points of V which converges to P_0 such that for each P_ν has a neighborhood U_ν with $dim T_P^{1,0}(V \cap U_\nu) \cap \mathcal{N}_P = q$ then $1 \notin I^p(P_0)$.*

To prove the theorem we have to find $W \subset b\Omega$. We know that if such a W exists then $W \subset \mathcal{V}(I^q(P))$, when $P \in W$. To each $f \in I^q(P)$ we will associate a holomorphic function $H_P[f]$, defined below, such that $H_P[f](P) = 0$ and $H_P[f]$ vanishes on any complex manifold that contains P and is contained in $\{Q \in \mathbb{C}^n \mid f(Q) = 0\}$.

Definition: If f is a real analytic function in a neighborhood of $P \in \mathbb{C}^n$ we define the holomorphic function $\mathcal{H}_P[f]$ by

$$\mathcal{H}_P[f] = \sum \frac{1}{\alpha!} D_z^\alpha f(P)(z - P)^\alpha.$$

Note that $\mathcal{H}_P[f]$ does not depend on the choice of coordinates, it is characterized as the unique holomorphic function h with the property that $D_z^\alpha h|_P = D_z^\alpha f|_P$, for all $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Remark: If W is a complex analytic manifold on which a real analytic function f vanishes and if $P \in W$ then $\mathcal{H}_P[f]$ vanishes on W. This is true because: if a complex curve through P contained in W is parametrized by $t \mapsto z(t) = (z_1(t), \ldots, z_n(t))$ with $t \in \{\mathbb{C} \mid |t| < 1\}$, $z(0) = P$, and $f(z(t)) = 0$ for all $t \in \{\mathbb{C} \mid |t| < 1\}$ then

$$D_t^m f(z(t))|_0 = D_t^m \mathcal{H}_P[f](z(t))|_0 = 0,$$

for all m. Hence $\mathcal{H}_P[f](z(t)) = 0$ for all $t \in \{\mathbb{C} \mid |t| < 1\}$.

The first step in the proof of the theorem is to define a subset $A \subset V^q(P_0)$ with $P_0 \in \bar{A}$. We will prove that if U is any neighborhood of P_0 and if $P \in U \cap A$ then there exists a complex analytic manifold W with $P \in W \subset V^q(P) \subset b\Omega$ with $dim W = q$. A is defined as follows. Let $g_1, \ldots, g_N$ be generators of $I^q(P_0)$.

1. $A_1 = U \cap regV^q(P_0)$, where $regV^q(P_0)$ are the regular points of $V^q(P_0)$.

2. $A_2 = \{P' \in A_1$ such that there exists a neighborhood $U' \subset \bar{U}' \subset U$ of P' such that $U' \cap regV^q(P_0)$ is a real analytic manifold$\}$.

3. $A_3 = \{P' \in A_2$ such that $dim(U' \cap regV^q(P_0))$ is maximal$\}$.

4. $A_4 = \{P' \in A_3$ such that there exists a neighborhood $U'' \subset \bar{U}'' \subset U'$ of P' with $dim(T^{1,0}_{P'}(U'' \cap regV^q(P_0)) = dim(T^{1,0}_{P''}(U'' \cap regV^q(P_0))$ for all $P'' \in U'' \cap regV^q(P_0)\}$.

5. $A = \{P'' \in A_4$ such that there exists a neighborhood $U''' \subset \bar{U}''' \subset U''$ of P'' with $dim(T^{1,0}_{P''}(U''' \cap V^q(P_0) \cap \mathcal{N}_{P''})) = dim(T^{1,0}_{P''}(U''' \cap V^q(P_0) \cap \mathcal{N}_{P'''}))$ for all $P''' \in U''' \cap regV^q(P_0)\}$.

Fix $P \in A$ and abusing notation we will denote the neighborhood U''' corresponding to P by U. We will assume that q is as large as possible, so that

$$1 \notin I^q(P_0) \text{ but } 1 \in I^{q+1}(P_0). \tag{7}$$

Set $m = dim T^{1,0}_P(V^q(P_0)$. Then, from lemma 2, it follows that $q = dim(T^{1,0}_P(V^q(P_0) \cap \mathcal{N}_P)$.

By a coherence theorem for ideals of real analytic functions (see [N]) the ideal $I^q(P)$ is generated by elements of $I^q(P_0)$. Hence there exist $f_{m+1}, \ldots, f_n \in I^q(P_0)$ such that $\partial f_{m+1} \wedge, \ldots, \wedge \partial f_n \neq 0$ in a neighborhood of P. Without loss of generality, we assume that $f_n = r$ and that the

$f_{m+1}, \dots, f_{n-1}$ are real. Let $z_1, \dots, z_n$ be coordinates with origin at P such that:

1. $z_i = \mathcal{H}_P(f_i)$ for $i = m+1, \dots, n$.

Then $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_m}$ is a basis of $T^{1,0}_P(V^q(P_0)$. Then by replacing the $z_1, \dots, z_n$ and the $f_{m+1}, \dots, f_{n-1}$ by appropriate linear combinations we obtain:

2. $r_{z_i \bar{z}_j}(0) = \lambda_i \delta_{ij}$ for $i, j = 1, \dots, n-1$. Since in a neighborhood U of P the rank of the Levi form on $T^{1,0}(U \cap V^q(P_0)) \cap \mathcal{N}_P$ is constant and since $1 \in I^{q+1}(P_0)$ we have $\lambda_1 = \cdots = \lambda_q = 0$ and $\lambda_i > 0$ when $q+1 \leq i \leq m$. From $V^q(P)$-convexiy it follows that $\lambda_i > 0$ when $m+1 \leq i \leq n-1$.

Note that $f_{l z_j}(0) = \delta_{lj}$ for $l = m+1, \dots, n$. Furthermore, $\partial^\alpha_z f_l(0) = \partial^\alpha_z r(0) = 0$ when $|\alpha| > 1$.

We define new holomorphic coordinates $\{w_1, \dots, w_n\}$ on a small neighborhood U' of P by

$$\begin{cases} w_i = z_i, & \text{if } 1 \leq i \leq q \text{ and } i = n; \\ w_j = z_j + \frac{1}{\lambda_j} \sum_{\alpha \in \mathbb{A}} \frac{\partial^\alpha_z \partial_{\bar{z}_j} r(0)}{\alpha!} z^\alpha, & \text{if } q+1 \leq j \leq m, \end{cases}$$

where $\mathbb{A}$ is the set of all $\alpha = \alpha_1 \dots \alpha_n$ such that $\alpha_{q+1} = \cdots = \alpha_n = 0$. It is important to note here the following.

Note: The sum above can be taken only over $|\alpha| \geq 2$ since $r_{z_i \bar{z}_j}(0) = 0$ when $1 \leq i \leq q$.

The main theorem will be proved by showing that if W is the q-dimensional complex manifold W defined by

$$W = \{Q \in U' \mid w_i(Q) = 0, \text{ when } q+1 \leq i \leq n\}$$

then $W \subset V^q \subset b\Omega$. That is, we will show if $f \in I^q$ then we have $f(w_1, \dots, w_q, 0, \dots, 0) = 0$. This will be established by proving that:

$$\partial^\alpha_w \partial^\beta_{\bar{w}} f(0) = 0, \tag{8}$$

when $\alpha, \beta \in \mathbb{A}$. To prove (8) it will suffice to prove

$$\partial^\alpha_w \partial^\beta_{\bar{w}} r(0) = 0, \tag{9}$$

when $\alpha, \beta \in \mathbb{A}$, since (9) implies that $W \subset b\Omega$ and hence, by a previous remark $W \subset V^q$.

The first step towards proving (9) is:

Lemma 3 *:*

$$\partial_w^\alpha \partial_{\bar{w}_j} r(0) = \partial_{\bar{w}}^\alpha \partial_{w_j} r(0) = 0 \tag{10}$$

when $\alpha \in \mathbb{A}$ *and* $q+1 \leq j \leq m$.

Proof: Expanding r in a Taylor series in the z and $\bar{z}$ coordinates we get:

$$\begin{aligned} r(z) &= \sum_{\alpha,\beta} \frac{1}{\alpha!\beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta r(0) z^\alpha \bar{z}^\beta \\ &= 2Re(z_n) + \sum_1^{n-1} \lambda_j |z_j|^2 + 2Re(\sum_{|\alpha|>0} \frac{1}{\alpha!} \partial_z^\alpha \partial_{\bar{z}_j} r(0) z^\alpha \bar{z}_j \\ &\quad + \sum_{|\alpha|>0, |\beta|>0} \frac{1}{\alpha!\beta!} \partial_z^\alpha \partial_{\bar{z}}^\beta r(0) z^\alpha \bar{z}^\beta \end{aligned}$$

expressing the z's in terms of the w's we have

$$\begin{cases} z_i = w_i, & \text{if } 1 \leq i \leq q \text{ and } i = n; \\ z_j = w_j - \frac{1}{\lambda_j} \sum_{\alpha \in \mathbb{A}} \frac{\partial_z^\alpha \partial_{\bar{z}_j} r(0)}{\alpha!} w^\alpha, & \text{if } q+1 \leq j \leq m, \end{cases}$$

then, with $q+1 \leq j \leq m$, we obtain

$$\begin{aligned} \lambda_j |z_j|^2 &= r_{z_j \bar{z}_j}(0) |w_j - \frac{1}{\lambda_j} \sum_{\alpha \in A} \frac{\partial_z^\alpha \partial_{\bar{z}_j} r(0)}{\alpha!} w^\alpha|^2 \\ &= -2Re\Big(\sum_{\alpha \in \mathbb{A}} \frac{\partial_z^\alpha \partial_{\bar{z}_j} r(0)}{\alpha!} w^\alpha \bar{w}_j \Big) + \sum_{m+1}^{n} \lambda_j |w_j|^2 \\ &\quad + \sum_{\alpha \in \mathbb{A}} \frac{1}{\lambda_j} | \frac{\partial_z^\alpha \partial_{\bar{z}_j} r(0)}{\alpha!} |^2 |w^\alpha|^2 \end{aligned}$$

substituting this in the above expansion of r it follows that the coefficients of the $w^\alpha \bar{w}_j$ and the $\bar{w}^\alpha w_j$ vanish when $\alpha \in \mathbb{A}$ and $q+1 \leq j \leq m$, which proves (10).

From now on we will fix P. Then, to simplify notation, we will write I^q and V^q instead of $I^q(P)$ and $V^q(P)$, respectively.

Definitions:

1. $\mathcal{R}_\tau$ is the ideal of germs of functions at 0 generated by $\{\partial_w^\alpha \partial_{\bar{w}}^\beta r\}$ with $\beta \in \mathbb{A}$ and $|\alpha| + |\beta| \leq \tau$.

2. I_τ^q is the ideal of germs of functions at 0 generated by $\{\partial_w^\alpha \partial_{\bar{w}}^\beta I^q\}$ with $\beta \in \mathbb{A}$ and $|\alpha| + |\beta| \leq \tau$. (This is the ideal $I^q(P)_\tau$ it should not be confused with the previously defined $I_\tau^q(P)$.)

3. If $\mathcal{D}$ is an ideal of germs of real analytic functions at 0 and if $\alpha, \beta \in \mathbb{A}$ then we set

$$\mathcal{D}_\tau = \sum_{|\alpha|+|\beta|\leq\tau} \partial_w^\alpha \partial_{\bar{w}}^\beta \mathcal{D},$$

$$\tilde{\mathcal{D}}_{\tau+1} = \sum_{j=q+1}^{m} \frac{\partial}{\partial w_j} \mathcal{D}_\tau,$$

$$\tilde{\tilde{\mathcal{D}}}_{\tau+1} = \sum_{j=m+1}^{n-1} \frac{\partial}{\partial w_j} \mathcal{D}_\tau,$$

$$\mathcal{D}_{[\tau]} = \sum_{|\alpha|\leq\tau} \partial_w^\alpha \mathcal{D},$$

and

$$\mathcal{D}_{[\bar{\tau}]} = \sum_{|\alpha|\leq\tau} \partial_{\bar{w}}^\alpha \mathcal{D}.$$

4. $\mathcal{M}_0$ is the ideal of germs of functions at 0 which vanish at 0.

Remarks:

1. Note that $\{I^q, I_1^q, \mathcal{R}_2, \tilde{\mathcal{R}}_2, \tilde{\tilde{\mathcal{R}}}_2, \mathcal{R}_{[\tau]}, \mathcal{R}_{[\bar{\tau}]}, (\tilde{\mathcal{R}}_1)_{[\tau]}, (\tilde{\mathcal{R}}_1)_{[\bar{\tau}]}), I_{[\tau]}^q, I_{[\bar{\tau}]}^q\} \subset \mathcal{M}_0$, and $r_{w_i\bar{w}_j} \in \mathcal{M}_0$ when $i \neq j$.

2. If $g \in I^q$ then ∂g is a combination of the ∂f_h hence $I_1^q = (f_{m+1}, \ldots, f_n)_1$ so that if $\tau \geq 1$ then $I_\tau^q = (f_{m+1}, \ldots, f_n)_\tau$.

Suppose that U is a sufficiently small neighborhood of 0 so that there is a basis of $T^{1,0}(U \cap b\Omega)$ consisting of the vector fields $L_1, \ldots, L_{n-1}$ on U such that:

1. If $Q \in U \cap V^q$ then $L_1|_Q, \ldots, L_m|_Q$ is a basis of $T_Q^{1,0}(V^q)$.

2. If $Q \in U \cap V^q$ then $L_1|_Q, \ldots, L_q|_Q$ is a basis of $T_Q^{1,0}(U) \cap \mathcal{N}_Q$.

3. $L_i|_0 = \frac{\partial}{\partial w_i}|_0$, for $i = 1, \ldots, n-1$.

Renumbering and taking appropriate linear combinations we have:

$$L_i = \frac{\partial}{\partial w_i} + \sum_{j=q+1}^{m} a_i^j \frac{\partial}{\partial w_j} + \sum_{j=m+1}^{n} b_i^j \frac{\partial}{\partial w_j} \tag{11}$$

for $i = 1, \ldots, q$,

$$L_i = \frac{\partial}{\partial w_i} + \sum_{j=i+1}^{n} c_i^j \frac{\partial}{\partial w_j} \tag{12}$$

for $i = q+1, \ldots, m$, and

$$L_i = \frac{\partial}{\partial w_i} - \frac{r_{w_i}}{r_{w_n}} \frac{\partial}{\partial w_n}. \tag{13}$$

for $i = m+1, \ldots, n-1$.

Definition: Let $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ be ideals generated by the $\{a_i^j\}$ with $1 \le i \le q$ and $q+1 \le j \le m$, the $\{b_i^j\}$ with $1 \le i \le q$ and $m+1 \le j \le n$, and the $\{c_i^j\}$ with $q+1 \le i \le m$ and $i+1 \le j \le n$, respectively.

Remark: Note that $\mathcal{A} + \mathcal{B} + \mathcal{C} \subset \mathcal{M}_0$.

Lemma 4 *With the notation given above*

$$\mathcal{B}_{[\tau]} \subset \mathcal{M}_0 + \mathcal{A}_{[\tau-1]}, \tag{14}$$

$$\mathcal{B}_\tau = I^q_{\tau+1} \quad mod\Big(\mathcal{M}_0 + \mathcal{A}_{\tau-1}\Big), \tag{15}$$

and

$$\partial_w^\alpha \partial_{\bar{w}}^\beta b_i^n \in \mathcal{R}_{\tau+1} + \mathcal{M}_0 + \mathcal{A}_{\tau-1}. \tag{16}$$

$$\partial_w^\alpha \partial_{\bar{w}}^\beta c_h^n \in \tilde{\mathcal{R}}_{\tau+1} + \mathcal{M}_0 + \tilde{\tilde{\mathcal{R}}}_\tau. \tag{17}$$

with $1 \le i \le q$, $q+1 \le h \le m$, *and* $|\alpha| + |\beta| = \tau$.

Proof: For $i = 1, \dots, q$ we have:

$$L_i(f_l) = f_{lw_i} + \sum_{j=q+1}^{m} a_i^j f_{lw_j} + \sum_{j=m+1}^{n} b_i^j f_{lw_j}$$

hence $L_i(f_l) \in (f_l)$, where (f_l) denotes the ideal generated by f_l. Then

$$f_l^l b_i^l \equiv -f_{lw_i} \quad mod\big(\mathcal{M}_0(\mathcal{A} + \mathcal{B})\big).$$

Since $f_{lw_l} \neq 0$ we obtain

$$\mathcal{B}_{[\tau]} = I^q_{[\tau+1]} \quad mod\Big(\mathcal{M}_0(\mathcal{A}_{[\tau]} + \mathcal{B}_{[\tau]}) + \mathcal{A}_{[\tau-1]} + \mathcal{B}_{[\tau-1]}\Big)$$

after applying ∂_w^α and taking the ideals generated when $\{\alpha \in \mathbb{A} \mid |\alpha| \leq \tau\}$, $\{i \mid 1 \leq i \leq q\}$, and $\{l \mid m+1 \leq l \leq n\}$. Then (14) follows by induction. A similar argument gives (15) and since $f_n = r$ we obtain (16). To prove (17) consider

$$L_h(r) = r_{w_h} + \sum_{k=q+1}^{n} c_h^k r_{w_k}.$$

Since $L_h(r) = 0$ on $r = 0$ we have $L_h(r) \in (r)$. We can solve the above for c_h^n, because $r_{w_n} \neq 0$, and obtain

$$c_h^n \in \tilde{\mathcal{R}}_1 + \mathcal{M}_0 \tilde{\tilde{\mathcal{R}}}_1.$$

Then, by differentiating τ times and by induction we obtain (17) concluding the proof of the lemma.

Lemma 5 *: Then we have for $\tau \geq 1$:*

(18) $$\tilde{\tilde{\mathcal{R}}}_{\tau+2} \subset M_0 + \mathcal{R}_\tau + \mathcal{A}_{\tau-1} + \overline{\mathcal{A}}_{\tau-1},$$

and

(19) $$\mathcal{B}_\tau \subset \mathcal{M}_0 + \mathcal{R}_{\tau+1} + \mathcal{A}_{\tau-1} + \overline{\mathcal{A}}_{\tau-1}.$$

Proof: Let L in $T^{1,0}(U \cap b\Omega)$ whose restriction to $U \cap V^q$ is in $T^{1,0}(U \cap V^q) \cap \mathcal{N})$. Then $\mathcal{L}(L, \bar{L}) \geq 0$ on $U \cap b\Omega$ and $\mathcal{L}(L, \bar{L}) = 0$ on $U \cap V^q$. Hence $L_k \mathcal{L}(L, \bar{L}) = 0$ on $U \cap V^q$. So that if $L = \sum_1^q \zeta_i L_i$ we have

$$L_k \mathcal{L}(L, \bar{L}) = \sum_{i,j=1}^{q} \bar{L}_k \mathcal{L}(L_i, \bar{L}_j) \zeta_i \bar{\zeta}_j = 0$$

on $U \cap V^q$. Hence $L_k \mathcal{L}(L_i, \bar{L}_j) = 0$ on $U \cap V^q$ so that $L_k \mathcal{L}(L_i, \bar{L}_j) \in I^q$ for $i = 1, \ldots, q$ and $k = m+1, \ldots, n-1$. The Levi form $\mathcal{L}(L_i, \bar{L}_k)$ is given by

$$(20) \qquad \begin{aligned} \mathcal{L}(L_i, \bar{L}_k) \equiv\ & r_{w_i \bar{w}_k} - r_{w_i \bar{w}_n} \frac{r_{\bar{w}_k}}{r_{\bar{w}_n}} + \sum_{j=q+1}^{m} r_{w_j \bar{w}_k} a_i^j \\ & - \sum_{j=q+1}^{m} r_{w_j \bar{w}_n} a_i^j \frac{r_{\bar{w}_k}}{r_{\bar{w}_n}} + \sum_{j=m+1}^{n} r_{w_j \bar{w}_k} b_i^j \\ & - \sum_{j=m+1}^{n} r_{w_j \bar{w}_n} b_i^j \frac{r_{\bar{w}_k}}{r_{\bar{w}_n}} \quad mod(I^q). \end{aligned}$$

Since L_i is in the null space of the Levi form on V^q we have $\mathcal{L}(L_i, \bar{L}_k) = 0$ on V^q and hence $\mathcal{L}(L_i, \bar{L}_k) \in I^q$. Then

$$(21) \qquad \tilde{\tilde{R}}_2 \subset I^q + \tilde{\tilde{R}}_1 + \mathcal{M}_0 \mathcal{A} + \mathcal{B}.$$

Applying ∂_w^α and summing over $\{\alpha \in \mathbb{A} \mid |\alpha| \leq \tau\}$, we obtain

$$(\tilde{\tilde{R}}_1)_{[\tau+1]} \subset I^q_{[\tau]} + (\tilde{\tilde{R}}_1)_{[\tau]} + \mathcal{M}_0 \mathcal{A}_{[\tau]} + \mathcal{A}_{[\tau-1]} + \mathcal{B}_{[\tau]}$$

then from (14) we obtain

$$(22) \qquad (\tilde{\tilde{R}}_1)_{[\tau+1]} \subset \mathcal{M}_0 + \mathcal{A}_{[\tau-1]}$$

by induction. Next, since $r_{w_j \bar{w}_k}(0) = \lambda_k \delta_{jk}$ with $\lambda_k > 0$, we solve (20) for b_i^k and then using (16) we get

$$\mathcal{B} \subset I^q + \mathcal{R}_1 + \tilde{\tilde{\mathcal{R}}}_2 + \mathcal{M}_0 \Big(\mathcal{A} + \mathcal{B} \Big).$$

Then

$$\mathcal{B}_\tau \subset I^q_\tau + \mathcal{R}_{\tau+1} + \tilde{\tilde{\mathcal{R}}}_{\tau+2} + \mathcal{M}_0 + \mathcal{A}_{\tau-1}.$$

From (15) we get

$$\mathcal{B}_\tau \subset \mathcal{B}_{\tau-1} + \mathcal{R}_{\tau+1} + \tilde{\tilde{\mathcal{R}}}_{\tau+2} + \mathcal{M}_0 + \mathcal{A}_{\tau-1}.$$

hence we obtain

$$\mathcal{B}_\tau \subset \mathcal{R}_{\tau+1} + \tilde{\tilde{\mathcal{R}}}_{\tau+2} + \mathcal{M}_0 + \mathcal{A}_{\tau-1}. \tag{23}$$

Now

$$\begin{aligned} \mathcal{L}(L_i, \bar{L}_j) &\equiv r_{w_i\bar{w}_j} + \sum_{h=q+1}^{m} \big(r_{w_h\bar{w}_j} a_i^h + r_{w_i\bar{w}_h}\bar{a}_j^h\big) \\ &+ \sum_{h=m+1}^{n-1} \big(r_{w_h\bar{w}_j} b_i^h + r_{w_i\bar{w}_h}\bar{b}_j^h\big) \\ &\quad mod\big((b_i^n, b_j^n) + \mathcal{A}\overline{\mathcal{A}} + \mathcal{A}\overline{\mathcal{B}} + \mathcal{B}\overline{\mathcal{A}} + \mathcal{B}\overline{\mathcal{B}}\big). \end{aligned} \tag{24}$$

Hence, solving (24) for $r_{w_i\bar{w}_j}$, we get

$$\mathcal{R}_2 \subset I^q + \mathcal{M}_0\big(\mathcal{A} + \overline{\mathcal{A}} + \mathcal{B} + \overline{\mathcal{B}}\big).$$

Then applying L_k to $\mathcal{L}(L_i, \bar{L}_j)$ in (24) we have $L_k\mathcal{L}(L_i, \bar{L}_j) \in I^q$ and solving for $r_{w_i\bar{w}_j w_k}$ we obtain

$$\tilde{\tilde{\mathcal{R}}}_3 \subset I^q + \mathcal{M}_0\big(\mathcal{A}_1 + \overline{\mathcal{A}}_1 + \mathcal{B}_1 + \overline{\mathcal{B}}_1\big) + \mathcal{A} + \overline{\mathcal{A}} + \mathcal{B} + \overline{\mathcal{B}} + \tilde{\tilde{\mathcal{R}}}_2$$

Then

$$\tilde{\tilde{\mathcal{R}}}_{\tau+3} \subset \mathcal{M}_0 + \mathcal{R}_{\tau+1} + \mathcal{A}_\tau + \overline{\mathcal{A}}_\tau + \mathcal{B}_\tau + \overline{\mathcal{B}}_\tau.$$

Using (23) and its conjugate, we obtain (18) after substituting $\tau - 1$ for τ. Then (19) follows from (18) and (23), thus concluding the proof of the lemma.

Lemma 6 *The ideals $\mathcal{A}, \mathcal{R}$, and $\tilde{\mathcal{R}}$ satisfy the following.*

$$\mathcal{A}_{[\tau]} \subset \mathcal{M}_0 \tag{25}$$

and

$$\mathcal{A}_\tau \subset \mathcal{M}_0 + \mathcal{R}_\tau + \tilde{\mathcal{R}}_{\tau+2}. \tag{26}$$

Proof: If $1 \le i \le q$ and $q+1 \le h \le m$ then we have $\mathcal{L}(L_i, \bar{L}_h) = 0$ on V^q so that $\mathcal{L}(L_i, \bar{L}_h) \in I^q$ and

$$\begin{aligned} \mathcal{L}(L_i, \bar{L}_h) &= r_{w_i\bar{w}_h} + \sum_{k=q+1}^{n} r_{w_i\bar{w}_k}\bar{c}_h^k + \sum_{k=q+1}^{m} r_{w_k\bar{w}_h} a_i^k \\ &+ \sum_{k=m+1}^{n} r_{w_k\bar{w}_h}\bar{b}_i^k + \big(\mathcal{A} + \mathcal{B}\big)\overline{\mathcal{C}}. \end{aligned} \tag{27}$$

Since when $q+1 \leq k \leq m$ we have $r_{w_k\bar{w}_k} \neq 0$ we solve (27) for a_i^k and obtain

$$a_i^k \in I^q + \mathcal{R} + \tilde{\mathcal{R}}_2 + \mathcal{M}_0\tilde{\tilde{\mathcal{R}}}_2.$$

Applying ∂_w^α with $|\alpha| \leq \tau$ and recalling that $I_{[\tau]}^q \subset \mathcal{M}_0$ and that (8) implies that $(\tilde{\mathcal{R}}_1)_{[\tau+1]} \subset \mathcal{M}_0$, we get

$$\mathcal{A}_{[\tau]} \subset \mathcal{M}_0 + \mathcal{R}_{[\tau]} + (\tilde{\tilde{\mathcal{R}}}_1)_{[\tau]}.$$

Then (22) implies

$$\mathcal{A}_{[\tau]} \subset \mathcal{M}_0 + \mathcal{R}_{[\tau]} + \mathcal{A}_{[\tau-2]}$$

then, since $\mathcal{R}_{[\tau]} \subset \mathcal{M}_0$, this proves (25) by induction. Next we get

$$\mathcal{A}_\tau \subset I_\tau^q + \mathcal{R}_\tau + \tilde{\mathcal{R}}_{\tau+2} + \mathcal{M}_0\tilde{\tilde{\mathcal{R}}}_{\tau+2} + \tilde{\tilde{\mathcal{R}}}_{\tau+1}.$$

From (15) and (18) we get

$$\mathcal{A}_\tau \subset \mathcal{M}_0 + \mathcal{R}_\tau + \tilde{\mathcal{R}}_{\tau+2} + \mathcal{A}_{\tau-2} + \overline{\mathcal{A}}_{\tau-2}.$$

Hence

$$\overline{\mathcal{A}}_\tau \subset \mathcal{M}_0 + \mathcal{R}_\tau + \tilde{\mathcal{R}}_{\tau+2} + \mathcal{A}_{\tau-2} + \overline{\mathcal{A}}_{\tau-2}$$

and by induction we prove (26).

To finish the proof of the proposition we still have to prove that $\tilde{\mathcal{R}}_\tau \subset \mathcal{M}_0$ and that $\mathcal{R}_\tau \subset \mathcal{M}_0$. We proceed as follows. For $1 \leq i,j \leq q$ and $q+1 \leq h \leq m$ we have $L_h\mathcal{L}(L_i, \bar{L}_j) \in I^q$ and hence, from (24) we obtain

$$r_{w_i\bar{w}_jw_h} \in I^q + \mathcal{M}_0\big(\mathcal{A}_1 + \overline{\mathcal{A}}_1 + \mathcal{B}_1 + \overline{\mathcal{B}}_1\big) + \mathcal{A} + \overline{\mathcal{A}} + \mathcal{B} + \overline{\mathcal{B}} + \tilde{\mathcal{R}}_2$$

Thus with $|\alpha| + |\beta| \leq \tau + 3$ and $\beta \neq 0$ we obtain

$$\begin{aligned} \partial_w^\alpha \partial_{\bar{w}}^\beta r \quad &\in \quad I_\tau^q + \mathcal{M}_0\big(\mathcal{A}_{\tau+1} + \overline{\mathcal{A}}_{\tau+1} + \mathcal{B}_{\tau+1} + \overline{\mathcal{B}}_{\tau+1}\big) \\ &\quad + \mathcal{A}_\tau + \overline{\mathcal{A}}_\tau + \mathcal{B}_\tau + \overline{\mathcal{B}}_\tau + \tilde{\mathcal{R}}_{\tau+2} \end{aligned}$$

Then applying (15), (19), (26), and (25) we get

$$\tilde{\mathcal{R}}_{\tau+3} \subset \mathcal{M}_0 + \mathcal{R}_\tau + \tilde{\mathcal{R}}_{\tau+2}$$

and thus by induction we get

$$\tilde{\mathcal{R}}_{\tau+3} \subset \mathcal{M}_0 + \mathcal{R}_\tau.$$

Consider $\mathcal{L}(L_i, \bar{L}_j)$ with $1 \leq i, j \leq q$ then $\mathcal{L}(L_i, \bar{L}_j) \in I^q$ and from (24) we get

$$\mathcal{R}_2 \subset \tilde{\mathcal{R}}_2\mathcal{A} + \tilde{\tilde{\mathcal{R}}}_2\mathcal{B} + \sum_{i=1}^{q}(b_i^n) \subset \mathcal{R}_1 + \tilde{\mathcal{R}}_2\mathcal{A} + \tilde{\tilde{\mathcal{R}}}_2\mathcal{B}.$$

Hence

$$\mathcal{R}_{\tau+2} \subset \mathcal{R}_{\tau+1} + \mathcal{M}_0 + \mathcal{A}_{\tau-1} + \mathcal{B}_{\tau-1},$$

then, by induction, we get $\mathcal{R}_{\tau+2} \subset \mathcal{M}_0$. Combing all the above we conclude that all the ideals mentioned are contained in $\mathcal{M}_0$. In particular $I_\tau^q \subset \mathcal{M}_0$, therefore $W \subset V^q$ which completes the proof of the theorem.

References

[BF] BEDFORD, E. and FORNAESS, J. E., *Complex manifolds in pseudoconvex boundaries*, Duke Math. J., 48(1981), 279-288.

[C1] CATLIN, D., *Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem*, Ann. Math., 120(1983), 147-171.

[C2] CATLIN, D., *Subelliptic estimates for the $\bar{\partial}$-Neumann problem on pseudoconvex domains*, Ann. Math., 126(1987), 131-191.

[CS] CHEN, S.-C. and SHAW M.-C., *Partial Differential Equations in Several Complex Variables*, AMS/IP Studies in Advanced Mathematics, vol. 19 (2000), International Press.

[D'A] D'ANGELO, J. P., *Real hypersurfaces, order of contact, and applications*, Ann. Math., 115(1982), 615-637.

[DF] DIEDERICH, K. and FORNAESS, J. E., *Pseudoconvex domains with real analytic boundary*, Ann. Math., 107(1978), 371-384.

[G] GREINER, P. C., *On subelliptic estimates of the $\bar{\partial}$-Neumann problem in $\mathbb{C}^2$*, J. Differential Geom. 9(1974), 239-250.

[K1] KOHN, J. J., *Harmonic integrals on strongly pseudoconvex manifolds I*, Ann. Math.., 78(1963), 112-148, *II*, Ann. Math. 79(1964), 450-472.

[K2] KOHN, J. J., *Boundary behavior of $\bar{\partial}$ on weakly pseudo-convex manifolds of dimension two*, J. Differential Geom., 6(1972), 523-542.

[K3] KOHN, J. J., *Subellipticity of the $\bar{\partial}$-Neumann problem on pseudoconvex domains: sufficient conditions*, Acta Math., 142(1979), 79-122.

[KN] KOHN, J. J. and NIRENBERG, L., *Noncoercive boundary value problems*, Comm. Pure Appl. Math., 18(1965), 443-492.

[M] MORREY, C. B., *The analytic embedding of abstract real analytic manifolds*, Ann. Math. 40(1958), 62-70.

[N] NARASIMHAN, R., *Introduction to the theory of analytic spaces*, Lecture notes in Math. No. 25, Springer Verlag, 1966.

[Si] SIU, Y.-T., *Effective termination of Kohn's algorithm for subeliptic multipliers*, arXiv; 0706.411v2 [math CV] 11 Jul 2008.

[St] STRAUBE, E. J., *Lectures on the L^2-Sobolev Theory of the $\bar{\partial}$-Neumann Problem*, preprint (2009).