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of Pseudo–Holomorphic Discs****Alexandre Sukhov
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Deformations and transversality of pseudo-holomorphic discs

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Abstract. We prove analogs of Thom's transversality theorem and Whitney's theorem on immersions for pseudo-holomorphic discs. We also prove that pseudo-holomorphic discs form a Banach manifold.

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1 Introduction

Classical Thom's transversality theorem and Whitney's approximation theorems (see, e. g., [6]) reflect the flexibility of smooth maps. They are essentially local because the global results follow by means of cut-offs and the like. In complex analytic category, in the absence of cut-offs, the corresponding results hold under global restrictions on the manifolds. Kaliman and Zaidenberg [8] prove the results for Stein manifolds. Forstnerič [3] proves results on flexibility of holomorphic maps under more general hypotheses. In almost complex setting, the problem makes sense only when the source manifold has complex dimension one, because holomorphic maps of almost complex manifolds of higher dimension generally do not exist. Compact pseudo-holomorphic curves might not admit any perturbations for a fixed almost complex structure. In this paper we prove versions of the theorems of Thom and Whitney for pseudo-holomorphic discs.

Theorem 1.1 *Let (M, J) be a C^∞ -smooth almost complex manifold of complex dimension $n \geq 2$ and let $f_0 : \mathbb{D} \rightarrow M$ be a J -holomorphic disc of class $C^\infty(\overline{\mathbb{D}})$; here $\mathbb{D} \subset \mathbb{C}$ is the unit disc. Then there exists a J -holomorphic immersion $f : \mathbb{D} \rightarrow M$ arbitrarily close to f_0 in $C^\infty(\overline{\mathbb{D}})$. Furthermore, f is homotopic to f_0 within the set of smooth J -holomorphic discs in M close to f_0 .*

This result is a version of Whitney's approximation theorem for pseudo-holomorphic discs. We need it in our work [2] on pseudo-holomorphic discs in Stein domains. We hope that Theorem 1.1 will find other applications, in particular, in the theory of intersections and moduli spaces of pseudo-holomorphic curves (see [11]).

For almost complex manifolds of complex dimension 2, McDuff [10] proves local approximation of a pseudo-holomorphic map by an immersion near a singular point. The proof in [10] uses a normal form of a holomorphic map at the singular point (see also [12, 14]), hence it uses the C^∞ smoothness of (M, J) . Although we state the result for C^∞ , our proof goes through for finite smoothness. McDuff [10] also proves that a J -holomorphic map $X \rightarrow (M, J)$ of a compact Riemann surface X can be approximated by a J' -holomorphic immersion for some J' close to J . In our Theorem 1.1, the structure J remains fixed. Moreover, we do not separately analyze singular points of the map. Following the proof in the smooth category, we deduce Theorem 1.1 from the following pseudo-holomorphic analog of Thom's transversality theorem.

Theorem 1.2 *Let (M, J) be a C^∞ -smooth almost complex manifold. Let S be a smooth locally closed real manifold of the space $J^k(\mathbb{C}, M)$ of k -jets of J -holomorphic maps $\mathbb{C} \rightarrow M$. Then the set of J -holomorphic discs $f : \overline{\mathbb{D}} \rightarrow M$ of class $C^\infty(\overline{\mathbb{D}})$ transverse to S (that is, the k -jet $j^k f$ is transverse to S) is dense in the space of all J -holomorphic discs of class $C^\infty(\overline{\mathbb{D}})$.*

We point out that S is a real submanifold of $J^k(\mathbb{C}, M)$ with no regard to the almost complex structures on M or $J^k(\mathbb{C}, M)$ (see [9]).

Perturbations of small J -holomorphic discs are essentially described by the implicit function theorem. For big discs, Ivashkovich and Rosay [7] prove that the center and direction of a J -holomorphic disc admit arbitrary small perturbations. Ivashkovich and Rosay [7] use the result in the study of the Kobayashi-Royden metric on an almost complex manifold.

The proof of Theorem 1.2 uses the Fredholm property of the linearized Cauchy-Riemann operator to parametrize J -holomorphic maps by holomorphic ones. The main difficulty here is that in general the Fredholm operator of the linearized problem has a non-trivial kernel. We modify it by adding a small holomorphic term to obtain an invertible operator. The modification is inspired by an important work of Bojarski [1]. The same idea yields another natural result.

Theorem 1.3 *Let (M, J) be a C^∞ -smooth almost complex manifold. The set of J -holomorphic discs of class $C^{k,\alpha}(\overline{\mathbb{D}})$, $k \geq 1$, $0 < \alpha < 1$ (resp. $W^{k,p}(\overline{\mathbb{D}})$, $k \geq 1$, $2 < p < \infty$) forms a C^∞ -smooth Banach manifold modelled on the space of holomorphic functions $\mathbb{D} \rightarrow \mathbb{C}^n$ of the same class.*

For the usual complex structure, Forstnerič [4] proved the result for holomorphic mappings of a strongly pseudoconvex domain in a Stein manifold. Finally, we point out that the manifold of J -holomorphic discs carries a natural almost complex structure only if J is integrable.

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2 Almost complex structures and J -holomorphic discs

We recall basic notions concerning almost complex manifolds and pseudo-holomorphic discs. Denote by \mathbb{D} the unit disc in \mathbb{C} and by J_{st} the standard complex structure of \mathbb{C}^n ; the value of n will be clear from the context. Let (M, J) be an almost complex manifold. A smooth map $f : \mathbb{D} \rightarrow M$ is called J -holomorphic if

$$df \circ J_{\text{st}} = J \circ df \quad (1)$$

We also call such a map a J -holomorphic disc or a pseudo-holomorphic disc.

In local coordinates $z \in \mathbb{C}^n$, an almost complex structure J is represented by a \mathbb{R} -linear operator $J(z) : \mathbb{C}^n \rightarrow \mathbb{C}^n$, $z \in \mathbb{C}^n$ such that $J(z)^2 = -I$, I being the identity. Then the Cauchy-Riemann equations (1) for a J -holomorphic disc $z : \mathbb{D} \rightarrow \mathbb{C}^n$ can be written in the form

$$z_\eta = J(z)z_\xi, \quad \zeta = \xi + i\eta \in \mathbb{D}.$$

We represent J by a complex $n \times n$ matrix function $A = A(z)$ and obtain the equivalent equations

$$z_{\bar{\zeta}} = A(z)\bar{z}_{\bar{\zeta}}, \quad \zeta \in \mathbb{D}. \quad (2)$$

We first recall the relation between J and A for fixed z . Let $J : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be a \mathbb{R} -linear map so that $\det(J_{\text{st}} + J) \neq 0$, where $J_{\text{st}}v = iv$. Set

$$Q = (J_{\text{st}} + J)^{-1}(J_{\text{st}} - J).$$

One can show that $J^2 = -I$ if and only if $QJ_{\text{st}} + J_{\text{st}}Q = 0$, that is, Q is a complex anti-linear operator. We introduce

$$\mathcal{J} = \{J : \mathbb{C}^n \rightarrow \mathbb{C}^n : J \text{ is } \mathbb{R}\text{-linear, } J^2 = -I, \det(J_{\text{st}} + J) \neq 0\},$$

$$\mathcal{A} = \{A \in \text{Mat}(n, \mathbb{C}) : \det(I - A\bar{A}) \neq 0\}.$$

Let $J \in \mathcal{J}$. Since the map Q is anti-linear, then there is a unique matrix $A \in \text{Mat}(n, \mathbb{C})$ such that

$$Av = Q\bar{v}, \quad v \in \mathbb{C}^n.$$

It turns out that the map $J \mapsto A$ is a birational homeomorphism $\mathcal{J} \rightarrow \mathcal{A}$ (see [13, 15]), and the inverse map $A \mapsto J$ has the form

$$Ju = i(I - A\bar{A})^{-1}[(I + A\bar{A})u - 2A\bar{u}]. \quad (3)$$

Let J be an almost complex structure in a domain $\Omega \subset \mathbb{C}^n$. Suppose $J(z) \in \mathcal{J}$, $z \in \Omega$. Then J defines a unique complex matrix function A in Ω such that $A(z) \in \mathcal{A}$, $z \in \Omega$. We call A the *complex matrix* of J . The matrix A has the same regularity properties as J . A function $f : \Omega \rightarrow \mathbb{C}$ is (J, J_{st}) -holomorphic if and only if it satisfies the Cauchy-Riemann equations

$$f_{\bar{z}} + f_z A = 0, \quad (4)$$

where $f_{\bar{z}}$ and f_z are considered row-vectors. Such non-constant functions generally do not exist unless J is integrable; the integrability condition in terms of $A = (a_{jk})$ has the form

$$N_{jkl} = N_{jlk}, \quad N_{jkl} := (a_{jk})_{\bar{z}_l} + \sum_s (a_{jk})_{z_s} a_{sl}. \quad (5)$$

The main analytic tool in the study of pseudo-holomorphic discs is the Cauchy-Green integral

$$Tu(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{u(\omega) d\omega \wedge d\bar{\omega}}{\omega - \zeta}.$$

As usual, we denote by $C^{k,\alpha}(\overline{\mathbb{D}})$ the space of functions in $\overline{\mathbb{D}}$ whose partial derivatives to order k satisfy a Hölder condition with exponent $0 < \alpha < 1$. We denote by $W^{k,p}(\mathbb{D})$ the Sobolev space of functions with derivatives to order k in $L^p(\mathbb{D})$. We will use the following regularity properties of the Cauchy-Green integral (see, e.g., [16]).

Proposition 2.1 *(i) Let $p > 2$ and $\alpha = (p-2)/p$. Then the linear operator $T : L^p(\mathbb{D}) \rightarrow W^{1,p}(\mathbb{D})$ is bounded. The inclusion $W^{1,p}(\mathbb{D}) \subset C^\alpha(\mathbb{D})$ is bounded, hence, the operator $T : L^p(\mathbb{D}) \rightarrow C^\alpha(\mathbb{D})$ is bounded, and $T : L^p(\mathbb{D}) \rightarrow L^\infty(\mathbb{D})$ is compact. If $f \in L^p(\mathbb{D})$, then $\partial_{\bar{\zeta}} T f = f$, $\zeta \in \mathbb{D}$, as a Sobolev derivative.*

(ii) Let $1 < p < 2$ and $s = 2p(2-p)^{-1}$. Then $T : L^p(\mathbb{D}) \rightarrow L^s(\mathbb{D})$ is bounded.

(iii) Let $k \geq 0$ be integer and let $0 < \alpha < 1$. Then $T : C^{k,\alpha}(\overline{\mathbb{D}}) \rightarrow C^{k+1,\alpha}(\mathbb{D})$ is bounded.

Using the Cauchy-Green operator we can replace the equations (2) by the equivalent integral equation

$$z = T(A(z)\bar{z}_{\bar{z}}) + \phi \quad (6)$$

where $\phi : \mathbb{D} \rightarrow \mathbb{C}^n$ is an arbitrary holomorphic vector function. In a small coordinate chart, we can assume that the matrix A is small with derivatives. By the implicit function theorem, for every small holomorphic vector function ϕ , the equation (6) has a unique solution. In particular, all J -holomorphic discs close to a given small disc are parametrized by small holomorphic vector functions. We would like to establish a similar correspondence for J -holomorphic discs which are not necessarily small.

3 Fredholm theory

Let B_j , $j = 1, 2$ be $n \times n$ matrix functions on \mathbb{D} of class $L^p(\mathbb{D})$, $p > 2$. Solutions of the equation

$$u_{\bar{z}} = B_1 u + B_2 \bar{u} \quad (7)$$

in the class $W^{1,p}(\mathbb{D})$ are called *generalized holomorphic vectors*. The equation (7) arises as the linearized equation (2). We introduce

$$T_0 u = T u - T u(0),$$

$$P u = u - T_0(B_1 u + B_2 \bar{u}).$$

Let $r > 2p(p-2)^{-1}$. Then $s = (1/p + 1/r)^{-1} > 2$. If $u \in L^r$, then $B_1 u, B_2 \bar{u} \in L^s$. Then $T(B_1 u + B_2 \bar{u}) \in W^{1,s}(\mathbb{D})$ is continuous, and $T_0(B_1 u + B_2 \bar{u})$ makes sense. Thus the operator

$$P : L^r(\mathbb{D}) \rightarrow L^r(\mathbb{D})$$

is bounded. The equation (7) is equivalent to the equation

$$P u = \phi \quad (8)$$

where ϕ is a usual \mathbb{C}^n -valued holomorphic function on the unit disc with $\phi(0) = u(0)$. In the scalar case $n = 1$ this equation admits a solution for every ϕ since the kernel of P is trivial. This is a fundamental results of the theory of generalized analytic functions [16]. However, for $n > 1$ this is no longer true (see [1]). The equation (8) in general does not necessarily give a one-to-one correspondence between generalized holomorphic vectors and usual holomorphic ones.

By Proposition 2.1, the operator $u \mapsto T_0(B_1 u + B_2 \bar{u})$ is compact and P is Fredholm. Hence the kernel of P is finite-dimensional. We modify the operator P by adding a small holomorphic term to obtain an operator with trivial kernel.

Theorem 3.1 *Let B_j , $j = 1, 2$ be $n \times n$ matrices of class $L^p(\mathbb{D})$, $p > 2$.*

- (i) *Let w_1, \dots, w_d form a basis of $\ker P$ over \mathbb{R} . There exist holomorphic polynomial vectors p_1, \dots, p_d with $p_1(0) = \dots = p_d(0) = 0$ such that the operator $\tilde{P} : L^r(\mathbb{D}) \rightarrow L^r(\mathbb{D})$ of the form*

$$\tilde{P} u = P u + \sum_{j=1}^d (\operatorname{Re}(u, w_j)) p_j \quad (9)$$

has trivial kernel. The polynomials p_j can be chosen to be arbitrarily small.

- (ii) *If $B_j \in W^{k,p}(\mathbb{D})$, $k \geq 0$, $2 < p < \infty$ (resp. $C^{k,\alpha}(\mathbb{D})$, $k \geq 0$, $0 < \alpha < 1$), then \tilde{P} is an invertible bounded operator in the space $W^{k+1,p}(\mathbb{D})$ (resp. $C^{k+1,\alpha}(\mathbb{D})$), in particular \tilde{P}^{-1} is bounded. The function $\phi = \tilde{P} u$ is holomorphic if and only if u satisfies (7). Furthermore, $\tilde{P} u(0) = u(0)$.*

We point out that a simpler version of Theorem 3.1 holds for the operator T in place of T_0 , but without the conclusion $\tilde{P}u(0) = u(0)$. We need the latter in the proof of Theorem 5.1 below.

In order to study the range of the operator P we make use of the adjoint P^* . For vector functions $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ we introduce the usual inner product

$$(u, v) = \sum_{j=1}^n \frac{i}{2} \int_{\mathbb{D}} u_j \bar{v}_j d\zeta \wedge d\bar{\zeta}.$$

We consider the adjoints with respect to the real inner product $\operatorname{Re}(\bullet, \bullet)$. The operator P^* is defined on $L^q(\mathbb{D})$ with $q = r(r-1)^{-1}$. Given a function χ denote by S_χ the operator

$$S_\chi u = \frac{1}{2\pi i} \int_{\mathbb{D}} \chi(\zeta) u(\zeta) d\zeta \wedge d\bar{\zeta}$$

We write S_1 for $\chi = 1$. The following lemma is immediate.

Lemma 3.2 (i) *The adjoint of the operator of matrix multiplication $u \mapsto B_j u$ has the form $u \mapsto B_j^* u$, where B_j^* denotes the hermitian transpose;*

(ii) $T^* = -\bar{T}$, here $\bar{T}u := \overline{T(\bar{u})}$;

(iii) *the conjugation operator $\sigma : u \mapsto \bar{u}$ is self-adjoint: $\sigma^* = \sigma$;*

(iv) $S_\chi^* = \bar{\chi} S_1$;

(v) $T_0^* = -\bar{T} - \bar{\zeta}^{-1} S_1$; and finally

(vi) $P^* = I + \bar{\zeta}^{-1} B_1^* \bar{T} \bar{\zeta} + \zeta^{-1} \bar{B}_2^* T \zeta \sigma$.

Denote by H_0 the space of all holomorphic vector functions $h \in L^r(\mathbb{D})$ such that $h(0) = 0$. Then H_0 is a closed subspace of $L^r(\mathbb{D})$.

Lemma 3.3 $H_0 + \operatorname{Range} P = L^r(\mathbb{D})$.

Proof : Let $v \in L^q(\mathbb{D})$, $q = r(r-1)^{-1}$ be orthogonal to both H_0 and $\operatorname{Range} P$. We prove that $v = 0$.

Since $(\operatorname{Range} P)^\perp = \ker P^*$, we have $P^*v = 0$, i.e.

$$\bar{\zeta} v = -B_1^* \overline{T(\zeta \bar{v})} - \bar{\zeta} \zeta^{-1} \bar{B}_2^* T(\zeta \bar{v}). \quad (10)$$

By “bootstrapping” we prove $\zeta \bar{v} \in L^p(\mathbb{D})$. Indeed, if $\zeta \bar{v} \in L^t(\mathbb{D})$, $1 < t < 2$, then $T(\zeta \bar{v}) \in L^s(\mathbb{D})$ with $s = 2t(2-t)^{-1}$. Then by (10) $\zeta \bar{v} \in L^k$ with

$$k = \frac{1}{p^{-1} + s^{-1}} = \frac{t}{1 - \beta t} > \frac{t}{1 - \beta}$$

where $\beta = \frac{1}{2} - \frac{1}{p}$, $0 < \beta < \frac{1}{2}$. Starting from $\zeta\bar{v} \in L^q(\mathbb{D})$ and applying the above argument finitely many times, we get $\zeta\bar{v} \in L^m(\mathbb{D})$ for $m > 2$. Finally, applying (10) one more time if necessary, we get $\zeta\bar{v} \in L^p(\mathbb{D})$. Put $u = T(\zeta\bar{v})$. Then $u \in W^{1,p}(\mathbb{D}) \subset C^\alpha(\overline{\mathbb{D}})$ with $\alpha = (p-2)p^{-1}$.

On the other hand, since v is orthogonal to H_0 , then the function $u = T(\zeta\bar{v})$ vanishes on $\mathbb{C} \setminus \overline{\mathbb{D}}$ and, in particular, on $b\mathbb{D}$. By (10), the function u in \mathbb{D} satisfies

$$u_{\bar{z}} = -\bar{B}_1^* u - \zeta \bar{\zeta}^{-1} B_2^* \bar{u}.$$

By the similarity principle ([5], Theorem 3.12; see also [11], Theorem 2.3.5), for $\zeta_0 \in b\mathbb{D}$, say $\zeta_0 = 1$, there exist a continuous nonsingular matrix S and a holomorphic vector ϕ in a neighborhood of ζ_0 in $\overline{\mathbb{D}}$ such that $u = S\phi$. By the boundary uniqueness theorem $\phi \equiv 0$, hence $u \equiv 0$ in a neighborhood of ζ_0 . (In [5], ζ_0 is an interior point, but the proof goes through for a boundary point.) By the connectedness argument, $u \equiv 0$ in \mathbb{D} , hence $v \equiv 0$ in \mathbb{D} . The lemma is proved.

Proof of Theorem 3.1: Part (i). By Lemma 3.3 there exist $p_1, \dots, p_d \in H_0$ such that

$$\text{Span}_{\mathbb{R}}(p_1, \dots, p_d) \oplus \text{Range } P = L^r(\mathbb{D}) \quad (11)$$

By polynomial approximation, we can choose p_j to be polynomial. We now show that the operator \hat{P} defined by (9) has trivial kernel. Let $\hat{P}u = 0$. Then by (11) we have $Pu = 0$ and $\text{Re}(u, w_j) = 0$, $j = 1, \dots, d$. Since the functions w_1, \dots, w_d form a basis of $\ker P$ over \mathbb{R} , we get $u = 0$. This proves part (i). Part (ii) is immediate. The theorem is proved.

The following statement is not new, and we do not need it in the rest of the paper. We include it because it is important by itself, and it is an immediate consequence of Theorem 3.1.

Corollary 3.4 *Let $B_1, B_2 \in L^p(\mathbb{D})$, $p > 2$. Then for every $\psi \in L^p(\mathbb{D})$ the non-homogeneous equation*

$$u_{\bar{z}} = B_1 u + B_2 \bar{u} + \psi$$

has a solution in $W^{1,p}(\mathbb{D})$.

Proof : By Theorem 3.1, $u = \tilde{P}^{-1}T\psi \in W^{1,p}(\mathbb{D})$ is a solution.

4 Manifold of J -holomorphic discs

We are ready to prove Theorem 1.3. Let $f_0 : \mathbb{D} \rightarrow M$ be a J -holomorphic disc. We would like to parametrize all J -holomorphic discs close to f_0 by holomorphic functions.

Following [7], by replacing each disc in M by its graph in $\overline{\mathbb{D}} \times M$, we can reduce the proof to the case, in which the disc f_0 is an embedding and contained in a single chart in $\mathbb{C} \times M$. We further specify the choice of that chart.

For every point $p \in M$ there exists a chart $h : U \subset M \rightarrow \mathbb{C}^n$ such that $h(p) = 0$ and $h_*J(0) = J_{st}$. Since \mathbb{D} is contractible, we can choose such charts h^ζ for every point $p = f_0(\zeta)$ smoothly depending on $\zeta \in \overline{\mathbb{D}}$. Put $H(\zeta, q) = (\zeta, h^\zeta(q))$, $\zeta \in \overline{\mathbb{D}}$, $q \in M$. Then

$$H : \tilde{U} \subset \overline{\mathbb{D}} \times M \rightarrow \overline{\mathbb{D}} \times \mathbb{C}^n$$

defines coordinates $(\zeta, z) \in \mathbb{C} \times \mathbb{C}^n$ in a neighborhood \tilde{U} of $f_0(\overline{\mathbb{D}})$. Then the push-forward $\tilde{J} = H_*(J_{st} \otimes J)$ is defined in a neighborhood of $\overline{\mathbb{D}} \times \{0\} \subset \overline{\mathbb{D}} \times \mathbb{C}^n$. By the definition of \tilde{J} , we have $\tilde{J}|_{\overline{\mathbb{D}} \times \{0\}} = J_{st}$, hence \tilde{J} has complex matrix \tilde{A} such that $\tilde{A}(\zeta, 0) = 0$. Note that the projection $(\zeta, z) \mapsto \zeta$ is (\tilde{J}, J_{st}) -holomorphic, hence by (4) the matrix \tilde{A} has the form

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ b & A \end{pmatrix}.$$

The matrix $A(\zeta, \bullet)$ is the complex matrix of the push-forward $h_*^\zeta J$; $A(\zeta, 0) = 0$, $b(\zeta, 0) = 0$.

A map $f : \mathbb{D} \rightarrow M$ close to f_0 is J -holomorphic if and only if $\zeta \mapsto (\zeta, f(\zeta))$ is $J_{st} \otimes J$ -holomorphic. The latter is equivalent to $\zeta \mapsto (\zeta, h^\zeta(f(\zeta)))$ being \tilde{J} -holomorphic. Finally, it holds if and only if the map $\zeta \mapsto g(\zeta) := h^\zeta(f(\zeta)) \in \mathbb{C}^n$ satisfies the equation

$$g_{\bar{\zeta}} = A\bar{g}_{\bar{\zeta}} + b. \quad (12)$$

This equation is equivalent to

$$g = T_0(A\bar{g}_{\bar{\zeta}} + b) + \phi,$$

where ϕ is holomorphic. We would like to apply the inverse function theorem to the map

$$g \mapsto F(g) = \phi = g - T_0(A\bar{g}_{\bar{\zeta}} + b)$$

to obtain a one-to-one correspondence $f \leftrightarrow g \leftrightarrow \phi = F(g)$. Note that $f_0 \leftrightarrow 0$ and $F(0) = 0$.

The Fréchet derivative $F'(0) : \dot{g} \mapsto \dot{\phi} = F'(0)\dot{g}$ of the map F at 0 has the form

$$\dot{\phi} = \dot{g} - T_0(B_1\dot{g} + B_2\bar{\dot{g}}).$$

Here $(B_1)_{kl} = \sum_{j=1}^n (a_{kj})_{z_l} (\bar{g}_j)_{\bar{\zeta}} + (b_k)_{z_l}$ and $(B_2)_{kl} = \sum_{j=1}^n (a_{kj})_{\bar{z}_l} (\bar{g}_j)_{\bar{\zeta}} + (b_k)_{\bar{z}_l}$, and the derivatives of $A = (a_{kj})$ and $b = (b_k)$ are evaluated at $(\zeta, 0)$.

In general the operator $F'(0)$ may be degenerate. We modify the map F (keeping the same notation F) using Theorem 3.1. Then we have

$$F(g) = \phi = g - T_0(A\bar{g}_{\bar{\zeta}} + b) + \sum_{j=1}^d \operatorname{Re}(w_j, g)p_j, \quad (13)$$

$$F'(0)(u) = u - T_0(B_1u + B_2\bar{u}) + \sum_{j=1}^d \operatorname{Re}(w_j, u)p_j.$$

By Theorem 3.1 one can choose w_j, p_j so that for the modified operator we have $\ker F'(0) = 0$. By the inverse function theorem F^{-1} is well defined and smooth in a neighborhood of zero

in the space of holomorphic vector functions of each class $C^{k,\alpha}$ (resp. $W^{k,p}$), hence it gives a chart in the set of J -holomorphic discs f close to f_0 .

The transition maps between the charts are smooth automatically because the set of J -holomorphic discs is a subset of the manifold of all discs in each smoothness class, and the map F is invertible as a map defined on all discs close to 0.

In conclusion we note that in this proof we could use the operator T instead of T_0 because a version of Theorem 3.1 holds for the operator T . Theorem 1.3 is proved.

Let \mathcal{M} be the manifold of all J -holomorphic discs in M in a smoothness class admitted by Theorem 1.3. It seems reasonable to look for a suitable almost complex structure on \mathcal{M} . However, we show that a natural almost complex structure exists on \mathcal{M} only if J is integrable.

Let $\Phi^\zeta : \mathcal{M} \ni f \mapsto f(\zeta) \in M$ be the evaluation map at $\zeta \in \mathbb{D}$.

Proposition 4.1 *Suppose there is an almost complex structure \mathcal{J} on \mathcal{M} such that the evaluation maps Φ^ζ are (\mathcal{J}, J) -holomorphic. Then J is integrable.*

Proof : Note that if \mathcal{J} exists, then it is unique. It suffices to prove the result for small discs in a coordinate chart. Let $f \in \mathcal{M}$, that is $f_{\bar{\zeta}} = A\bar{f}_{\bar{\zeta}}$. Without loss of generality $f(0) = 0$ and $A(0) = 0$. A tangent vector u at f is a solution of the linearized equation, which implies

$$u_{\bar{\zeta}}(0) = (A_z u + A_{\bar{z}} \bar{u}) \bar{f}_{\bar{\zeta}}(0). \quad (14)$$

The hypothesis about \mathcal{J} means that for every such u , the map Ju also satisfies that equation. Using the description (3) of J in terms on A and $A(0) = 0$, we get $Ju(0) = iu(0)$ and

$$(Ju)_{\bar{\zeta}}(0) = i(u_{\bar{\zeta}} - 2A_z \bar{f}_{\bar{\zeta}} \bar{u})(0).$$

Applying (14) to Ju and comparing the result with (14) again we get

$$A_{\bar{z}} \bar{f}_{\bar{\zeta}} \bar{u}(0) = A_{\bar{z}} \bar{u} \bar{f}_{\bar{\zeta}}(0).$$

Since $u(0)$ and $\bar{f}_{\bar{\zeta}}(0)$ are arbitrary, we have

$$(a_{jk})_{\bar{z}_i}(0) = (a_{jl})_{\bar{z}_k}(0).$$

Since $A(0) = 0$, then the integrability conditions (5) hold at 0, and the proposition follows.

5 Parametrization of jets and evaluation map

We first prove the existence of solutions of (7) with given jet at a fixed point.

Theorem 5.1 *Let $B_1, B_2 \in C^{k-1,\alpha}(\mathbb{D})$, $k \geq 1$, $0 < \alpha < 1$. Then for every $a_0, \dots, a_k \in \mathbb{C}^n$ there exists a solution u of (7) such that $u \in C^{k,\alpha}(\mathbb{D})$ and*

$$\partial_{\bar{\zeta}}^j u(0) = a_j, \quad 0 \leq j \leq k. \quad (15)$$

Proof : As a basis of induction, put $u = \tilde{P}^{-1}a_0$. By Theorem 3.1, $u \in C^{k,\alpha}(\mathbb{D})$ is a solution of (7) with $u(0) = a_0$. Assume by induction, that $u_1 \in C^{k,\alpha}(\mathbb{D})$ is a solution of (7) with

$$\partial_{\zeta}^j u_1(0) = a_j, \quad 0 \leq j \leq k-1.$$

We look for the desired solution of (7) in the form

$$u = u_1 + \zeta^k v \tag{16}$$

Hence v must satisfy the following conditions:

$$v_{\bar{\zeta}} = B_1 v + B_2 \zeta^{-k} \bar{\zeta}^k \bar{v}, \tag{17}$$

$$v(0) = b, \quad b = \frac{a_k - \partial_{\zeta}^k u_1(0)}{k!}.$$

Let \tilde{P}_0 be the operator constructed in Theorem 3.1 for the equation (17). Define $v = \tilde{P}_0^{-1}b$. We show that u given by (16) is in $C^{k,\alpha}(\mathbb{D})$ and satisfies (15).

By Theorem 3.1, $v \in W^{1,p}(\mathbb{D})$ for all $2 < p < \infty$, hence $v \in C^{\alpha}(\mathbb{D})$. By bootstrapping, the equation $\tilde{P}_0 v = b$ implies $v \in C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}_{1/2})$.

Since $w = \zeta^k v$ satisfies (7), then

$$w = T(B_1 w + B_2 \bar{w}) + \phi, \tag{18}$$

where ϕ is holomorphic. Since $w = \zeta^k v \in C^{k,\alpha}(\mathbb{D} \setminus \mathbb{D}_{1/2})$, then by (18) $\phi \in C^{k,\alpha}(b\mathbb{D})$. Since ϕ is holomorphic, then $\phi \in C^{k,\alpha}(\mathbb{D})$. Finally by bootstrapping, the equation (18) yields $w \in C^{k,\alpha}(\mathbb{D})$. Hence $u \in C^{k,\alpha}(\mathbb{D})$.

Since $v \in C^{\alpha}(\mathbb{D})$, then Taylor's expansion of w at 0 has the form

$$w(\zeta) = b\zeta^k + O(|\zeta|^{k+\alpha}).$$

Hence, $\partial_{\zeta}^k w(0) = k!b$ and $\partial_{\zeta}^j w(0) = 0$ for $0 \leq j < k$. Hence, $u = u_1 + \zeta^k v$ satisfies (15). The proof is complete.

Let $j_{\zeta}^k : (C^k(\overline{\mathbb{D}}))^n \rightarrow (\mathbb{C}^n)^{k+1}$ be holomorphic k -jet evaluation map at $\zeta \in \overline{\mathbb{D}}$ defined by

$$j_{\zeta}^k(u) = \{\partial_{\zeta}^j u(\zeta) : 0 \leq j \leq k\}.$$

Let $\mathcal{V} = \mathcal{V}^{k,\alpha}$ denote the space of all solutions of (7) of class $C^{k,\alpha}(\mathbb{D})$.

Proposition 5.2 *Let $B_1, B_2 \in C^{k-1,\alpha}(\mathbb{D})$, $k \geq 1$, $0 < \alpha < 1$. There exists a subspace $V \subset \mathcal{V}$, $\dim V < \infty$, such that for every $\zeta \in \overline{\mathbb{D}}$ the restriction $j_{\zeta}^k|_V : V \rightarrow (\mathbb{C}^n)^{k+1}$ is surjective.*

Proof : By Theorem 5.1 there exists a subspace $V_0 \subset \mathcal{V}$ such that $j_0^k|_{V_0}$ is bijective. The evaluation point 0 in Theorem 5.1 can be replaced by any other point $\zeta \in \overline{\mathbb{D}}$ by extending B_j to all of \mathbb{C} and applying Theorem 5.1 to a bigger disc with center at ζ . Let $V(\zeta) \subset \mathcal{V}$ denote such a subspace that $j_\zeta^k|_{V(\zeta)}$ is bijective. By continuity, every $\zeta_0 \in \overline{\mathbb{D}}$ has a neighborhood $U(\zeta_0) \subset \mathbb{D}$ so that for every $\zeta \in U(\zeta_0)$ the map $j_\zeta^k|_{V(\zeta)}$ is bijective. By compactness, we find a finite covering $\cup_{i=1}^N U(\zeta_i)$ of $\overline{\mathbb{D}}$ and then $V = \sum_{i=1}^N V(\zeta_i)$ has the needed property. The proof is complete.

6 Transversality of J -holomorphic discs

In this section we prove Theorems 1.1 and 1.2. Let (M, J) be an almost complex manifold. We denote by $J^k(\mathbb{C}, M)$ the manifold of k -jets of J -holomorphic maps $\mathbb{C} \rightarrow M$. Let $U \subset M \rightarrow \mathbb{C}^n$ be a coordinate chart in which J is defined by the complex matrix A . Let $f : D \subset \mathbb{C} \rightarrow U$ be a J -holomorphic map. Then all the derivatives $\partial_\zeta^p \partial_\zeta^q f(\zeta)$, $p + q \leq k$, are uniquely determined by $\partial_\zeta^p f(\zeta)$, $p \leq k$ from the equation (2). Hence for every chart $U \subset M$, in which J is defined by its complex matrix A , and open subset $D \subset \mathbb{C}$, there is a chart $\tilde{U} \subset J^k(\mathbb{C}, M) \rightarrow D \times U \times \mathbb{C}^{mk}$ so that the k -jet extension of a J -holomorphic map $f : D \rightarrow U$ has the form

$$j^k f(\zeta) = (\zeta, f(\zeta), \partial_\zeta^1 f(\zeta), \dots, \partial_\zeta^k f(\zeta)).$$

Proof of Theorem 1.2. Let $f_0 : \mathbb{D} \rightarrow M$ be a J -holomorphic disc. We would like to find a J -holomorphic disc $f : \mathbb{D} \rightarrow M$ close to f_0 so that $j^k f$ is transverse to S .

Following the proof of Theorem 1.3, we will use the one-to-one correspondence $f \leftrightarrow g$ between the J -holomorphic discs f close to f_0 and the solutions $g(\zeta) = h^\zeta(f(\zeta))$ of the equation (12) close to 0. We will also use the map $F : g \mapsto \phi$ defined by (13).

Introduce $\tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$ as the manifold of k -jets of solutions of (12). Then $\tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$ consists of a single chart $\tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n) \ni j^k g(\zeta) \mapsto (\zeta, g(\zeta), \partial_\zeta^1 g(\zeta), \dots, \partial_\zeta^k g(\zeta)) \in \mathbb{C}^{kn+1}$.

The correspondence $f \mapsto g$ gives rise to a diffeomorphism

$$\Psi : J^k(\overline{\mathbb{D}}, M) \supset U \rightarrow \tilde{U} \subset \tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$$

defined in a neighborhood $U \supset j^k f_0(\overline{\mathbb{D}})$. Then $j^k f$ is transverse to S in $J^k(\overline{\mathbb{D}}, M)$ if and only if $j^k g$ is transverse to $\tilde{S} := \Psi(S)$ in $\tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$.

Let V be the space of solutions of the equation (7) provided by Proposition 5.2. Let u_1, \dots, u_N form a basis of V . Let $\phi_j = F'(f_0)(u_j)$. For $s = (s_1, \dots, s_N) \in \mathbb{R}^N$ put $\phi_s = \sum_l s_l \phi_l$. Then the map

$$\begin{aligned} \Phi : \overline{\mathbb{D}} \times \mathbb{R}^N &\rightarrow \tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n), \\ \Phi(\zeta, s) &= j^k F^{-1}(\phi_s)(\zeta) \end{aligned}$$

is defined for small $s \in \mathbb{R}^N$. By Proposition 5.2, the map Φ is a submersion for $s = 0$, $\zeta \in \overline{\mathbb{D}}$. Then $X = \Phi^{-1}(\tilde{S})$ is a smooth manifold in a neighborhood of $\overline{\mathbb{D}} \times \{0\} \subset \overline{\mathbb{D}} \times \mathbb{R}^N$.

Let $\tau : \overline{\mathbb{D}} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be the projection. Then $\Phi(\bullet, s)$ is transverse to $\tilde{S} \subset \tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$ if and only if s is a regular value of $\tau|_X$. (See the proof of Thom's transversality theorem in [6].) By Sard's theorem the set of critical values has measure zero. Hence there exists $s \in \mathbb{R}^N$ arbitrarily close to 0 such that $g = F^{-1}(\phi_s)$ has k -jet extension transverse to \tilde{S} in $\tilde{J}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$. Then the corresponding $j^k f$ is transverse to S in $J^k(\overline{\mathbb{D}}, M)$. Theorem 1.2 is proved.

Alternatively, for a disc f_0 in a single coordinate chart, we could prove Theorem 1.2 without the correspondence $f \leftrightarrow g$ and the equation (12). Then we would repeat some of the arguments in the proof of Theorem 1.3 for the map $F(f) = f - f_0 - T(A\bar{f}_z)$. The general case reduces to the case of a single chart by passing from each disc to its graph.

Proof of Theorem 1.1. Theorem 1.1 follows from Theorem 1.2 in the same way that the classical Whitney theorem follows from that of Thom (see [6]). Let $\sigma \in J^1(\mathbb{C}, M)$. Then there are $\zeta \in \mathbb{C}$, open set $U \ni \zeta$, and a J -holomorphic map $f : U \rightarrow M$ such that $j^1 f(\zeta) = \sigma$. Define $\text{rank } \sigma := \text{rank } df(\zeta)$;

$$S = \{\sigma \in J^1(\mathbb{C}, M) : \text{rank } \sigma = 0\}.$$

Then a J -holomorphic map $f : \mathbb{D} \rightarrow M$ is an immersion if and only if $j^1 f(\overline{\mathbb{D}}) \cap S = \emptyset$. We have $\dim_{\mathbb{R}} J^1(\mathbb{C}, M) = 4n + 2$ and $\dim_{\mathbb{R}} S = 2n + 2$. If $j^1 f$ is transverse to S , then indeed $j^1 f(\overline{\mathbb{D}}) \cap S = \emptyset$, because otherwise $\dim_{\mathbb{R}} \mathbb{D} + \dim_{\mathbb{R}} S \geq \dim_{\mathbb{R}} J^1(\mathbb{C}, M)$ implies $n \leq 1$. The desired result now follows by Theorem 1.2. The homotopy statement follows by Theorem 1.3. The proof is complete.

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