

**Rigidity for Local Holomorphic Conformal Embeddings**  
**from  $B^n$  into  $B^{N_1} \times \dots \times B^{N_m}$**

**Yuan Yuan**  
**Yuan Zhang**

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# Rigidity for local holomorphic conformal embeddings from $\mathbb{B}^n$ into $\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}$

Yuan Yuan and Yuan Zhang

## Abstract

In this article, we study local holomorphic conformal embeddings from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}$  with respect to the normalized Bergman metrics. Assume that each conformal factor is smooth Nash algebraic. Then each component of the map is a multi-valued holomorphic map between complex Euclidean spaces by the algebraic extension theorem derived along the lines of Mok and Mok-Ng. Applying holomorphic continuation and analyzing real analytic subvarieties carefully, we show that each component is either a constant map or a proper holomorphic map between balls. Applying a linearity criterion of Huang, we conclude the total geodesy of non-constant components.

## 1 Introduction

Write  $\mathbb{B}^n := \{z \in \mathbb{C}^n : |z| < 1\}$  for the unit ball in  $\mathbb{C}^n$ . Denote by  $ds_n^2$  the normalized Bergman metric on  $\mathbb{B}^n$  defined as follows:

$$ds_n^2 = \sum_{j,k \leq n} \frac{1}{(1 - |z|^2)^2} ((1 - |z|^2)\delta_{jk} + \bar{z}_j z_k) dz_j \otimes d\bar{z}_k. \quad (1)$$

Let  $U \subset \mathbb{B}^n$  be a connected open subset. Consider a holomorphic conformal embedding

$$F = (F_1, \dots, F_m) : (U, ds_n^2) \rightarrow (\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}, \bigoplus_{j=1}^m ds_{N_j}^2) \quad (2)$$

with conformal factors  $\{\lambda(z, \bar{z}); \lambda_1(z, \bar{z}), \dots, \lambda_m(z, \bar{z})\}$  in the sense that

$$\lambda(z, \bar{z}) ds_n^2 = \sum_{j=1}^m \lambda_j(z, \bar{z}) F_j^*(ds_{N_j}^2).$$

Here and in what follows, the conformal factors  $\lambda(z, \bar{z}), \lambda_j(z, \bar{z})$  ( $j = 1, \dots, m$ ) are assumed to be positively real-valued smooth Nash algebraic functions over  $\mathbb{C}^n$ . Moreover, for each  $j$  with  $1 \leq j \leq m$ ,  $ds_{N_j}^2$  denotes the corresponding normalized Bergman metric of  $\mathbb{B}^{N_j}$  and  $F_j$  is a holomorphic map from  $U$  to  $\mathbb{B}^{N_j}$ . We write  $F_j = (f_{j,1}, \dots, f_{j,l}, \dots, f_{j,N_j})$ , where  $f_{j,l}$  is the  $l$ -th component of  $F_j$ . In this paper, we prove the following rigidity theorem:

**Theorem 1.1.** *Suppose  $n \geq 2$ . Under the above notation and assumption, we then have, for each  $j$  with  $1 \leq j \leq m$ , that either  $F_j$  is a constant map or  $F_j$  extends to a totally geodesic holomorphic embedding from  $(\mathbb{B}^n, ds_n^2)$  into  $(\mathbb{B}^{N_j}, ds_{N_j}^2)$ . Moreover, we have the following identity*

$$\sum_{F_j \text{ is not a constant}} \lambda_j(z, \bar{z}) = \lambda(z, \bar{z}).$$

In particular, when  $\lambda_j(z, \bar{z}), \lambda(z, \bar{z})$  are positive constant functions, we have the following rigidity result for local isometric embeddings:

**Corollary 1.2.** *Let*

$$F = (F_1, \dots, F_m) : (U \subset \mathbb{B}^n, \lambda ds_n^2) \rightarrow (\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}, \bigoplus_{j=1}^m \lambda_j ds_{N_j}^2) \quad (3)$$

*be a local holomorphic isometric embedding in the sense that  $\lambda ds_n^2 = \sum_{j=1}^m \lambda_j F_j^*(ds_{N_j}^2)$ . Assume that  $n \geq 2$  and  $\lambda, \lambda_j$  are positive constants. We then have, for each  $j$  with  $1 \leq j \leq m$ , that either  $F_j$  is a constant map or  $F_j$  extends to a totally geodesic holomorphic embedding from  $(\mathbb{B}^n, ds_n^2)$  into  $(\mathbb{B}^{N_j}, ds_{N_j}^2)$ . Moreover, we have the following identity*

$$\sum_{F_j \text{ is not a constant}} \lambda_j = \lambda.$$

Recall that a function  $h(z, \bar{z})$  is called a Nash algebraic function over  $\mathbb{C}^n$  if there is an irreducible polynomial  $P(z, \xi, X)$  in  $(z, \xi, X) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}$  with  $P(z, \bar{z}, h(z, \bar{z})) \equiv 0$  over  $\mathbb{C}^n$ . We mention that a holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^N$  is a totally geodesic embedding with respect to the normalized Bergman metric if and only if there are a (holomorphic) automorphism  $\sigma \in \text{Aut}(\mathbb{B}^n)$  and an automorphism  $\tau \in \text{Aut}(\mathbb{B}^N)$  such that  $\tau \circ F \circ \sigma(z) \equiv (z, 0)$ . Also, we mention that by the work of Mok [Mo1], the result in Corollary 1.2 does not hold anymore when  $n = 1$ . (See also many examples and related classification results in the work of Ng ([Ng1]).

The study of the global extension and rigidity problem for local isometric embedding was first carried out in a paper of Calabi [Ca]. After [Ca], there appeared quite a few papers along these lines of research (see [Um], for instance). In 2003, motivated by problems from Arithmetic Algebraic Geometry, Clozel and Ullmo [CU] took up again the problem by considering the rigidity problem for a local isometric embedding with a certain symmetry from  $\mathbb{B}^1$  into  $\mathbb{B}^1 \times \dots \times \mathbb{B}^1$ . More recently, Mok carried out a systematic study of this problem in a very general setting. Many far reaching deep results have been obtained by Mok and later by Ng and Mok-Ng. (See [Mo1] [Mo2] [MN] [Ng1-3] and the references therein). Here, we would like to mention that our result was already included in the papers by Calabi when  $m = 1$  [Ca], by Mok [Mo1] [Mo2] when  $N_1 = \dots = N_m$ , and by Ng [Ng1] [Ng3] when  $m = 2$  and  $N_1, N_2 < 2n$ .

As in the work of Mok [Mo1], our proof of the theorem is also based on the similar algebraic extension theorem derived in [Mo2] and Mok-Ng [MN]. However, different from the case considered in [Mo1] [Ng2], the properness of a factor of  $F$  does not immediately imply the linearity

of that factor; for the classical linearity theorem does not hold anymore for proper rational mappings from  $\mathbb{B}^n$  into  $\mathbb{B}^N$  with  $N > 2n - 2$ . (See [Hu1]). Hence, the cancelation argument as in [Mok1] [Ng3] seems to be difficult to apply in our setting.

In our proof of Theorem 1.1, a major step is to prove that a non-constant component  $F_j$  of  $F$  must be proper from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_j}$ , using the multiple-valued holomorphic continuation technique. This then reduces the proof of Theorem 1.1 to the case when all components are proper. Unfortunately, due to the non-constancy for the conformal factors  $\lambda_j(z, \bar{z})$  and  $\lambda(z, \bar{z})$ , it is not immediate that each component must also be conformal (and thus  $\lambda_j$  must be a constant multiple of  $\lambda$ ) with respect to the normalized Bergman metric. However, we observe that the blowing-up rate for the Bergman metric of  $\mathbb{B}^n$  with  $n \geq 2$  in the complex normal direction is twice of that along the complex tangential direction, when approaching the boundary. From this, we will be able to derive an equation regarding the CR invariants associated to the map at the boundary of the ball. Lastly, a linearity criterion of Huang in [Hu1] can be applied to simultaneously conclude the linearity of all components.

We mention that in the context of Corollary 1.2, namely, when each conformal factor is assumed to be constant, the proof used to prove Theorem 1.1 can be further simplified as told us by Mok and Ng in their private communications. In this case, one can work directly on the Kähler potential functions instead of the hyperbolic metrics. However, when the conformal factors are not constant, and  $\partial\bar{\partial}$ -lemma can not be applied and the metric equation (which can be regarded as differential equations on the map) does not lead to the functional equation on the components of the map. We appreciate very much many valuable comments of Mok and Ng to the earlier version of this paper, especially, for telling us how to essentially simplify the proof of a key lemma (Lemma 2.2) through the consideration of the metric potential functions. Their very helpful comments lead to the present version.

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## 2 Bergman metric and proper rational maps

Let  $\mathbb{B}^n$  and  $ds_n^2$  be the unit ball and its normalized Bergman metric, respectively, as defined before. Denote by  $\mathbb{H}^n \subset \mathbb{C}^n$  the Siegel upper half space. Namely,  $\mathbb{H}^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} :$

$\Im w - |z|^2 > 0$ }. Here, for  $m$ -tuples  $a, b$ , we write dot product  $a \cdot b = \sum_{j=1}^m a_j b_j$  and  $|a|^2 = a \cdot \bar{a}$ . Recall the following Cayley transformation

$$\rho_n(z, w) = \left( \frac{2z}{1-iw}, \frac{1+iw}{1-iw} \right). \quad (4)$$

Then  $\rho_n$  biholomorphically maps  $\mathbb{H}^n$  to  $\mathbb{B}^n$ , and biholomorphically maps  $\partial\mathbb{H}^n$ , the Heisenberg hypersurface, to  $\partial\mathbb{B}^n \setminus \{(0, 1)\}$ . Applying the Cayley transformation, one can compute the normalized Bergman metric on  $\mathbb{H}^n$  as follows:

$$\begin{aligned} ds_{\mathbb{H}^n}^2 &= \sum_{j,k < n} \frac{\delta_{jk}(\Im w - |z|^2) + \bar{z}_j z_k}{(\Im w - |z|^2)^2} dz_j \otimes d\bar{z}_k + \frac{dw \otimes d\bar{w}}{4(\Im w - |z|^2)^2} \\ &+ \sum_{j < n} \frac{\bar{z}_j dz_j \otimes d\bar{w}}{2i(\Im w - |z|^2)^2} - \sum_{j < n} \frac{z_j dw \otimes d\bar{z}_j}{2i(\Im w - |z|^2)^2}. \end{aligned} \quad (5)$$

One can easily check that

$$\begin{aligned} L_j &= \frac{\partial}{\partial z_j} + 2i\bar{z}_j \frac{\partial}{\partial w}, j = 1, \dots, n-1. \\ \bar{L}_j &= \frac{\partial}{\partial \bar{z}_j} - 2iz_j \frac{\partial}{\partial \bar{w}}, j = 1, \dots, n-1. \\ T &= 2\left(\frac{\partial}{\partial w} + \frac{\partial}{\partial \bar{w}}\right) \end{aligned}$$

span the complexified tangent vector bundle of  $\partial\mathbb{H}^n$ . (See, for instance, [BER], [Hu2], [Hu3] [HJX] [JX] [Mi].)

Let  $F$  be a rational proper holomorphic map from  $\mathbb{H}^n$  to  $\mathbb{H}^N$ . By a result of Cima-Suffridge [CS],  $F$  is holomorphic in a neighborhood of  $\partial\mathbb{H}^n$ . Assign the weight of  $w$  to be 2 and that of  $z$  1. Denote by  $o_{wt}(k)$  terms with weighted degree higher than  $k$  and by  $P^{(k)}$  a function of weighted degree  $k$ . For  $p_0 = (z_0, w_0) \in \partial\mathbb{H}^n$ , write  $\sigma_{p_0}^0 : (z, w) \rightarrow (z + z_0, w + w_0 + 2iz \cdot \bar{z}_0)$  for the standard Heisenberg translation. The following normalization lemma will be used here:

**Lemma 2.1.** [Hu2-3] *For any  $p \in \partial\mathbb{H}^n$ , there is an element  $\tau \in \text{Aut}(\mathbb{H}^{N+1})$  such that the map  $F_p^{**} = ((f_p^{**})_1(z), \dots, (f_p^{**})_{n-1}(z), \phi_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) = \tau \circ F \circ \sigma_p^0$  takes the following normal form:*

$$\begin{aligned} f_p^{**}(z, w) &= z + \frac{i}{2}a^{(1)}(z)w + o_{wt}(3), \\ \phi_p^{**}(z, w) &= \phi^{(2)}(z) + o_{wt}(2), \\ g_p^{**}(z, w) &= w + o_{wt}(4) \end{aligned}$$

with

$$(\bar{z} \cdot a^{(1)}(z))|z|^2 = |\phi^{(2)}(z)|^2. \quad (6)$$

In particular, write  $(f_p^{**})_l(z) = z_j + \frac{i}{2} \sum_{k=1}^{n-1} a_{lk} z_k w + o_{wt}(3)$ . Then,  $(a_{lk})_{1 \leq l, k \leq n-1}$  is an  $(n-1) \times (n-1)$  semi-positive Hermitian matrix. We next present the following key lemma for our proof of Theorem 1.1:

**Lemma 2.2.** *Let  $F$  be a proper rational map from  $\mathbb{B}^n$  to  $\mathbb{B}^N$ . Then*

$$X := ds_n^2 - F^*(ds_N^2), \quad (7)$$

*is a semi-positive real analytic symmetric (1,1)-tensor over  $\mathbb{B}^n$ , that extends also to a real analytic (1,1)-tensor in a small neighborhood of  $\partial\mathbb{B}^n$  in  $\mathbb{C}^n$ .*

**Proof of lemma 2.2:** Our original proof was largely simplified by Ng [Ng4] and Mok [Mo4] by considering the potential  $-\log(1 - \|F(z)\|^2)$  of the pull-back metric  $F^*(ds_N^2)$  as follows: Since  $1 - \|F(z)\|^2$  vanishes identically on  $\partial\mathbb{B}^n$  and since  $1 - \|z\|^2$  is a defining equation for  $\partial\mathbb{B}^n$ , one obtains

$$1 - \|F(z)\|^2 = (1 - \|z\|^2)\varphi(z)$$

for a real analytic function  $\varphi(z)$ .

Since  $\rho := \|F(z)\|^2 - 1$  is subharmonic over  $\mathbb{B}^n$  and has maximum value 0 on the boundary, applying the classical Hopf lemma, we conclude that  $\varphi(z)$  can not vanish at any boundary point of  $\mathbb{B}^n$ . Apparently,  $\varphi(z)$  can not vanish inside  $\mathbb{B}^n$ . Therefore,  $X = \sqrt{-1}\partial\bar{\partial}\log\varphi(z)$  is real analytic on an open neighborhood of  $\overline{\mathbb{B}^n}$ . The semi-positivity of  $X$  over  $\mathbb{B}^n$  is an easy consequence of the Schwarz lemma. ■

Applying the Cayley transformation (and also a rotation transformation when handling the regularity near  $(0, 1)$ ), we have the following corollary:

**Corollary 2.3.** *Let  $F$  be a rational proper holomorphic map from  $\mathbb{H}^n$  to  $\mathbb{H}^N$ . Then*

$$X := ds_{\mathbb{H}^n}^2 - F^*(ds_{\mathbb{H}^N}^2), \quad (8)$$

*is a semi-positive real analytic symmetric (1,1)-tensor over  $\mathbb{H}^n$ , that extends also to a real analytic (1,1)-tensor in a small neighborhood of  $\partial\mathbb{H}^n$  in  $\mathbb{C}^n$ .*

The boundary value of  $X$  is an intrinsic CR invariant associated with the equivalence class of the map  $F$ . Next, we compute  $X$  in the normal coordinates at the boundary point.

Write  $t = \Im w - |z|^2$  and  $H = \Im g - |\tilde{f}|^2$ . Here  $(\tilde{f}, g)$  denotes the map between Heisenberg hypersurfaces. Write  $o(k)$  for terms whose degrees with respect to  $t$  are higher than  $k$ . For a real analytic function  $h$  in  $(z, w)$ , we use  $h_z, h_w$  to denote the derivatives of  $h$  with respect to  $z, w$ . By replacing  $w$  by  $u + i(t + |z|^2)$ ,  $H$  can also be regarded as an analytic function on  $z, \bar{z}, u, t$ . The following lemma gives an asymptotic behavior of  $H$  with respect to  $t$ :

**Lemma 2.4.**  $H(z, \bar{z}, u, t) = (g_w - 2i\tilde{f}_w \cdot \bar{\tilde{f}})|_{t=0}t - (2|\tilde{f}_w|^2)|_{t=0}t^2 + \frac{1}{3}(-\frac{1}{2}g_{w^3} + 3i\tilde{f}_w \cdot \bar{\tilde{f}_{w^2}} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}})|_{t=0}t^3 + o(3)$ .

**Proof of Lemma 2.4:** Notice that  $H = H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))$ . Since  $F$  is proper,  $H$ , as a function of  $t$  with parameters  $\{z, u\}$ , can be written as  $P_1t + P_2t^2 + P_3t^3 + o(3)$ , where  $P_1, P_2, P_3$  are analytic in  $(z, \bar{z}, u)$ . Then

$$\begin{aligned} P_1 &= \left. \frac{\partial H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t} \right|_{t=0} \\ &= iH_w - iH_{\bar{w}} \Big|_{t=0} \\ &= \frac{1}{2}(g_w + \bar{g}_w) + i(\tilde{f} \cdot \bar{\tilde{f}_w} - \bar{\tilde{f}} \cdot \tilde{f}_w) \Big|_{t=0}, \end{aligned} \tag{9}$$

$$\begin{aligned} P_2 &= \left. \frac{1}{2} \frac{\partial^2 H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t^2} \right|_{t=0} \\ &= \frac{1}{2}(-H_{w^2} + 2H_{w\bar{w}} - H_{\bar{w}^2}) \Big|_{t=0} \\ &= \frac{1}{2} \left( \frac{i}{2}g_{w^2} - \frac{i}{2}\bar{g}_{w^2} - 2|\tilde{f}_w|^2 + \tilde{f}_{w^2} \cdot \bar{\tilde{f}} + \bar{\tilde{f}} \cdot \tilde{f}_{w^2} \right) \Big|_{t=0}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} P_3 &= \left. \frac{1}{6} \frac{\partial^3 H(z, \bar{z}, u + i(t + |z|^2), u - i(t + |z|^2))}{\partial t^3} \right|_{t=0} \\ &= \frac{1}{6}(-iH_{w^3} + 3iH_{w^2\bar{w}} - 3iH_{\bar{w}^2w} + iH_{\bar{w}^3}) \Big|_{t=0} \\ &= \frac{1}{6} \left( -\frac{1}{2}g_{w^3} - \frac{1}{2}\bar{g}_{w^3} + i\tilde{f}_{w^3} \cdot \bar{\tilde{f}} - i\bar{\tilde{f}} \cdot \tilde{f}_{w^3} - 3i\tilde{f}_{w^2} \cdot \bar{\tilde{f}_w} + 3i\bar{\tilde{f}_w} \cdot \tilde{f}_{w^2} \right) \Big|_{t=0}. \end{aligned} \tag{11}$$

On the other hand, applying  $T, T^2, T^3$  to the defining equation  $g - \bar{g} = 2i\tilde{f} \cdot \bar{\tilde{f}}$ , we have

$$g_w - \bar{g}_w - 2i(\tilde{f}_w \cdot \bar{\tilde{f}} + \tilde{f} \cdot \bar{\tilde{f}_w}) = 0, \tag{12}$$

$$g_{w^2} - \bar{g}_{w^2} - 2i(\tilde{f}_{w^2} \cdot \bar{\tilde{f}} + \bar{\tilde{f}_{w^2}} \cdot \tilde{f} + 2|\tilde{f}_w|^2) = 0, \tag{13}$$

$$g_{w^3} - \bar{g}_{w^3} - 2i(\tilde{f}_{w^3} \cdot \bar{\tilde{f}} + \bar{\tilde{f}_{w^3}} \cdot \tilde{f} + 3\tilde{f}_{w^2} \cdot \bar{\tilde{f}_w} + 3\bar{\tilde{f}_w} \cdot \tilde{f}_{w^2}) = 0. \tag{14}$$

over  $\Im w = |z|^2$ .

Substituting (12), (13) and (14) into (9), (10) and (11), we get

$$\begin{aligned}
P_1 &= g_w - 2i\tilde{f}_w \cdot \tilde{f} \Big|_{t=0}, \\
P_2 &= -2|\tilde{f}_w|^2 \Big|_{t=0}, \\
P_3 &= \frac{1}{3} \left( -\frac{1}{2}g_{w^3} + 3i\tilde{f}_w \cdot \overline{\tilde{f}_{w^2}} + i\tilde{f}_{w^3} \cdot \tilde{f} \right) \Big|_{t=0}. \quad \blacksquare
\end{aligned} \tag{15}$$

We remark that by the Hopf Lemma, it follows easily that  $P_1 \neq 0$  along  $\partial\mathbb{H}^n$ .

We next write  $X = X_{jk}dz_j \otimes d\bar{z}_k + X_{jn}dz_j \otimes d\bar{w} + X_{nj}dw \otimes d\bar{z}_j + X_{nn}dw \otimes d\bar{w}$ . By making use of Lemma 2.1, we shall compute in the next proposition the values of  $X$  at the origin. The proposition might be of independent interest, as the CR invariants in the study of proper holomorphic maps between Siegel upper half spaces are related to the CR geometry of the map.

**Proposition 2.5.** *Assume that  $F = (\tilde{f}, g) = (f_1, \dots, f_{N-1}, g) : \mathbb{H}^n \rightarrow \mathbb{H}^N$  is a proper rational holomorphic map, that satisfies the normalization (at the origin) stated in Lemma 2.1. Then*

$$\begin{aligned}
X_{jk}(0) &= -2i(f_k)_{z_j w}(0) = a_{kj}, \\
X_{jn}(0) &= \overline{X_{nj}}(0) = \frac{3i}{4} \overline{(f_j)_{w^2}}(0) + \frac{1}{8}g_{z_j w^2}(0), \\
X_{nn}(0) &= \frac{1}{6}g_{w^3}(0).
\end{aligned}$$

**Proof of Proposition 2.5:** Along the direction of  $dz_j \otimes d\bar{z}_k$ , collecting the coefficient of  $t^2$  in the Taylor expansion of  $H^2 X$  with respect to  $t$ , we get

$$\begin{aligned}
P_1^2 X_{jk}(0) &= \left[ (2P_1 P_2 \delta_{jk} + (P_2^2 + 2P_1 P_3) \bar{z}_j z_k) - \frac{1}{2} \{ 2iP_1 (\tilde{f}_{wz_j} \cdot \overline{\tilde{f}_{z_k}} - \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{wz_k}}) + 2P_2 \tilde{f}_{z_j} \cdot \overline{\tilde{f}_{z_k}} \right. \\
&\quad - (\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) + 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) \\
&\quad - 2(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}}) - (\tilde{f} \cdot \overline{\tilde{f}_{z_j}})(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}}) - 2(\tilde{f} \cdot \overline{\tilde{f}_{z_j w}})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) \\
&\quad + 2(\tilde{f} \cdot \overline{\tilde{f}_{z_j}})(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k w}}) - (\tilde{f}_{z_j w^2} \cdot \tilde{f})(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) + 2(\tilde{f} \cdot \overline{\tilde{f}_{z_j w}})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}}) \\
&\quad - (\tilde{f} \cdot \overline{\tilde{f}_{z_j}})(\tilde{f} \cdot \overline{\tilde{f}_{z_k w^2}}) - \frac{1}{4}g_{z_j w^2} \overline{g_{z_k}} + \frac{1}{2}g_{z_j w} \overline{g_{z_k w}} - \frac{1}{4}g_{z_j} \overline{g_{z_k w^2}} + \frac{i}{2}(\overline{\tilde{f}_{w^2}} \cdot \tilde{f}_{z_j}) \overline{g_{z_k}} \\
&\quad - i(\overline{\tilde{f}_w} \cdot \tilde{f}_{wz_j}) \overline{g_{z_k}} + i(\overline{\tilde{f}_w} \cdot \tilde{f}_{z_j}) \overline{g_{z_k w}} + \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_j w^2}}) \overline{g_{z_k}} - i(\tilde{f} \cdot \overline{\tilde{f}_{z_j w}}) \overline{g_{z_k w}} \\
&\quad + \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_j}}) \overline{g_{z_k w^2}} - \frac{i}{2}(\tilde{f}_{w^2} \cdot \overline{\tilde{f}_{z_k}}) g_{z_j} - i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k}}) g_{z_j w} + i(\tilde{f}_w \cdot \overline{\tilde{f}_{z_k w}}) g_{z_j} \\
&\quad \left. - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_k}}) g_{z_j w^2} + i(\tilde{f} \cdot \overline{\tilde{f}_{z_k w}}) g_{z_j w} - \frac{i}{2}(\tilde{f} \cdot \overline{\tilde{f}_{z_k w^2}}) g_{z_j} \right] \Big|_{t=0}.
\end{aligned}$$

Letting  $(z, w) = 0$  and applying the normalization condition as stated in Lemma 2.1, we have

$$X_{jk}(0) = \frac{\partial a_k^{(1)}(z)}{\partial z_j} = a_{kj}.$$

Similarly, considering the coefficient of  $t^2$  along  $dz_j \otimes d\bar{w}$  and  $dw \otimes d\bar{w}$ , respectively, we have

$$\begin{aligned} P_1^2 X_{jn}(0) = & \left[ (-iP_1P_3 - \frac{i}{2}P_2^2)\bar{z}_j - \frac{1}{2}\{2iP_1(\tilde{f}_{wz_j} \cdot \bar{f}_w - \tilde{f}_{z_j} \cdot \bar{f}_{w^2}) + 2P_2\tilde{f}_{z_j} \cdot \bar{f}_w \right. \\ & - (\bar{f}_{w^2} \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \bar{f}_w) + 2(\bar{f}_w \cdot \tilde{f}_{z_j})(\tilde{f}_w \cdot \bar{f}_w) + 2(\bar{f}_w \cdot \tilde{f}_{wz_j})(\tilde{f} \cdot \bar{f}_w) \\ & - 2(\bar{f}_w \cdot \tilde{f}_{z_j})(\tilde{f} \cdot \bar{f}_{w^2}) - (\tilde{f} \cdot \bar{f}_{z_j})(\bar{f}_{w^2} \cdot \bar{f}_w) - 2(\tilde{f} \cdot \bar{f}_{z_jw})(\tilde{f}_w \cdot \bar{f}_w) \\ & + 2(\tilde{f} \cdot \bar{f}_{z_j})(\tilde{f}_w \cdot \bar{f}_{w^2}) - (\tilde{f}_{z_jw^2} \cdot \tilde{f})(\tilde{f} \cdot \bar{f}_w) + 2(\tilde{f} \cdot \bar{f}_{z_jw})(\tilde{f} \cdot \bar{f}_{w^2}) - (\tilde{f} \cdot \bar{f}_{z_j})(\tilde{f} \cdot \bar{f}_{w^3}) \\ & - \frac{1}{4}g_{z_jw^2}\bar{g}_w + \frac{1}{2}g_{z_jw}\bar{g}_{w^2} - \frac{1}{4}g_{z_j}\bar{g}_{w^3} + \frac{i}{2}(\bar{f}_{w^2} \cdot \tilde{f}_{z_j})\bar{g}_w - i(\bar{f}_w \cdot \tilde{f}_{wz_j})\bar{g}_w + i(\bar{f}_w \cdot \tilde{f}_{z_j})\bar{g}_{w^2} \\ & + \frac{i}{2}(\tilde{f} \cdot \bar{f}_{z_jw^2})\bar{g}_w - i(\tilde{f} \cdot \bar{f}_{z_jw})\bar{g}_{w^2} + \frac{i}{2}(\tilde{f} \cdot \bar{f}_{z_j})\bar{g}_{w^3} - \frac{i}{2}(\bar{f}_{w^2} \cdot \bar{f}_w)g_{z_j} - i(\bar{f}_w \cdot \bar{f}_w)g_{z_jw} \\ & \left. + i(\bar{f}_w \cdot \bar{f}_{w^2})g_{z_j} - \frac{i}{2}(\tilde{f} \cdot \bar{f}_w)g_{z_jw^2} + i(\tilde{f} \cdot \bar{f}_{w^2})g_{z_jw} - \frac{i}{2}(\tilde{f} \cdot \bar{f}_{w^3})g_{z_j}\right] \Big|_{t=0} \end{aligned}$$

and

$$\begin{aligned} P_1^2 X_{nn}(0) = & \left[ \frac{1}{4}(2P_1P_3 + P_2^2) - \frac{1}{2}\{2iP_1(\tilde{f}_{w^2} \cdot \bar{f}_w - \tilde{f}_w \cdot \bar{f}_{w^2}) + 2P_2\tilde{f}_w \cdot \bar{f}_w - (\bar{f}_{w^2} \cdot \tilde{f}_w)(\tilde{f} \cdot \bar{f}_w) \right. \\ & + 2(\bar{f}_w \cdot \tilde{f}_w)(\tilde{f}_w \cdot \bar{f}_w) + 2(\bar{f}_w \cdot \tilde{f}_{w^2})(\tilde{f} \cdot \bar{f}_w) - 2(\bar{f}_w \cdot \tilde{f}_w)(\tilde{f} \cdot \bar{f}_{w^2}) \\ & - (\tilde{f} \cdot \bar{f}_w)(\bar{f}_{w^2} \cdot \tilde{f}_w) - 2(\tilde{f} \cdot \bar{f}_{w^2})(\bar{f}_w \cdot \tilde{f}_w) + 2(\tilde{f} \cdot \bar{f}_w)(\tilde{f}_w \cdot \bar{f}_{w^2}) - (\tilde{f}_{w^3} \cdot \tilde{f})(\tilde{f} \cdot \bar{f}_w) \\ & + 2(\tilde{f} \cdot \bar{f}_{w^2})(\tilde{f} \cdot \bar{f}_{w^2}) - (\tilde{f} \cdot \bar{f}_w)(\tilde{f} \cdot \bar{f}_{w^3}) - \frac{1}{4}g_{w^3}\bar{g}_w + \frac{1}{2}g_{w^2}\bar{g}_{w^2} - \frac{1}{4}g_w\bar{g}_{w^3} \\ & + \frac{i}{2}(\bar{f}_{w^2} \cdot \tilde{f}_w)\bar{g}_w - i(\bar{f}_w \cdot \tilde{f}_{w^2})\bar{g}_w + i(\bar{f}_w \cdot \tilde{f}_w)\bar{g}_{w^2} + \frac{i}{2}(\tilde{f} \cdot \bar{f}_{w^3})\bar{g}_w - i(\tilde{f} \cdot \bar{f}_{w^2})\bar{g}_{w^2} \\ & + \frac{i}{2}(\tilde{f} \cdot \bar{f}_w)\bar{g}_{w^3} - \frac{i}{2}(\bar{f}_{w^2} \cdot \tilde{f}_w)g_w - i(\tilde{f}_w \cdot \bar{f}_w)g_{w^2} \\ & \left. + i(\tilde{f}_w \cdot \bar{f}_{w^2})g_w - \frac{i}{2}(\tilde{f} \cdot \bar{f}_w)g_{w^3} + i(\tilde{f} \cdot \bar{f}_{w^2})g_{w^2} - \frac{i}{2}(\tilde{f} \cdot \bar{f}_{w^3})g_w\right] \Big|_{t=0}. \end{aligned}$$

Let  $(z, w) = 0$ . It follows that

$$\begin{aligned} X_{jn}(0) &= \frac{3i}{4}(\bar{f}_j)_{w^2}(0) + \frac{1}{8}g_{z_jw^2}(0), \\ X_{nn}(0) &= \frac{1}{6}g_{w^3}(0), \end{aligned}$$

for  $g_{w^3}(0) = \overline{g_{w^3}(0)}$  by (14).  $\blacksquare$

Making use of the computation in Proposition 2.5, we give a proof of Theorem 1.1 in the case when each component extends as a proper holomorphic map. Indeed, we prove a slightly more general result than what we need later as following:

**Proposition 2.6.** *Let*

$$F = (F_1, \dots, F_m) : (\mathbb{B}^n, ds_n^2) \rightarrow (\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}, \bigoplus_{j=1}^m ds_{N_j}^2)$$

be a holomorphic conformal embedding with conformal factors  $\{\lambda(z, \bar{z}); \lambda_1(z, \bar{z}), \dots, \lambda_m(z, \bar{z})\}$  in the sense that

$$\lambda(z, \bar{z}) ds_n^2 = \sum_{j=1}^m \lambda_j(z, \bar{z}) F_j^*(ds_{N_j}^2).$$

Here for each  $j$ ,  $\lambda(z, z), \lambda_j(z, \bar{z})$  are positively real-valued  $C^2$ -smooth functions over  $\overline{\mathbb{B}^n}$ , and  $F_j$  is a proper rational map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_j}$  for each  $j$ . Then  $\lambda(z, \bar{z}) \equiv \sum_{j=1}^m \lambda_j(z, \bar{z})$  over  $\overline{\mathbb{B}^n}$ , and for any  $j$ ,  $F_j$  is a totally geodesic embedding from  $\mathbb{B}^n$  to  $\mathbb{B}^{N_j}$ .

**Proof of Proposition 2.6:** After applying the Cayley transformation and considering  $((\rho_{N_1})^{-1}, \dots, (\rho_{N_m})^{-1}) \circ F \circ \rho_n$  instead of  $F$ , we can assume, without loss of generality, that

$$F = (F_1, \dots, F_m) : (\mathbb{H}^n, ds_{\mathbb{H}^n}^2) \rightarrow (\mathbb{H}^{N_1} \times \dots \times \mathbb{H}^{N_m}, \bigoplus_{j=1}^m ds_{\mathbb{H}^{N_j}}^2)$$

is a conformal map with conformal factors  $\{\lambda(Z, \bar{Z}); \lambda_1(Z, \bar{Z}), \dots, \lambda_m(Z, \bar{Z})\}$ . Also, each  $F_j$  a proper rational map from  $\mathbb{H}^n$  into  $\mathbb{H}^{N_j}$ , respectively. Here we write  $Z = (z, w)$ . Moreover, we can assume, without loss of generality, that each component  $F_j$  of  $F$  satisfies the normalization condition as in Lemma 2.1. Since  $F$  is conformal, we have

$$\lambda(Z, \bar{Z}) ds_{\mathbb{H}^n}^2 = \sum_{j=1}^m \lambda_j(Z, \bar{Z}) F_j^*(ds_{\mathbb{H}^{N_j}}^2), \text{ or } (\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z})) ds_{\mathbb{H}^n}^2 + \sum_{j=1}^m \lambda_j(Z, \bar{Z}) X(F_j) = 0. \quad (16)$$

Here, we write  $X(F_j) = ds_{\mathbb{H}^n}^2 - F_j^*(ds_{\mathbb{H}^{N_j}}^2)$ . Collecting the coefficient of  $dw \otimes d\bar{w}$ , one has

$$\frac{\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z})}{4(\Im w - |z|^2)^2} + \sum_{j=1}^m \lambda_j(Z, \bar{Z}) (X(F_j))_{nn} = 0. \quad (17)$$

Since  $X(F_j)$  is smooth up to  $\partial\mathbb{H}^n$ , we see that  $\lambda(Z, \bar{Z}) - \sum_{j=1}^m \lambda_j(Z, \bar{Z}) = O(t^2)$  as  $Z = (z, w) \in \mathbb{H}^n \rightarrow 0$ , where  $t = \Im w - |z|^2$ . However, since the  $dz_l \otimes d\bar{z}_k$ -component of  $ds_{\mathbb{H}^n}^2$  blows up at the

rate of  $o(\frac{1}{l^2})$  as  $(z, w) \in \mathbb{H}^n \rightarrow 0$ , collecting the coefficients of the  $dz_l \otimes d\bar{z}_k$ -component in (16) and then letting  $(z, w) \in \mathbb{H}^n \rightarrow 0$ , we conclude that, for any  $1 \leq l, k \leq n-1$ ,

$$\sum_{j=1}^m \lambda_j(0, 0)(X(F_j))_{kl}(0) = 0.$$

By Proposition 2.5, we have  $\sum_{j=1}^m \lambda_j(0, 0)a_{lk}^j(0) = 0$ , where  $a_{lk}^j$  is associated with  $F_j$  in the expansion of  $F_j$  at 0 as in Lemma 2.1. Since the matrix  $(a_{kl}^j)_{1 \leq l, k \leq n-1}$  is a semi-positive matrix and since  $\lambda_j(0, 0) > 0$ , it follows immediately that  $a_{lk}^j(0) = 0$  for all  $j, k, l$ . Namely,  $F_j = (z, w) + O_{wt}(3)$  for each  $j$ .

Next, for each  $p \in \partial\mathbb{H}^n$ , let  $\tau_j \in \text{Aut}(\mathbb{H}^n)$  be such that  $(F_j)_p^{**} = \tau_j \circ F_j \circ \sigma_p^0$  has the normalization as in Lemma 2.1. Let  $\tau = (\tau_1, \dots, \tau_m)$ . Notice that  $F_p^{**} := ((F_1)_p^{**}, \dots, (F_m)_p^{**}) = \tau \circ F \circ \sigma_p^0$  is still a conformal map satisfying the condition as in the proposition. Applying the just presented argument to  $F_p^{**}$ , we conclude that  $(F_j)_p^{**} = (z, w) + O_{wt}(3)$ . By Theorem 4.2 of [Hu2], this implies that  $F_j = (Z, 0)$ . Namely,  $F_j$  is a totally geodesic embedding. In particular, we have  $X(F_j) \equiv 0$ . This also implies that  $\lambda \equiv \sum_{j=1}^m \lambda_j$  over  $\mathbb{B}^n$ . The proof of Proposition 2.6 is complete. ■

### 3 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. As in the theorem, we let  $U \subset \mathbb{B}^n$  be a connected open subset. Let

$$F = (F_1, \dots, F_m) : (U, ds_n^2) \rightarrow (\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}, \bigoplus_{j=1}^m ds_{N_j}^2)$$

be a holomorphic conformal embedding with conformal factors  $\{\lambda(z, \bar{z}); \lambda_1(z, \bar{z}), \dots, \lambda_m(z, \bar{z})\}$  in the sense that

$$\lambda(z, \bar{z})ds_n^2 = \sum_{j=1}^m \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2).$$

Here  $\lambda_j(z, \bar{z}), \lambda(z, \bar{z}) > 0$  are smooth Nash algebraic functions;  $ds_n^2$  and  $ds_{N_j}^2$  are the Bergman metrics of  $\mathbb{B}^n$  and  $\mathbb{B}^{N_j}$ , respectively; and  $F_j$  is a holomorphic map from  $U$  to  $\mathbb{B}^{N_j}$  for each  $j$ . For the proof of Theorem 1.1, we can assume without loss of generality that none of the  $F_j$ 's is a constant map. Following the idea in [MN], we can show that  $F$  extends to an algebraic map over  $\mathbb{C}^n$ . (For the convenience of readers, we include the detailed argument in the appendix.) Namely, for each (non-constant) component  $f_{j,l}$  of  $F_j$ , there is an irreducible polynomial  $P_{j,l}(z, X) = a_{j,l}(z)X^{m_{j,l}} + \dots$  in  $(z, X) \in \mathbb{C}^n \times \mathbb{C}$  of degree  $m_{j,l} \geq 1$  in  $X$  such that  $P_{j,l}(z, f_{j,l}) \equiv 0$  for  $z \in U$ .

We will proceed to show that, for each  $j$ ,  $F_j$  extends to a proper rational map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_j}$ . For this purpose, we let  $R_{j,l}(z)$  be the resultant of  $P_{j,l}$  in  $X$  and let  $E_{j,l} = \{R_{j,l} \equiv 0, a_{j,l} \equiv 0\}$ ,  $E = \cup E_{j,l}$ . Then  $E$  defines a proper complex analytic variety in  $\mathbb{C}^n$ . For any continuous

curve  $\gamma : [0, 1] \rightarrow \mathbb{C}^n \setminus E$  where  $\gamma(0) \in U$ ,  $F$  can be continued holomorphically along  $\gamma$  to get a germ of holomorphic map at  $\gamma(1)$ . Also, if  $\gamma_1$  is homotopic to  $\gamma_2$  in  $\mathbb{C}^n \setminus E$ ,  $\gamma_1(0) = \gamma_2(0) \in U$  and  $\gamma_1(1) = \gamma_2(1)$ , then continuations of  $F$  along  $\gamma_1$  and  $\gamma_2$  are the same at  $\gamma_1(1) = \gamma_2(1)$ . Now let  $p_0 \in U$  and  $p_1 \in \partial\mathbb{B}^n \setminus E$ . Let  $\gamma(t)$  be a smooth simple curve connecting  $p_0$  to  $p_1$  and  $\gamma(t) \notin \partial\mathbb{B}^n$  for  $t \in (0, 1)$ . Then each  $F_j$  defines a holomorphic map in a connected neighborhood  $V_\gamma$  of  $\gamma$  by continuing along  $\gamma$  the initial germ of  $F_j$  at  $p_0$ . (We can also assume that  $V_\gamma \cap \mathbb{B}^n$  is connected.) Let

$$S_\gamma = \{p \in V_\gamma : \|F_j(p)\| = 1 \text{ for some } j\}.$$

Then  $S_\gamma$  is a real analytic (proper) subvariety of  $V_\gamma$ . We first prove

**Claim 3.1.** *When  $V_\gamma$  is sufficiently close to  $\gamma$ ,  $\dim(S_\gamma \cap \mathbb{B}^n) \leq 2n - 2$ .*

**Proof of Claim 3.1:** Seeking a contradiction, suppose not. Assume that  $t_0 \in (0, 1]$  is the first point such that for a certain  $j$ , the local variety defined by  $\|F_j(z)\|^2 = 1$  near  $p^* = \gamma(t_0)$  has real dimension  $2n - 1$  at  $p^*$ . Since any real analytic subset of codimension two inside a connected open set does not affect the connectivity, by slightly changing  $\gamma$  without changing its homotopy type and terminal point, we can assume that  $\gamma(t) \notin S_\gamma$  for any  $t < t_0$ . Hence,  $p^*$  also lies on the boundary of the connected component  $\hat{V}$  of  $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$ , that contains  $\gamma(t)$  for  $t < t_0$ . Then a certain small open piece  $\Sigma$  of  $S_\gamma \cap \mathbb{B}^n$  containing  $p^*$  lies in the boundary of  $\hat{V}$ . Now, for any  $p \in \Sigma$ , let  $q(\in \hat{V}) \rightarrow p$ , we have along  $\{q\}$ ,

$$\lambda(z, \bar{z}) ds_n^2 = \sum_j \lambda_j(z, \bar{z}) F_j^*(ds_{N_j}^2).$$

Suppose that  $j^\sharp$  is such that  $\|F_{j^\sharp}(p)\| = 1$  and  $\|F_j(z)\| < 1$  for any  $j$ ,  $p \in \Sigma$  and  $z \in \hat{V}$ . Since  $p \in \Sigma \subset \mathbb{B}^n$ ,  $ds_n^2|_p < \infty$ , we must have

$$\overline{\lim}_{q \rightarrow p} F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)|_q < \infty.$$

On the other hand,

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2) = \frac{\sum_{l,k} \{\delta_{lk}(1 - \|F_{j^\sharp}\|^2) + \bar{f}_{j^\sharp,l} f_{j^\sharp,k}\} df_{j^\sharp,l} \otimes d\bar{f}_{j^\sharp,k}}{(1 - \|F_{j^\sharp}\|^2)^2}.$$

For any vector  $v \in \mathbb{C}^n$  with  $\|v\| = 1$ ,

$$F_{j^\sharp}^*(ds_{N_{j^\sharp}}^2)(v, v)(q) = \frac{\|\sum_\xi \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi\|^2}{(1 - \|F_{j^\sharp}(q)\|^2)} + \frac{|\sum_{l,\xi} \overline{f_{j^\sharp,l}(q)} \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(q) v_\xi|^2}{(1 - \|F_{j^\sharp}(q)\|^2)^2}. \quad (18)$$

Letting  $q \rightarrow p$ , since  $1 - \|F_{j^\sharp}(q)\|^2 \rightarrow 0^+$ , we get

$$\left\| \sum_\xi \frac{\partial f_{j^\sharp,l}}{\partial z_\xi}(p) v_\xi \right\|^2 = 0$$

for any  $v = (v_1, \dots, v_\xi, \dots, v_n)$  with  $\|v\| = 1$ . Thus

$$\frac{\partial f_{j^\#, l}(p)}{\partial z_\xi} = 0, \quad \text{for } l = 1, \dots, N_{j^\#}.$$

Hence, we see  $dF_{j^\#} = 0$  in a certain open subset of  $\Sigma$ . Since any open subset of  $\Sigma$  is a uniqueness set for holomorphic functions, we see  $F_{j^\#} \equiv \text{const.}$  This is a contradiction. ■

Now, since  $\dim(S_\gamma \cap \mathbb{B}^n) \leq 2n - 2$ , we can always slightly change  $\gamma$  without changing the homotopy type of  $\gamma$  in  $V_\gamma \setminus E$  and the end point of  $\gamma$  so that  $\gamma(t) \notin S_\gamma$  for any  $t \in (0, 1)$ . Since  $\lambda(z, \bar{z})ds_n^2 = \sum \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2)$  in  $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$  and since  $ds_n^2$  blows up when  $q \in V_\gamma \cap \mathbb{B}^n$  approaches to  $\partial\mathbb{B}^n$ , we see that for each  $q \in V_\gamma \cap \partial\mathbb{B}^n$ ,  $\|F_{j_q}(q)\| = 1$  for some  $j_q$ . Hence, we can assume without loss of generality, that there is a  $j_0 \geq 1$  such that each of the  $F_1, \dots, F_{j_0}$  maps a certain open piece of  $\partial\mathbb{B}^n$  into  $\partial\mathbb{B}^{N_1}, \dots, \partial\mathbb{B}^{N_{j_0}}$ , but for  $j > j_0$ ,

$$\dim\{q \in \partial\mathbb{B}^n \cap V_\gamma : \|F_j(q)\| = 1\} \leq 2n - 2.$$

By the Hopf lemma, we must have  $N_j \geq n$  for  $j \leq j_0$ . By the result of Forstneric [Fo] and Cima-Suffridge [CS],  $F_j$  extends to a rational proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_j}$  for  $j \leq j_0$ . Now, we must have

$$\lambda(z, \bar{z})ds_n^2 - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2) = \sum_{j=j_0+1}^m \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2)$$

in  $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$ , that is connected by Claim 3.1. Let  $q \in (V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma \rightarrow p \in \partial\mathbb{B}^n \cap V_\gamma$ . Notice

$$\left( \lambda(z, \bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z}) \right) ds_n^2 \Big|_q + \sum_{j=1}^{j_0} \lambda_j(z, \bar{z}) (ds_n^2 - F_j^*(ds_{N_j}^2)) \Big|_q = \sum_{j=j_0+1}^m \lambda_j(z, \bar{z}) F_j^*(ds_{N_j}^2) \Big|_q.$$

By Lemma 2.2,  $X_j := ds_n^2 - F_j^*(ds_{N_j}^2)$  is smooth up to  $\partial\mathbb{B}^n$  for  $j \leq j_0$ . We also see, by the choice of  $j_0$  and Claim 3.1, that for a generic point in  $\partial\mathbb{B}^n \cap V_\gamma$ ,  $F_j^*(ds_{N_j}^2)$  are real analytic in a small neighborhood of  $p$  for  $j \geq j_0 + 1$ . Thus by considering the normal component as before in the above equation, we see that  $\lambda(z, \bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})$  has double vanishing order in an open set of the unit sphere. Since  $\lambda(z, \bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})$  is real analytic over  $\mathbb{C}^n$ , we obtain

$$\lambda(z, \bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z}) = (1 - |z|^2)^2 \psi(z, \bar{z}). \quad (19)$$

Here  $\psi(z, \bar{z})$  is a certain real analytic function over  $\mathbb{C}^n$ . Next, write

$$Y = \left( \lambda(z, \bar{z}) - \sum_{j=1}^{j_0} \lambda_j(z, \bar{z}) \right) ds_n^2.$$

Then  $Y$  extends real analytically to  $\mathbb{C}^n$ . Write  $X = \sum_1^{j_0} \lambda_j(z, \bar{z})X_j$ . From what we argued above, we easily see that there is a certain small neighborhood  $\mathcal{O}$  of  $q \in \partial\mathbb{B}^n$  in  $\mathbb{C}^n$  such that (1): we can holomorphically continue the initial germ of  $F$  in  $U$  through a certain simple curve  $\gamma$  with  $\gamma(t) \in \mathbb{B}^n$  for  $t \in (0, 1)$  to get a holomorphic map, still denoted by  $F$ , over  $\mathcal{O}$ ; (2):  $\|F_j\| < 1$  for  $j > j_0$  and  $\|F_j(z)\| > 1$  for  $j \leq j_0$  over  $\mathcal{O} \setminus \mathbb{B}^n$ ; and (3):

$$X = \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})X_j = \sum_{j=j_0+1}^m \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2) - Y. \quad (20)$$

We mention that we are able to make  $\|F_j\| < 1$  for any  $z \in \mathcal{O}$  and  $j > j_0$  in the above due to the fact that  $(V_\gamma \cap \mathbb{B}^n) \setminus S_\gamma$ , as defined before, is connected.

Now, let  $\mathcal{P}$  be the union of the poles of  $F_1, \dots, F_{j_0}$ . Fix a certain  $p^* \in \mathcal{O} \cap \partial\mathbb{B}^n$  and let  $\tilde{E} = E \cup \mathcal{P}$ . Then for any  $\gamma : [0, 1] \rightarrow (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$  with  $\gamma(0) = p^*$ ,  $F_j$  extends holomorphically to a small neighborhood  $U_\gamma$  of  $\gamma$  that contracts to  $\gamma$ . Still denote the holomorphic continuation of  $F_j$  (from the initial germ of  $F_j$  at  $p^* \in \mathcal{O}$ ) over  $U_\gamma$  by  $F_j$ . If for some  $t \in (0, 1)$ ,  $\|F_j(\gamma(t))\| = 1$ , then we similarly have

**Claim 3.2.** *Shrinking  $U_\gamma$  if necessary, we then have  $\dim\{p \in U_\gamma : \|F_j(p)\| = 1 \text{ for some } j\} \leq 2n - 2$ .*

**Proof of the Claim 3.2:** Still seek for a contradiction, if we suppose not. Define  $S_\gamma$  in a similar way. Without loss of generality, we assume that  $t_0 \in (0, 1)$  is the first point such that for a certain  $j_{t_0}$ , the local variety defined by  $\|F_{j_{t_0}}(z)\|^2 = 1$  near  $\gamma(t_0)$  has real dimension  $2n - 1$  at  $\gamma(t_0)$ . Then, as before, we have

$$X = \sum_{j=1}^{j_0} \lambda_j(z, \bar{z})(ds_n^2 - F_j^*(ds_{N_j}^2)) = \sum_{j=j_0+1}^m \lambda_j(z, \bar{z})F_j^*(ds_{N_j}^2) - Y \quad (21)$$

in a connected component  $W$  of  $U_\gamma \setminus S_\gamma$  that contains  $\gamma(t)$  for  $t \ll 1$  with  $\gamma(t_0) \in \partial W$ . Now, for any  $q \in W \rightarrow p \in \partial W$  near  $p_0 = \gamma(t_0)$  and  $v \in \mathbb{C}^n$  with  $\|v\| = 1$ , we have the following:

$$\begin{aligned} & \sum_{j=1}^{j_0} \lambda_j(q, \bar{q}) \left( ds_n^2(v, v) \Big|_q - \frac{\|\sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi\|^2}{(1 - \|F_j(q)\|^2)} - \frac{|\sum_{l,\xi} \overline{f_{j,l}(q)} \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi|^2}{(1 - \|F_j(q)\|^2)^2} \right) \\ &= \sum_{j=j_0+1}^m \lambda_j(q, \bar{q}) \left( \frac{\|\sum_\xi \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi\|^2}{(1 - \|F_j(q)\|^2)} + \frac{|\sum_{l,\xi} \overline{f_{j,l}(q)} \frac{\partial f_{j,l}}{\partial z_\xi}(q)v_\xi|^2}{(1 - \|F_j(q)\|^2)^2} \right) - Y(v, v) \Big|_q. \end{aligned} \quad (22)$$

Now, if  $p_0$  is not a point of codimension one for any local variety defined by  $\|F_j(z)\|^2 = 1$  near  $p_0$  for  $j \leq j_0$ , then it has to be a codimension one point for a certain local variety  $S_{j'}$  defined by  $\|F_{j'}(z)\|^2 = 1$  near  $p_0$  for  $j' > j_0$ . Let  $J$  be the collection of all such  $j'$ . Let  $S^0$  be an open small piece of the boundary of  $W$  near  $p_0$ . Then for a generic  $p \in S^0$ , the left hand side of

(22) remains bounded as  $q \rightarrow p \in S^0$ . For a term on the right hand side with index  $j \in J$ , if  $S^0 \cap S_j$  contains an open neighborhood of  $\partial W$  near  $p_0$ , then it approaches to  $+\infty$  for a generic  $p$  unless  $F_j = \text{constant}$  as argued in the proof of Claim 3.1. The other terms on the right hand side remain bounded as  $q \rightarrow p$  for a generic  $p$ . This is a contradiction to the assumption that none of the  $F_j$  for  $j > j_0$  is constant. Hence, we can assume that  $p_0$  is a point of codimension one for a local variety defined by  $\|F_j(z)\|^2 = 1$  near  $p_0$  for a certain  $j \leq j_0$ . Let  $J$  be the set of indices such that for  $j' \in J$ , we have  $j' \leq j_0$  and  $S_{j'} := \{\|F_{j'}\| = 1\}$  is a local real analytic variety of codimension one near  $p_0$ . For  $j > j_0$ , since  $\|F_j(z)\| < 1$  for  $z \in U_\gamma \approx p_0$  and since  $t_0$  is the first point we have some  $j^*$  with  $\|F_{j^*}\| = 1$  defining a variety of real codimension one, we see that  $\|F_j(z)\| < 1$  for  $z \in W \approx p_0$ . Define  $S^0$  similarly, as an open piece of  $\partial W$ . Hence, as  $q \in W \rightarrow p \in S^0$ , the right hand side of (22) remains to be bounded. On the other hand, in the left hand side of (22), for any  $j' \in J$  with  $S_{j'} \cap S^0$  containing an open piece of  $\partial W$  near  $p_0$ , if the numerator  $|\sum_{l,\xi} \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q) v_\xi|^2$  of the last term does not go to 0 for some vectors  $v$ , then the term with index  $j'$  on the left hand side would go to  $-\infty$  for a generic  $p \in S^0$ . If this happens to such  $j'$ , the left hand side would approach to  $-\infty$ . Notice that all other terms on the right hand side remain bounded as  $q \rightarrow p \in S^0$  for a generic  $p$ . This is impossible. Therefore we must have for some  $j' \in J$  that  $|\sum_l \overline{f_{j',l}(q)} \frac{\partial f_{j',l}}{\partial z_\xi}(q)|^2 = \frac{\partial \sum_l |f_{j',l}|^2}{\partial z_\xi}(q) = \frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(q) \rightarrow 0$  and thus  $\frac{\partial \|F_{j'}\|^2}{\partial z_\xi}(p) = 0$  for all  $\xi$  and  $p \in S_{j'}$ . This immediately gives the equality  $d(\|F_{j'}\|^2) = 0$  along  $S_{j'}$ . Assume, without loss of generality, that  $p_0$  is also a smooth point of  $S_{j'}$ . If  $S_{j'}$  has no complex hypersurface passing through  $p_0$ , by a result of Trepreau [Tr], the union of the image of local holomorphic disks attached to  $S_{j'}$  passing through  $p_0$  fills in an open subset. Since  $F_{j'}$  is not constant, there is a small holomorphic disk smooth up to the boundary  $\phi(\tau) : \mathbb{B}^1 \rightarrow \mathbb{C}^n$  such that  $\phi(\partial\mathbb{B}^1) \subset S_{j'}$ ,  $\phi(1) = p_0$  and  $F_{j'}$  is not constant along  $\phi$ . Since  $\partial\mathbb{B}^{N_{j'}}$  does not contain any non-trivial complex curves,  $r = (\|F_{j'}\|^2 - 1) \circ \phi \neq 0$ . Applying the maximum principle and then the Hopf lemma to the subharmonic function  $r = (\|F_{j'}\|^2 - 1) \circ \phi$ , we see that the outward normal derivative of  $r$  at  $\tau = 1$  is positive. This contradicts to the fact that  $d(\|F_{j'}\|^2) = 0$  along  $S_{j'}$ . We can argue in the same way for points  $p \in S_{j'}$  near  $p_0$  to conclude that for any  $p \in S_{j'}$  near  $p_0$ , there is a complex hypersurface contained in  $S_{j'}$  passing through  $p$ . Namely,  $S_{j'}$  is Levi flat, foliated by a family of smooth complex hypersurfaces denoted by  $Y_\eta$  with real parameter  $\eta$  near  $p_0$ . Let  $Z$  be a holomorphic vector field along  $Y_\eta$ . We then easily see that  $0 = \bar{Z}Z(\|F_{j'}\|^2 - 1) = \sum_{k=1}^{N_{j'}} |Z(f_{j',k})|^2$ . Thus, we see that  $F_{j'}$  is constant along each  $Y_\eta$ . Hence,  $F_{j'}$  can not be a local embedding at each point of  $S_{j'}$ . On the other hand, notice that  $F_{j'}$  is a proper holomorphic map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_{j'}}$ , then  $F_{j'}$  is a local embedding near  $\partial\mathbb{B}^n$ . Hence, the set of points where  $F_{j'}$  is not a local embedding can be at most of complex codimension one (and thus real codimension two). This is a contradiction. This proves Claim 3.2.

Hence, we see that  $\mathcal{E} = \{p \in (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E} : \text{some branch, obtained by the holomorphic continuation through curves described before, of } F_j \text{ for some } j \text{ maps } p \text{ to } \partial\mathbb{B}^{N_j}\}$  is a real analytic variety of real dimension at most  $2n - 2$ . Now, for any  $p \in (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$ , any curve  $\gamma : [0, 1] \rightarrow (\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$  with  $\gamma(0) = p^* \in \mathcal{O} \cap \partial\mathbb{B}^n$  and  $\gamma(1) = p$ , we can homotopically change

$\gamma$  in  $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$  (but without changing the terminal point) such that  $\gamma(t) \notin \mathcal{E}$  for  $t \in (0, 1)$ . Now, the holomorphic continuation of the initial germ of  $F_j$  from  $p^*$  never cuts  $\partial\mathbb{B}^{N_j}$  along  $\gamma(t)$  ( $0 < t < 1$ ). We thus see that  $\|F_j(p)\| \leq 1$  for  $j > j_0$ .

Let  $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$  be all possible (distinct) germs of holomorphic functions that we can get at  $p$  by the holomorphic continuation, along curves described above in  $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$ , of  $f_{j,l}$ . Let  $\sigma_{j,l,\tau}$  be the fundamental symmetric function of  $\{(f_{j,l})_{k;p}\}_{k=1}^{n_{jl}}$  of degree  $\tau$ . Then  $\sigma_{j,l,\tau}$  well defines a holomorphic function over  $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ .  $|\sigma_{j,l,\tau}|$  is bounded in  $(\mathbb{C}^n \setminus \overline{\mathbb{B}^n}) \setminus \tilde{E}$ . By the Riemann removable singularity theorem,  $\sigma_{j,l,\tau}$  is holomorphic over  $\mathbb{C}^n \setminus \overline{\mathbb{B}^n}$ . By the Hartogs extension theorem,  $\sigma_{j,l,\tau}$  extends to a bounded holomorphic function over  $\mathbb{C}^n$ . Hence, by the Liouville theorem,  $\sigma_{j,l,\tau} \equiv \text{const}$ . This forces  $(f_{j,l})_k$  and thus  $F_j$  for  $j > j_0$  to be constant. We obtain a contradiction. This proves that each  $F_j$  extends to a proper rational map from  $\mathbb{B}^n$  into  $\mathbb{B}^{N_j}$ . Together with Proposition 2.6, we complete the proof of the main Theorem. ■

We conclude the section with a remark: The regularity of  $\lambda_j, \lambda$  can be reduced to be only real analytic in the complement of a certain real codimension two subset. Also, we need only to assume that they are positive outside a real analytic variety of codimension two. This is obvious from our proof for Theorem 1.1.

## 4 Appendix: algebraic extension

In this appendix, we prove the algebraicity of the local holomorphic conformal embedding. As in the theorem, we let  $U \subset \mathbb{B}^n$  be a connected open subset. Let

$$F = (F_1, \dots, F_m) : (U, ds_n^2) \rightarrow (\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}, \oplus_{j=1}^m ds_{N_j}^2)$$

be a holomorphic conformal embedding with conformal factors  $\{\lambda(z, \bar{z}); \lambda_1(z, \bar{z}), \dots, \lambda_m(z, \bar{z})\}$ . Here  $\lambda_j(z, \bar{z}) > 0, \lambda(z, \bar{z}) > 0$  are smooth Nash algebraic functions in  $\mathbb{C}^n$ , and  $ds_n^2$  and  $ds_{N_j}^2$  are the Bergman metrics of  $\mathbb{B}^n$  and  $\mathbb{B}^{N_j}$ , respectively. We further assume without loss of generality that none of the  $F_j$ 's is a constant map. Our proof uses exactly the same method employed in the paper of Mok-Ng [MN], following a suggestion of Yum-Tong Siu. Namely, we use the Grauert tube technique to reduce the problem to the algebraicity problem for CR mappings. However, different from the consideration in [MN], the Grauert tube constructed by using the unit sphere bundle over  $\mathbb{B}^{N_1} \times \dots \times \mathbb{B}^{N_m}$  with respect to the metric  $\oplus_{j=1}^m ds_{N_j}^2$ , up to conformal factors, may have complicated geometry and may not even be pseudoconvex anymore in general. To overcome the difficulty, we bend the target hypersurface to make it sufficiently positively curved along the tangential direction of the source domain. For the convenience of the reader, we give the proof when  $m = 2$  for the simplicity of our notation. The argument for general  $m$  is exactly the same as for  $m = 2$  and is skipped.

Let  $K > 0$  be a large constant to be determined. Consider  $S_1 \subset TU$  and  $S_2 \subset U \times T\mathbb{B}^{N_1} \times T\mathbb{B}^{N_2}$  as follows:

$$S_1 := \{(t, \zeta) \in TU : (1 + K|t|^2)\lambda(t, \bar{t})ds_n^2(t)(\zeta, \zeta) = 1\}, \quad (23)$$

$$S_2 := \{(t, z, \xi, w, \eta) \in U \times T\mathbb{B}^{N_1} \times T\mathbb{B}^{N_2} : (1 + K|t|^2)[\lambda_1(t, \bar{t})ds_{N_1}^2(z)(\xi, \xi) + \lambda_2(t, \bar{t})ds_{N_2}^2(w)(\eta, \eta)] = 1\}. \quad (24)$$

The defining functions  $\rho_1, \rho_2$  of  $S_1, S_2$  are, respectively, as follows:

$$\rho_1 = (1 + K|t|^2)\lambda(t, \bar{t})ds_n^2(t)(\zeta, \zeta) - 1,$$

$$\rho_2 = (1 + K|t|^2)[\lambda_1(t, \bar{t})ds_{N_1}^2(z)(\xi, \xi) + \lambda_2(t, \bar{t})ds_{N_2}^2(w)(\eta, \eta)] - 1.$$

Then one can easily check that the map  $(id, F_1, dF_1, F_2, dF_2)$  maps  $S_1$  to  $S_2$  according to the metric equation

$$\lambda(t, \bar{t})ds_n^2 = \lambda_1(t, \bar{t})F_1^*(ds_{N_1}^2) + \lambda_2(t, \bar{t})F_2^*(ds_{N_2}^2).$$

**Lemma 4.1.**  *$S_1, S_2$  are both real algebraic hypersurfaces. Moreover for  $K$  sufficiently large,  $S_1$  is smoothly strongly pseudoconvex. For any  $\xi \neq 0, \eta \neq 0, (0, 0, \xi, 0, \eta) \in S_2$  is a smooth strongly pseudoconvex point when  $K$  is sufficiently large, where  $K$  depends on the choice of  $\xi$  and  $\eta$ .*

*Proof.* It is immediate from the defining functions that  $S_1, S_2$  are smooth real algebraic hypersurfaces. We show the strong pseudoconvexity of  $S_2$  at  $(0, 0, \xi, 0, \eta)$  as follows: (The strong pseudoconvexity of  $S_1$  follows from the same computation.)

By applying  $\partial\bar{\partial}$  to  $\rho_2$  at  $(0, 0, \xi, 0, \eta)$ , we have the following Hessian matrix

$$\begin{bmatrix} A & 0 & D_1 & 0 & D_2 \\ 0 & B_1 & 0 & 0 & 0 \\ \bar{D}_1 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ \bar{D}_2 & 0 & 0 & 0 & C_2 \end{bmatrix} \quad (25)$$

where  $A, B_j, C_j, D_j, j = 1, 2$  are function-valued matrices with the following (in)equalities:

$$A := \left( \partial_{t_i} \partial_{\bar{t}_j} \rho_2 \right) = \left( K(\lambda_1(0)|\xi|^2 + \lambda_2(0)|\eta|^2)\delta_{i\bar{j}} + \partial_{t_i} \partial_{\bar{t}_j} \lambda_1(0)|\xi|^2 + \partial_{t_i} \partial_{\bar{t}_j} \lambda_2(0)|\eta|^2 \right) \geq \delta K(|\xi|^2 + |\eta|^2)I_n, \quad (26)$$

$$B_1 := \left( \partial_{z_k} \partial_{\bar{z}_l} \rho_2 \right) = \left( -\lambda_1(0)R_{z_k z_l \mu \bar{\nu}}(0)\xi^\mu \xi^{\bar{\nu}} \right) \geq \delta|\xi|^2 I_{N_1}, \quad B_2 := \left( \partial_{w_k} \partial_{\bar{w}_l} \rho_2 \right) \geq \delta|\eta|^2 I_{N_2}, \quad (27)$$

$$C_1 := \left( \partial_{\xi_k} \partial_{\xi_l} \rho \right) = \left( \lambda_1(0) \delta_{kl} \right) \geq \delta I_{N_1}, \quad C_2 := \left( \partial_{\eta_k} \partial_{\eta_l} \rho \right) \geq \delta I_{N_2}, \quad (28)$$

$$D_1 := \left( \partial_{t_i} \partial_{\xi_l} \rho \right) = \left( \partial_{t_i} \lambda_1(0) \xi^l \right)_{i \leq n}^{l \leq N_1}, \quad D_2 := \left( \partial_{t_i} \partial_{\eta_l} \rho \right) = \left( \partial_{t_i} \lambda_2(0) \xi^l \right)_{i \leq n}^{l \leq N_2} \quad (29)$$

at  $(0, 0, \xi, 0, \eta)$  for some  $\delta > 0$ .

Let  $(e, r, r', s, s') = (e_1, \dots, e_n, r_1, \dots, r_{N_1}, r'_1, \dots, r'_{N_1}, s_1, \dots, s_{N_2}, s'_1, \dots, s'_{N_2}) \neq 0$ . It holds that

$$\begin{aligned} & \begin{bmatrix} e & r & r' & s & s' \end{bmatrix} \begin{bmatrix} A & 0 & D_1 & 0 & D_2 \\ 0 & B_1 & 0 & 0 & 0 \\ \bar{D}_1 & 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & B_2 & 0 \\ \bar{D}_2 & 0 & 0 & 0 & C_2 \end{bmatrix} \begin{bmatrix} \bar{e}^t \\ \bar{r}^t \\ \bar{r}'^t \\ \bar{s}^t \\ \bar{s}'^t \end{bmatrix} \\ & \geq \delta K (|\xi|^2 + |\eta|^2) |e|^2 + \delta (|\xi|^2 |r|^2 + |\eta|^2 |s|^2) + \delta (|r'|^2 + |s'|^2) \\ & \quad - 2 \left| \sum_{i \leq n, l \leq N_1} e_i \partial_{t_i} \lambda_1(0) \xi^l \bar{r}'_l \right| - 2 \left| \sum_{i \leq n, l \leq N_2} e_i \partial_{t_i} \lambda_2(0) \eta^l \bar{s}'_l \right| \\ & \geq (\delta K - M) (|\eta|^2 + |\xi|^2) |e|^2 + \delta (|\xi|^2 |r|^2 + |\eta|^2 |s|^2) + (\delta - \epsilon) |r'|^2 + (\delta - \epsilon) |s'|^2 \\ & > 0. \end{aligned} \quad (30)$$

Here the second inequality holds as

$$\left| \sum_{i,l} e_i \partial_{t_i} \lambda_1(0) \xi^l \bar{r}'_l \right| \leq M_1 |e| |\xi \cdot r| \leq M_1 |e| |\xi| |r| \leq \frac{M}{2} |e|^2 |\xi|^2 + \frac{\epsilon}{2} |r|^2$$

by the standard Cauchy-Schwarz inequality and  $M = \frac{M_1^2}{\epsilon}$ . The last strict inequality holds as  $\xi \neq 0$  and  $\eta \neq 0$  by letting  $\epsilon < \delta$  and raising  $K$  sufficiently large.  $\square$

**Theorem 4.2.** *Under the assumption of Theorem 1.1,  $F$  is Nash algebraic.*

*Proof.* Without loss of generality, one can arrange that  $F(0) = 0$  by composing elements from  $Aut(\mathbb{B}^n)$  and  $Aut(\mathbb{B}^{N_1}) \times Aut(\mathbb{B}^{N_2})$ . Furthermore, since  $F_1, F_2$  are not constant maps, we can assume that  $dF_1|_0 \neq 0$  and  $dF_2|_0 \neq 0$ . Therefore, there exists  $0 \neq \zeta \in T_0 \mathbb{B}^n$ , such that  $dF_1(\zeta) \neq 0$  and  $dF_2(\zeta) \neq 0$ . After scaling, we assume that  $(0, \zeta) \in S_1$ . Notice that both the fiber of  $S_1$  over  $0 \in U$  and the fiber of  $S_2$  over  $(0, 0) \in U \times (\mathbb{B}^{N_1} \times \mathbb{B}^{N_2})$  are independent of the choice of  $K$ . Now the theorem follows from the algebraicity theorem of Huang [Hu1] and Lemma 4.1 applied to the map  $(id, F_1, dF_1, F_2, dF_2)$  from  $S_1$  into  $S_2$ .  $\square$

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Yuan Yuan, [yuan yuan@math.rutgers.edu](mailto:yuan yuan@math.rutgers.edu), Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA.

Yuan Zhang, [yuz009@math.ucsd.edu](mailto:yuz009@math.ucsd.edu), Department of Mathematics, University of California at San Diego, La Jolla, CA 92093, USA.