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A TRANSVERSAL FREDHOLM PROPERTY FOR THE $\bar{\partial}$-NEUMANN PROBLEM ON G-BUNDLES

DEDICATED TO M.A. SHUBIN ON HIS 65TH BIRTHDAY

JOE J PEREZ

ABSTRACT. Let M be a strongly pseudoconvex complex manifold which is also the total space of a principal G-bundle with compact base M/G. Assume also that G acts on M by holomorphic transformations. For such M, we provide a simple condition on forms α sufficient for the regular solvability of $\Box u = \alpha$ and other problems related to the $\bar{\partial}$-Neumann problem on M. Similar properties are shared by $\Box_b$.

1. INTRODUCTION

Let G be a connected Lie group and M be a manifold which is the total space of a principal bundle

$$G \longrightarrow M \longrightarrow X$$

with X compact. With respect to a G-invariant measure on M, define the Hilbert space $L^2(M)$. This decomposes essentially uniquely as

$$L^2(M) \cong L^2(G) \otimes L^2(X), \tag{1.1}$$

with an invariant measure on G and the quotient measure on X. By convention the action of G is from the right, thus in the above decomposition, $t \in G$ acts unitarily in $L^2(M)$ by $t \to \rho_t \otimes \mathbf{1}_{L^2(X)}$. The von Neumann algebra of operators on $L^2(G)$ commuting with right translations is denoted by $\mathcal{L}_G$ and the corresponding algebra of bounded linear operators on $L^2(M)$ that commute with the action of G we will denote by $\mathcal{B}(L^2(M))^G$. This algebra has a decomposition itself as follows,

$$\mathcal{B}(L^2(M))^G \cong \mathcal{B}(L^2(G) \otimes L^2(X))^G \cong \mathcal{L}_G \otimes \mathcal{B}(L^2(X))),$$

and with respect to this decomposition we formulate

Definition 1.1. Let M be a G-manifold with quotient $X = M/G$ and let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces of sections of bundles over M. A closed, densely defined, linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ commuting with the action of G is called *transversally Fredholm* if there exist finite-rank projections $P_{L^2(X)}, P'_{L^2(X)} \in \mathcal{B}(L^2(X))$ such that $\ker A \subset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P_{L^2(X)})$ and $\operatorname{im} A \supset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P'_{L^2(X)})^{\perp}$.

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Remark 1.2. When dealing with algebras of operators acting in bundles E over M, we will systematically suppress notation referring to that bundle, for example shortening $\mathcal{B}(L^2(M,E))$ to simply $\mathcal{B}(L^2(M))$.

This note will provide an example in which the transversal Fredolm property holds and an application in the following setting. Let M be a strongly pseudoconvex complex manifold which is also the total space of a G-bundle $G \longrightarrow M \longrightarrow X$ with X compact. Furthermore, assume that G acts on M by holomorphic transformations. With respect to a G-invariant measure and Riemannian structure on M, define the Hilbert spaces of (p,q)-forms $L^2(M,\Lambda^{p,q})$.

On M, consider Kohn's Laplacian, $\Box$ in $L^2(M,\Lambda^{p,q})$, $(q > 0)$ and its spectral decomposition, $\Box = \int_0^\infty \lambda dE_\lambda$. It was shown in [P1] that if $\delta \geq 0$, then the Schwartz kernel K of the spectral projection $P_\delta = \int_0^\delta dE_\lambda$ belongs to $C^\infty(\bar{M} \times \bar{M})$. Choosing a piecewise smooth section $X \hookrightarrow M$, we may write points in M as pairs $(t,x) \in G \times X$. It follows that almost everywhere K takes the form

$$K(t,x;s,y) = K(ts^{-1},x;e,y) =: \kappa(ts^{-1};x,y),$$

where we have used the G-invariance of P_δ. Consider κ's Fourier expansion

$$\kappa(t;x,y) = \sum_{kl} \psi_k(x) h_{kl}(t) \bar{\psi}_l(y) \tag{1.2}$$

with $(\psi_k)_k$ an orthonormal basis of $L^2(X)$. The proof of Lemma 6.2 in [P1] implies that the functions h_{kl} are smooth in G with $\sum_{kl} \|h_{kl}\|^2_{L^2_R(G)} < \infty$, where $L^2_R(G)$ consists of the functions on G that are square-integrable with respect to right-Haar measure.

The main result of the present paper asserts that when κ corresponds to P_δ, the sum in equation (1.2) is finite. This means that the spectral projections of $\Box$ are subordinate to simple projections of the form $P = \mathbf{1}_{L^2(G)} \otimes P_{L^2(X)}$ with $P_{L^2(X)}$ the projection onto the space spanned by the ψ_k that appear in the sum. Since there are finitely many of these, we have that $\operatorname{rank} P_{L^2(X)} < \infty$. Thus our main result is

Theorem 1.3. *Let M be a strongly pseudoconvex complex manifold which is also the total space of a G-bundle $G \longrightarrow M \longrightarrow X$ with X compact. Furthermore, assume that G acts on M by holomorphic transformations. It follows that for $q > 0$, the Laplacian $\Box$ in $L^2(M,\Lambda^{p,q})$ is transversally Fredholm.*

We will also show that the problem $\Box u = \alpha$ has regular solutions for $\alpha \in \operatorname{im} P^\perp$ and sketch an application to the $\bar{\partial}$-Neumann problem.

As well as sharpening the results in [P1], the results of this note will be useful in studying the $\bar{\partial}$-Neumann problem and its consequences for G-manifolds with nonunimodular structure group; in [P1], G was always assumed unimodular. These more general G-manifolds, among others, occur naturally as complexifications of group actions, as shown in [HHK]. Together with the amenability property introduced in [P2], the present results lead to a deeper understanding of two important exemplary nonunimodular G-manifolds discussed in [GHS]. One of these has a large space of L^2-holomorphic functions while the other has $L^2\mathcal{O} = \{0\}$; these, as well as the complexifications constructed in [HHK], are Stein manifolds.

Remark 1.4. All the results in this note remain valid for weakly pseudoconvex M satisfying a subelliptic estimate, and for the boundary Laplacian, $\Box_b$, [P3].

2. Invariant operators in $L^2(M)$

Here we briefly sketch the construction of the Schwartz kernel (1.2) of P_δ. We will continue to simplify notation by suppressing the operators' acting in bundles; some additional details are in [P1].

On the group alone, the projection P_L onto a right-translation-invariant subspace $L \subset L^2(G)$ is a left-convolution operator with distributional kernel h,

$$(P_L u)(t) = (\lambda_h u)(t) = \int_G ds\ h(ts^{-1})u(s), \qquad (u \in L^2(G)),$$

where ds is the right-invariant Haar measure.

On $L^2(M)$ we take the decomposition (2) a step further. Letting $(\psi_k)_k$ be an orthonormal basis for $L^2(X)$, we may write

$$L^2(M) \cong L^2(G) \otimes L^2(X) \cong \bigoplus_k L^2(G) \otimes \psi_k,$$

and with respect to this decomposition write matrix representations for operators in $L^2(M)$ as

$$\mathcal{B}(L^2(M)) \ni P \longleftrightarrow [P_{kl}]_{kl}, \qquad P_{kl} \in \mathcal{B}(L^2(G)).$$

When $P \in \mathcal{B}(L^2(M))^G$ each of the P_{kl} is an operator commuting with the right action of G and thus is a left-convolution operator. Thus $P_{kl} = \lambda_{h_{kl}}$ for distributions h_{kl} on G, as in the expansion (1.2). When P is a self-adjoint projection, we find that the matrix of convolution operators $H = [\lambda_{h_{kl}}]_{kl}$ is an idempotent in that $\sum_k H_{jk}H_{kl} = H_{jk}$ and the operator P^* has matrix representation $[\lambda^*_{h_{lk}}]_{kl}$.

3. Regularity of the $\bar\partial$-Neumann problem on G-manifolds

We provide a brief list of the properties of the $\bar\partial$-Neumann problem relevant to our work here and refer the reader to [FK, GHS, P1, P3] for more detail. With the invariant measure and Riemannian structure on M define the Sobolev spaces $H^s(M, \Lambda^{p,q})$ of (p,q)-forms on M. Note that the G-invariance of the structures and the compactness of X imply that any two such choices of the structures yield equivalent Sobolev spaces. A word on notation: we will write $A \lesssim B$ to mean that there exists a $C > 0$ such that $|A(u)| \leq C|B(u)|$ uniformly for u in a set that will be made clear in the context.

Lemma 3.1. *Suppose that M is strongly pseudoconvex and $U \subset \bar{M}$ with compact closure. Assume also that $\zeta, \zeta_1 \in C_c^\infty(U)$ for which $\zeta_1|_{\mathrm{supp}(\zeta)} = 1$. If $q > 0$ and $\alpha|_U \in H^s(U, \Lambda^{p,q})$, then $\zeta(\Box + 1)^{-1}\alpha \in H^{s+1}(\bar{M}, \Lambda^{p,q})$ and*

$$\|\zeta(\Box + 1)^{-1}\alpha\|_{s+1}^2 \lesssim \|\zeta_1\alpha\|_s^2 + \|\alpha\|_0^2. \tag{3.1}$$

Proof. This is Prop. 3.1.1 from [FK] extended to the noncompact case in [E]. □

It follows easily (Corollary 4.3, [P1]) that the image of the Laplacian's spectral projection P_δ is contained in $C^\infty(\bar{M}, \Lambda^{p,q})$.

In order to derive properties of the Schwartz kernel of P_δ, we will need global Sobolev estimates strengthening the previous result. The following assertion (Theorem 4.5 of [P1]) provides global *a priori* Sobolev estimates on M, *cf.* Prop. 3.1.11, [FK]. Note that this crucially uses the uniformity on M guaranteed by the G-action and the compactness of X.

Lemma 3.2. *Let $q > 0$. For every integer $s \geq 0$, the following estimate holds uniformly,*

$$\|u\|_{s+1}^2 \lesssim \|\Box u\|_s^2 + \|u\|_0^2, \qquad (u \in \mathrm{dom}\,(\Box) \cap C^\infty(\bar{M}, \Lambda^{p,q})).$$

The previous two lemmata give

Corollary 3.3. *For $q > 0$, let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian $\Box$ in $L^2(M, \Lambda^{p,q})$ and for $\delta \geq 0$, define $P_\delta = \int_0^\delta dE_\lambda$. It follows that* $\mathrm{im}\, P_\delta \subset H^\infty(M, \Lambda^{p,q})$.

Proof. The assertion follows from lemmata 3.1, 3.2 and the fact that $\mathrm{im}\, P_\delta \subset \mathrm{dom}(\Box^k)$ for all $k = 1, 2, \ldots$. Thus the estimates

$$\|\Box^{k-s}u\|_{s+1} \lesssim \|\Box^{k-s+1}u\|_s + \|\Box^{k-s}u\|_0, \quad (s = 1, 2, \ldots, k)$$

hold for $u \in \mathrm{im}\, P_\delta$. These can be reduced to the result. □

Remark 3.4. By results in [E, P3], these regularity properties essentially hold true for G-manifolds M that are weakly pseudoconvex but satisfy a subelliptic estimate. Similar results hold for the boundary Laplacian $\Box_b$ as indicated in [P1].

4. The finiteness result

In this section, we modify an ingenious lemma from [GHS] which asserts that on a regular covering space $\Gamma \to M \to X$, it is true that any closed, invariant subspace $L \subset L^2(M)$ that belongs to some $H^s(M)$ $(s > 0)$ has the following property. There exists an $N < \infty$ and a Γ-equivariant injection P_N such that

$$L \xhookrightarrow{P_N} L^2(\Gamma) \otimes \mathbb{C}^N.$$

This result has analogues in [A] and Theorem 8.10, [LL], gotten by different methods.

Here, we will use essentially the same proof as in [GHS] to obtain a similar result for G-bundles. We will need the following

Definition 4.1. For any positive integer s, let $H^{0,s}(G \times X) = L^2(G) \otimes H^s(X)$ be the completion of $C_c^\infty(G \times X)$ in the norm defined by

$$\|u\|^2_{H^{0,s}(G\times X)} = \int_G dt \; \|u(t,\cdot)\|^2_{H^s(X)}.$$

Clearly $\|\cdot\|_{H^{0,s}(G\times X)} \leq \|\cdot\|_{H^s(M)}$ and so $H^s(M) \subset H^{0,s}(G \times X)$.

The next two statements in this section follow [GHS] closely. Lemma 4.2 is taken verbatim and our Theorem 4.3 is a small variation on Prop. 1.5 of that article.

Lemma 4.2. *Let X be a compact Riemannian manifold, possibly with boundary and let $(\psi_k)_k$ be any orthonormal basis of $L^2(X)$. Then, for all $s > 0$ and $\delta > 0$ there exists an integer $N > 0$ such that for all $u \in H^s(X)$ in the L^2-orthogonal complement of $(\psi_k)_1^N$ we have the uniform estimate*

$$\|u\|_{L^2(X)} \leq \delta \|u\|_{H^s(X)}, \qquad (u \in H^s(X),\ u \perp \psi_k,\ k = 1, 2, \ldots, N).$$

Proof. Assuming the contrary, there exist $s > 0$ and $\delta > 0$ so that for each $N > 0$ there is an $u_N \in H^s(X)$ with $\langle u_N, \psi_k\rangle = 0$ for $k = 1, 2, \ldots, N$ and $\|u_N\|_s < 1/\delta\ \|u_N\|_0$. Without loss of generality we may rescale the u_N to unit length. By Sobolev's compactness theorem, the sequence $(u_N)_N$ is a compact subset of $L^2(X)$. By the requirement that each u_N be orthogonal to ψ_k for $k = 1, 2, \ldots, N$, the sequence converges weakly to zero, contradicting the choice of normalization. □

Theorem 4.3. *Assume that G is a Lie group and $G \to M \to X$ is a G-bundle with compact quotient, X. Let L be an L^2-closed, G-invariant subspace in $H^\infty(M)$, such that for $s \in \mathbb{N}$ sufficiently large, the estimate*

$$\|u\|_{H^s(M)} \lesssim \|u\|_{L^2(M)} \tag{4.1}$$

holds for $u \in L$. Then $L \subset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P_{L^2(X)})$ where $P_{L^2(X)}$ is a finite-rank projection in $L^2(X)$.

Proof. First, assume that $M \cong G \times X$ is a trivial bundle. For each fixed $t \in G$, define the *slice at t*, $S_t = \{(t,x) \in M \mid x \in X\}$, and note that by the trace theorem, the restrictions of elements of L to these slices are in $H^\infty(S_t)$. Note also that the invariance of L implies that all the restrictions $L|_{S_t}$ are identical as t varies in G. Choose an orthonormal basis $(\psi_j)_j$ for $L^2(S_e) \cong L^2(X)$. Let L satisfy the assumptions of the theorem and define a map $P_N : L \to L^2(G) \otimes \mathbb{C}^N$ by

$$(P_N u)(t) = (u_1(t), u_2(t), \ldots, u_N(t)),$$

where

$$u_j(t) = \langle u|_{S_t}, \psi_j\rangle_{L^2(X)}, \qquad (j = 1, 2, \ldots, N).$$

We will show that P_N is injective for large N, so let us assume that $u \in L$ with $P_N u = 0$. The smoothness of L implies that $(P_N u)(t) = 0$ identically in G and so Lemma (4.2) and invariance imply that there is a $\delta_N > 0$ such that

$$\|u|_{S_t}\|^2_{L^2(S_t)} \leq \delta_N^2 \|u|_{S_t}\|^2_{H^s(S_t)}, \quad (t \in G). \tag{4.2}$$

Integrating over $t \in G$ we obtain

$$\|u\|^2_{L^2(M)} \leq \delta_N^2 \|u\|^2_{H^{0,s}(G\times X)} \leq \delta_N^2 \|u\|^2_{H^s(M)}. \tag{4.3}$$

If this were possible for any N, this would contradict the estimate (4.1) unless $u = 0$, since $\delta_N \to 0$ as $N \to \infty$. To obtain the result for a trivial bundle, let N be the least integer for which P_N is injective and note that $L|_{S_e}$, considered as a subspace of $L^2(X)$, is in the span of $\psi_1, \psi_2, \ldots, \psi_N$. The result for a general bundle follows by a trivialization argument. □

Remark 4.4. Note that the assumptions are redundant. For L to be L^2-closed and in $H^\infty(M)$ implies the an estimate (4.1) for any s.

Corollary 4.5. *For $q > 0$, let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral resolution of the Laplacian in $L^2(M, \Lambda^{p,q})$. With respect to a piecewise smooth section $X \hookrightarrow M$, the spectral projection $P_\delta = \int_0^\delta dE_\lambda$ has a representation*

$$(P_\delta u)(t,x) = \sum_{kl=1}^{N} \int_{G\times X} dsdy\ \psi_k(x) h_{kl}(st^{-1}) \bar{\psi}_l(y) u(s,y), \tag{4.4}$$

where $(\psi_k)_k$ are an orthonormal basis of $L^2(X)$. Furthermore, $H = [h_{kl}]_{kl}$ is a self-adjoint, idempotent convolution operator in $\bigoplus_1^N L^2(G)$ with $h_{kl} \in C^\infty(G)$ and

$$\sum_{kl=1}^{N} \|h_{kl}\|^2_{L^2_R(G)} = \sum_{k=1}^{N} h_{kk}(e) < \infty.$$

Proof. By Corollary 3.3, the theorem applies. Thus the Fourier expansion (1.2) is finite. The remaining assertions follow from Sect. 2. □

Remark 4.6. In the case that G is unimodular, $\sum_{kl} \|h_{kl}\|^2_{L^2_R(G)} < \infty$ is the same as saying that P_δ is in the G-trace class, which we established in [P1] in the setting in which M is strongly pseudoconvex and in [P3] where M satisfies a subelliptic estimate. The new content of Corollary 4.5 is the finiteness of the sum (4.4), *etc.* This transverse dimension gives a meaningful (though much rougher) measure of the spectral subspaces of $\Box$ (and $\Box_b$) than the G-dimension when G is unimodular, but is also defined when the group is not assumed unimodular as in [HHK] and in important examples in [GHS]. Note that [HHK] also deals with the situation in which the G-action is only proper, rather than free as we assume here.

Another point that deserves mentioning here is that our Corollary 4.5 actually generalizes Prop. 2.7 (applied to the Laplacian) in [GHS]. In Theorem 6.6 of [P1] we obtained a formal generalization that does not reduce to the [GHS] statement when G was taken discrete.

5. Applications

We will give a version of the solution of the $\bar{\partial}$-Neumann problem, for our noncompact M. The version valid for M compact, *e.g.* Prop. 3.1.15 of [FK], is unlikely to remain valid in our setting because the Neumann operator on a noncompact space is usually unbounded.

As before, let $\Box = \int_0^\infty \lambda dE_\lambda$ be the spectral decomposition of the Laplacian in $L^2(M, \Lambda^{p,q})$ for $q > 0$ and for $\delta > 0$ put

$$L_\delta = \operatorname{im} \int_\delta^\infty dE_\lambda \quad \text{and} \quad P_\delta = \int_0^\delta dE_\lambda. \tag{5.1}$$

In this section we will show that $\Box u = \alpha$, and the corresponding $\bar{\partial}$-Neumann problem have regular solutions for $\alpha \in L_\delta$.

Lemma 5.1. *If $\alpha \in L_\delta \cap C^\infty(\bar{M})$, then the solution u of $\Box u = \alpha$ is smooth.*

Proof. Let $\alpha \in L_\delta \cap C^\infty(\bar{M})$ and solve $\Box u = \alpha$ in $L^2(M)$. Note that $\|u\|_{L^2(M)} \le (1/\delta)\|\alpha\|_{L^2(M)}$. Adding u to both sides of the equation, $(\Box + 1)u = \alpha + u$, we obtain that $(\Box+1)u = \Box u + u = \alpha + u$. Applying $(\Box+1)^{-1}$, the real estimate, Lemma 3.1 provides that

$$\|\zeta u\|_{s+1} \lesssim \|\zeta_1(\alpha+u)\|_s + \|\alpha+u\|_0 \le \|\zeta_1\alpha\|_s + \|\zeta_1 u\|_s + \|\alpha+u\|_0.$$

Nesting the supports of cutoff functions, concatenating and reducing these estimates for $s = 0, 1, \dots,$ we obtain that for each positive integer s we have

$$\|\zeta u\|_{s+1} \lesssim \|\zeta_1\alpha\|_s + \|\alpha+u\|_0 \le \|\zeta_1\alpha\|_s + (1 + 1/\delta)\,\|\alpha\|_0.$$

Thus $u \in C^\infty(\bar{M})$ by the Sobolev embedding theorem. □

Corollary 5.2. *In L_δ, the Laplacian satisfies the genuine estimate*

$$\|u\|_{s+1} \lesssim \|\Box u\|_s + \|u\|_0, \qquad (u \in L_\delta).$$

Proof. Let $(\alpha_k)_k \subset L_\delta \cap H^\infty$ and $\alpha_k \to \alpha \in H^s(M)$. The previous lemma implies that there exists a sequence $(u_k)_k \subset C^\infty$ solving $\Box u_k = \alpha_k$. Lemma 3.2 implies that $\|u_k\|_{s+1} \lesssim \|\Box u_k\|_s + \|u_k\|_0$ uniformly in k, so $(u_k)_k$ is Cauchy in the H^{s+1} norm. □

Lemma 5.3. *Suppose that $q > 0$, $\alpha \in L^2(M, \Lambda^{p,q}), \bar{\partial}\alpha = 0$, and $\alpha \in L_\delta$. Then there is a unique solution ϕ of $\bar{\partial}\phi = \alpha$ with $\phi \perp \ker(\bar{\partial})$. If $\alpha \in H^s(\bar{M}, \Lambda^{p,q})$, then $\phi \in H^s(\bar{M}, \Lambda^{p,q-1})$ and $\|\phi\|_s \lesssim \|\alpha\|_s$ for each s.*

Proof. Taking $\alpha \in L_\delta$, there is a unique solution to $\Box u = \alpha$ orthogonal to the kernel of $\Box$; in fact $u \in L_\delta \subset (\ker \Box)^\perp$. Since $\bar{\partial}\alpha = 0$, applying $\bar{\partial}$ to

$$\Box u = \bar{\partial}^*\bar{\partial}u + \bar{\partial}\bar{\partial}^* u = \alpha$$

gives that $\bar\partial\bar\partial^*\bar\partial u = 0$. This implies that $\langle\bar\partial\bar\partial^*\bar\partial u, \bar\partial u\rangle = 0$ which is equivalent to $\|\bar\partial^*\bar\partial u\|^2 = 0$. Thus $\bar\partial\bar\partial^* u = \alpha$ and we may take $\phi = \bar\partial^* u \in \operatorname{im}\bar\partial^*$. But $\operatorname{im}\bar\partial^* \subset (\ker\bar\partial)^\perp$. The regularity claim follows immediately from Corollary 5.2 and the order of $\bar\partial^*$. ☐

Putting these results together, we obtain

Corollary 5.4. *Let M be a strongly pseudoconvex complex manifold and $q > 0$. Assume also that M is the total space of a bundle $G \to M \to X$ with G a Lie group acting by holomorphic transformations with compact quotient $X = M/G$. With respect to a piecewise smooth section $X \hookrightarrow M$, define the slices S_t. Then there exists a finite-dimensional subspace $L|_{S_e} \subset L^2(X)$, such that the equation $\Box u = \alpha$ has solutions $u \in L^2(M, \Lambda^{p,q})$ satisfying uniform estimates, on the space of $\alpha \in L^2(M, \Lambda^{p,q})$ satisfying $\alpha|_{S_t} \perp L|_{S_e}$ for all $t \in G$.*

Proof. Choose $\delta > 0$. Corollary 4.5 implies that there exists a finite rank projection $P_{L^2(X)} \in \mathcal{B}(L^2(X))$ such that $P_\delta < \mathbf{1}_{L^2(G)} \otimes P_{L^2(X)}$. Since $L_\delta \supset \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P_{L^2(X)})^\perp = \operatorname{im}(\mathbf{1}_{L^2(G)} \otimes P^\perp_{L^2(X)})$, the Laplacian is regular on forms α whose restrictions to all slices are orthogonal to $\operatorname{im} P_{L^2(X)}$. ☐

Remark 5.5. A similar result holds for the $\bar\partial$-equation by Lemma 5.3.

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