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Boltzmanngasse 9
A-1090 Wien, Austria

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Sławomir Dinew
Sławomir Kołodziej

Vienna, Preprint ESI 2214 (2010) January 12, 2010

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

PLURIPOTENTIAL ESTIMATES ON COMPACT HERMITIAN MANIFOLDS

SŁAWOMIR DINEW, SŁAWOMIR KOŁODZIEJ

Dedicated to Professor Shing-Tung Yau on the occasion of his 60th birthday

Abstract. We discuss pluripotential aspects of the Monge-Ampère equations on compact Hermitian manifolds and prove L^∞ estimates for any metric, as well as the existence of weak solutions under an extra assumption.

1. INTRODUCTION

The complex Monge-Ampère equation is a fundamental tool in complex geometry. It appears in the problem of prescribing the Ricci curvature on compact Kähler manifold in a fixed Kähler class. The solution of Calabi conjecture by Shing-Tung Yau in 1976 ([30]), which says that the complex Monge-Ampère equation can always be solved for any smooth volume form, satisfying the necessary normalization condition, revolutionized complex differential geometry and is still an object of inspiration for future studies. The parabolic version of the (elliptic) Monge-Ampère equation, namely the Kähler-Ricci flow is one of the central themes in modern differential geometry. From the point of view of complex dynamics, in turn, the Monge-Ampère equation produces potentials for singular measures and it is an interesting problem to relate the regularity of such potentials to the regularity properties of the measure itself (see, for example [11]).

There are many modifications and generalizations in the existing literature. In particular, one can study the Monge-Ampère equation with respect to a general Hermitian metric instead of a Kähler one. In such a case the equation is not so geometric, since Hermitian metrics do not represent positive cohomology classes (see however [27] for some geometrical applications). On the other hand any compact complex manifold admits a Hermitian metric which is not the case for Kähler metrics. Thus for example Hopf manifolds are beyond the scope of the Kähler theory. Here we would like to mention the recent preprint of Streets and Tian [26], where a Hermitian version of the Kähler-Ricci flow is being considered, aiming at important geometrical applications. The study of non-Kähler metrics is also motivated by physics as it is, for instance, in Fu and Yau paper [14].

In complex dynamics positive non-closed currents are also studied. One can mention here the concept of a *harmonic current* which is not necessarily d closed but dd^c closed. Laminations associated to such currents are studied for instance in [13].

In the eighties and nineties some results regarding the Monge-Ampère equation in the Hermitian setting were obtained by Cherrier and Hanani ([5], [6], [7], [20], [21]). For the next few years there seems to be no activity on the subject until very recently, when the results were rediscovered and generalized by Guan-Li ([16],[17]) and Tosatti-Weinkove ([27], [28]). The PDE techniques applied in all these papers are similar to the a priori estimates due to S. T. Yau from the Kähler case ([30]). Therefore a natural question

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2000 *Mathematics Subject Classification.* 32U05, 32U40, 53C55.

Key words and phrases. Monge-Ampère operator, Hermitian manifold, pluripotential theory.

(posed in [16], [17] and [27]) arises whether one can obtain L^∞ estimates via suitably constructed pluripotential theory in this setting.

Indeed, there are obstructions for such a theory. First of all in the non-Kähler case there are no local potentials for the metric and hence the methods from [22] as well as [2] do not directly apply. However it is still possible to work globally as in [23], that is independently of the local potentials. In such an approach the main difficulty that arises in transplanting the theory directly to Hermitian setting is the comparison principle which says the following:

Let $\phi,\ \psi \in PSH(X,\omega)\cap L^\infty(X)$*. If* ω *is a Kähler metric then*

$$\int_{\{u<v\}} \omega_v^n \le \int_{\{u<v\}} \omega_u^n\,.$$

The aim of this paper is to develop some basic pluripotential techniques in the Hermitian setting and thereby prove the analogues of the results from [24]. As the theory is in many aspects similar to the Kählerian one, we stress mainly those points where either differences occur and/or arguments need suitable modification.

After some preliminary material we discuss different variants of the comparison principle in the Hermitian setting. In particular we show that the assumptions used in the Guan and Li paper [16] are perfectly suitable for a meaningful pluripotential theory, while the assumption of d-balanced metrics (from [27]) is not well adapted and causes some additional technical difficulties.

In Section 4 we obtain L^∞ a priori estimates for weak solutions to the Monge-Ampère equation with Monge-Ampère measure dominated by capacity, generalizing thus results in [22], [23] from the Kähler setting. In particular solutions, provided that they exist, are bounded for all measures with L^p densities ($p>1$). Another corollary is that some singular measures such as volumes of real hypersurfaces may also admit bounded potentials with respect to any hermitian metric. Since the details of the proof are technically involved below we sketch the main ideas.

Essentially we follow the proof for the Kähler case from [23] (see also [12] for a similar argument). We have to cope with the lack of a comparison principle for a general hermitian metric. The first main idea (Theorem 3.5) is that a comparison principle type estimate is available provided one can control the oscillation of the difference of the functions involved and additionally one has to allow some error terms. These error terms involve Hessian type operators of lower order (such as Laplacians with respect to the metric). Contrary to the smooth case ([27], [16]) one cannot control those terms pointwise. Instead, the crucial observation (Lemma 4.2) is that it is enough to work only with sublevel sets for levels close to infimum and then control the integrals of all such error terms over those sets by the Monge-Ampère mass. Combining those two results with a uniform estimate for the the capacity of sublevel sets (Proposition ! 2.5) one obtains a weak version of the fundamental Lemma 2.2 from [23] which is satisfactory for our needs.

In the next section we assume that a Hermitian metric ω satisfies

$$\forall u \in PSH(X,\omega)\cap L^\infty(X) \quad \int_X (\omega + dd^c u)^n = \int_X \omega^n\,. \tag{1.1}$$

This assumption is equivalent to validity of the usual comparison principle. We then show the existence of continuous solutions of the degenerate Monge-Ampère equation when the right hand side belongs to some Orlicz space, in particular when it is in L^p ($p>1$). This result for smooth data, and without the extra hypothesis, was recently obtained by Tosatti-Weinkove in [28], building on Cherrier's [5] who proved it for balanced metrics.

The last section contains some remarks regarding the uniqueness of solutions. In the smooth case this was very recently established by Tosatti and Weinkove ([28]). We propose alternative elementary proof which is in the spirit of pluripotential theory. We believe that our approach has the slight advantage that it could be more easily adapted to non smooth setting.

The tools developed in this paper make it possible to extend stability and Hölder regularity theorems from [23] [25] to the Hermitian case at least with the above assumption. They can also be applied for manifolds with a boundary considered in [16]. This will be done in subsequent papers.

Notation. As it is customary C will denote different constants that may vary line-to-line. In arguments where a lot of constants appear we shall either enumerate or indexate such C's in order to make a distinction between them. An exception is made for the special constant B (see (2.1) for its definition), since it is linked to the geometry of the manifold and plays crucial role in the proofs.

Acknowledgements. The research was done while the first named author was a Junior Research Fellow at the Erwin Schrödinger Institute, Vienna. He wishes to express his gratitude to this institution for the perfect working conditions as well as for the financial support.

The second author partially supported by EU grant MTKD-CT-2006-042360 and Polish ministerial grants 189/6 PR UE/2007/7, 3679/B/H03/2007/33.

2. PRELIMINARIES

Throughout the note X will be a fixed compact Hermitian manifold equipped with a fixed Hermitian metric ω and $n = dim_{\mathbb{C}}X$. Also $d = \partial + \overline{\partial}$ denotes the standard operator of exterior differentiation while $d^c := i/2\pi(\overline{\partial} - \partial)$.

Let the universal constant $B > 1$ satisfy the following inequalities:

$$-B\,\omega^2 \le n\,dd^c\,\omega \le B\,\omega^2, \quad -B\,\omega^3 \le n^2 d\,\omega \wedge d^c\,\omega \le B\,\omega^3\,. \tag{2.1}$$

We define the function class

$$PSH(X,\omega) := \{u \in L^1(X,\omega) : dd^c u \ge -\omega,\ u \in \mathcal{C}^{\uparrow}(X)\},$$

where $\mathcal{C}^{\uparrow}(X)$ denotes the space of upper semicontinuous functions and the inequality is understood in the weak sense of currents. We call the functions that belong to $PSH(X,\omega)$ ω-plurisubharmonic (ω-psh for short). We shall often use the handy notation $\omega_u := \omega + dd^c u$.

Contrary to the Kähler case the form ω need not have local potentials, nevertheless ω-psh functions are locally standard plurisubharmonic functions plus some smooth function. This follows from the fact that in normal coordinates (see e.g. [26] or [16]) at a given point $\omega = i\sum dz_j \wedge d\bar{z}_j$. We shall refer to that fact in the sequel as to *psh-like property* of ω-psh functions. This property allows to use some local results from pluripotential theory developed by Bedford and Taylor in [4]. In particular the Monge-Ampère operator

$$\omega_u^n := \omega_u \wedge \cdots \wedge \omega_u$$

is well defined for bounded ω-psh functions. Furthermore, if $u_j \in PSH(\omega) \cap L^\infty$ is either decreasing or increasing almost everywhere to u, then

$$(dd^c u_j + \omega)^n \to (dd^c u + \omega)^n$$

in the sense of currents. This follows from the convergence theorems in [4] via the following argument. Suppose ω is a Hermitian form in a ball B, and Ω a Kähler form such that $\omega < \Omega$. Write

$$dd^c u_j + \omega = dd^c u_j + \Omega - T, \quad T = (\Omega - \omega).$$

Then by the Newton expansion

$$(dd^c u_j + \omega)^n = (dd^c u_j + \Omega)^n - n(dd^c u_j + \Omega)^{n-1} \wedge T + ... \pm T^n. \tag{2.2}$$

By the convergence theorem for psh functions [4] all the terms on the right converge as currents, and the sum of their limits is

$$(dd^c u + \Omega)^n - n(dd^c u + \Omega)^{n-1} \wedge T + ... \pm T^n = (dd^c u + \omega)^n.$$

We note that *all* functions u in $PSH(X, \omega)$, normalized by the condition $sup_X u = 0$ are uniformly integrable. This follows from classical results in potential theory and psh-like property as in [22]. Since such results seem to be important in a more general setting (compare [29]) we give here a complete argument following quite closely the one in [18], where the authors treat the Kähler case.

Proposition 2.1. *Let $u \in PSH(X, \omega)$ be a function satisfying $sup_X u = 0$. Then there exists a constant C dependent only on X, ω such that*

$$\int_X |u|\, \omega^n \leq C.$$

Proof. Consider a double covering of X by coordinate balls $B_s^1 \subset\subset B_s^2 \subset X$, $s = 1, \cdots, N$. In each B_s^2 there exists a strictly plurisubharmonic potential ρ_s satisfying the following properties:

$$\begin{cases} \rho_s|_{\partial B_s^2} = 0 \\ inf_{B_s^2} \rho_s \geq -C_1 \\ dd^c\, \rho_s = \omega_{2,s} \geq \omega, \end{cases} \tag{2.3}$$

where C_1 is a constant dependent only on the covering and ω. Suppose now that there exists a sequence $u_j \in PSH(X, \omega), sup_X u_j = 0$ satisfying $lim_{j\to\infty} \int_X |u_j|\, \omega^n = \infty$. After choosing subsequence (which for the sake of brevity we still denote by u_j) we may assume that

$$\int_X |u_j|\, \omega^n \geq 2^j \tag{2.4}$$

and moreover a sequence of points x_j where u_j attains maximum is contained in some fixed ball B_s^1.

Note that $\rho_s + u_j$ is an ordinary plurisubharmonic function in B_s^2 and by the sub mean value property one has

$$\rho_s(x_j) = \rho_s(x_j) + u_j(x_j) \leq C_2 \int_{B_s^2} \rho_s(z) + u_j(z) d\lambda \leq C_2 \int_{B_s^2} u_j(z) d\lambda + C_3, \tag{2.5}$$

where $d\lambda$ is the Lebesgue measure in the local coordinate chart, while C_2, C_3 are constants dependent only on B_s^1 and B_s^2. Thus (2.5) implies that for some constant C_4 one has

$$\int_{B_s^2} |u_j(z)| d\lambda \leq C_4. \tag{2.6}$$

Consider the function $v := \sum_{j=1}^{\infty} \frac{u_j}{2^j}$. By classical potential theory this is again a ω-psh function or constantly $-\infty$. By (2.6), however, the integral of v over B_s^2 is finite, thus it is a true ω-psh function. By the same reasoning we easily obtain that $v \in L^1(B_t^1)$ for any $t \in 1, \cdots, N$ and hence $v \in L^1(X)$. This contradicts (2.4), and thus the existence of an uniform bound is established. $\square$

Next we define a capacity in the Hermitian setting ([23]):

Definition 2.2. *For any Borel set $E \subset X$ we define the capacity cap_ω by the formula*

$$cap_\omega(E) := sup\{ \int_E \omega_u^n \,|u \in PSH(X,\omega),\ 0 \leq u \leq 1\}.$$

Contrary to the Kähler case it is not obvious that this is always a finite quantity. This follows however from the next proposition.

Proposition 2.3. *For any $k \in \{0, \cdots, n\}$ the quantity*

$$sup\{ \int_X {\omega_u}^k \wedge \omega^{n-k} \,|u \in PSH(X,\omega),\ 0 \leq u \leq 1\}$$

is finite.

Proof. For $k = 0$ the statement trivially holds. Assume now $k > 0$ and let us fix a function u which is a competitor for the supremum. Then one has the following

$$\int_X {\omega_u}^k \wedge \omega^{n-k} = \int_X {\omega_u}^{k-1} \wedge \omega^{n-k+1} + \int_X dd^c\, u \wedge {\omega_u}^{k-1} \wedge \omega^{n-k}$$
$$= \int_X {\omega_u}^{k-1} \wedge \omega^{n-k+1} + \int_X u\, dd^c[{\omega_u}^{k-1} \wedge \omega^{n-k}] = \int_X {\omega_u}^{k-1} \wedge \omega^{n-k+1}$$
$$+ \int_X u[(k-1)\, dd^c\, \omega \wedge {\omega_u}^{k-2} \wedge \omega^{n-k} - (k-1)(k-2) d\,\omega \wedge d^c\,\omega \wedge {\omega_u}^{k-3} \wedge \omega^{n-k}$$
$$- (k-1)(n-k) d^c\,\omega \wedge {\omega_u}^{k-2} \wedge d\,\omega \wedge \omega^{n-k-1} + (k-1)(n-k) d\,\omega \wedge {\omega_u}^{k-2} \wedge d^c\,\omega \wedge \omega^{n-k-1}$$
$$- (n-k)(n-k-1)\, {\omega_u}^{k-1} \wedge d\,\omega \wedge d^c\,\omega \wedge \omega^{n-k-2} + (n-k)\, {\omega_u}^{k-1} \wedge dd^c\,\omega \wedge \omega^{n-k-1}].$$

This quantity can be estimated (recall u is uniformly bounded) by

$$C \int_X [{\omega_u}^{k-1} \wedge \omega^{n-k+1} + {\omega_u}^{k-2} \wedge \omega^{n-k+2} + {\omega_u}^{k-3} \wedge \omega^{n-k+3}]$$

(if $k < 3$ the terms with negative power of ω_u do not appear). Now the proof is finished by induction. $\square$

For ω Kähler the capacity is equicontinuous with the Bedford-Taylor capacity denoted in [23] or [24] by ${cap_\omega}'$. The latter can also be defined on non-Kähler manifolds and is equivalent to its non-Kähler counterpart by the decomposition (2.2). Finally, cap_ω and ${cap_\omega}'$ are equicontinuous in the Hermitian case by the same proof as in [23] except that in each strictly pseudoconvex domain V_s one considers two local Kähler forms $\omega_{1,s}$ and $\omega_{2,s}$ satisfying $\omega_{1,s} \leq \omega \leq \omega_{2,s}$ and works with the potentials of those metrics.

Coupling this fact with the argument from [23] (Lemma 4.3) one obtains the following corollary:

Corollary 2.4. *Let $p > 1$ and f be a non negative function belonging to $L^p(\omega^n)$. Then for some absolute constant C dependent only on (X, ω) and any compact $K \subset X$ one has*

$$\int_K f\,\omega^n \leq C(p,X)||f||_p\, cap_\omega(K) exp(-C\, {cap_\omega}^{-1/n}(K)),$$

with $C(p,X)$ a constant dependent on p and (X, ω).

As yet another consequence of psh-like property of $\omega - psh$ functions one gets the capacity estimate of sublevel sets of those functions.

Proposition 2.5. *Let $u \in PSH(X,\omega), sup_X u = 0$. Then there exists an independent constant C such that for any $s > 1$ $cap_\omega(\{ u < -t \}) \leq \frac{C}{t}$.*

Proof. We shall use the double covering introduced in Proposition 2.1. Fix a function $v \in PSH(X,\omega), 0 \leq v \leq 1$. Then we obtain

$$\int_{\{ u<-t \}} \omega_v^n \leq \frac{1}{t}\int_X -u\,\omega_v^n \leq \frac{1}{t}(\sum_{s=1}^{N}\int_{B_s^1} -u(\omega_{2,s} + dd^c\, v)^n)$$
$$\leq \frac{1}{t}(\sum_{s=1}^{N}\int_{B_s^1} -(u+\rho_s)(dd^c(\rho_s+v))^n).$$

Now by the generalized L^1 Chern-Levine-Nirenberg inequalities (see, for example [8], Proposition 3.11) applied to each pair $B_s^1 \subset\subset B_s^2$ one obtains that the last quantity can be estimated by

$$\frac{1}{t}\sum_{s=1}^{N} C_{B_s^1,B_s^2}||u+\rho_s||_{L^1(B_s^2)}||\rho_s+v||_{L^\infty(B_s^2)} \leq \frac{1}{t}max_s\{ C_{B_s^1,B_s^2}\}(C_5 N\int_X -u\,\omega^n + C)(C+1)^n,$$

where constants $C_{B_s^1,B_s^2}$ depend on the covering, while C_5 - only on (X,ω). By Proposition 2.1 this quantity is uniformly bounded and the statement follows. □

We finish this preliminary section with a lemma which shall be used throughout the note. It follows from the proof of the comparison principle by Bedford and Taylor in [3].

Lemma 2.6. *Let u, v be $PSH(X,\omega)$ functions and T a (positive but non necessarily closed) current of the form $\omega_{u_1} \wedge \cdots \wedge \omega_{u_{n-1}}$ for bounded functions u_i belonging to $PSH(X,\omega)$. Then*

$$\int_{\{u<v\}} dd^c(u-v)\wedge T \geq \int_{\{u<v\}} d^c(u-v)\wedge dT.$$

3. COMPARISON PRINCIPLES

In this section we establish the comparison principle in various forms in the non-Kähler case. It has the same form as for Kähler forms (comp. [23]) under an extra assumption (3.1) below. Otherwise we get some extra terms, but this general form is sufficient for later applications.

Note that since for any bounded ω-psh function u we have for a suitable constant C that $u - C < 0 < u + C$ a necessary condition for such an inequality to hold is the following one

$$\forall u \in PSH(X,\omega) \cap L^\infty(X) \quad \int_X (\omega + dd^c\, u)^n = \int_X \omega^n\,. \tag{3.1}$$

Below we show that (3.1) is also a sufficient condition:

Proposition 3.1. *If (3.1) holds, then for any u, $v \in PSH(X,\omega) \cap L^\infty(X)$ we have*

$$\int_{\{u<v\}} \omega_v^n \leq \int_{\{u<v\}} \omega_u^n\,.$$

Proof. It follows from the locality of the Monge-Ampère operator (which is independent of the underlying metric) [3] (see also [19], [12]) that

$$(\omega + dd^c \max(u,v))^n|_{\{ u>v \}} = (\omega + dd^c\, u)^n|_{\{ u>v \}}.$$

Repeating the argument from [19] we obtain for any $\epsilon > 0$

$$\begin{aligned}\int_{\{u-\epsilon<v\}} \omega_v^n &= \int_{\{u-\epsilon<v\}} (\omega + dd^c \max(u,v))^n \\ &= \int_X (\omega + dd^c \max(u,v))^n - \int_{\{u-\epsilon\geq v\}} (\omega + dd^c \max(u,v))^n \\ &\leq \int_X \omega_u^n - \int_{\{u-\epsilon>v\}} \omega_u^n = \int_{\{u-\epsilon\leq v\}} \omega_u^n,\end{aligned}$$

where we have used condition (3.1) and the positivity of the measure in passing from the second line to the last one.

Letting $\epsilon \searrow 0$ and using monotone convergence one obtains the claimed result. ☐

The condition used by Guan and Li in their papers [16] and [17], namely $dd^c\omega = 0$, $d^c\omega \wedge d\omega = 0$, implies (3.1), so comparison principle is true in this setting. In fact it is enough to have only inequalities in this condition as the next proposition shows.

Proposition 3.2. *Assume*

$$dd^c\omega \geq 0, \quad d^c\omega \wedge d\omega \geq 0.$$

Then the comparison principle from the preceding proposition holds.

Proof. Let u, v be smooth ω-psh. Then the general case will follow by the psh-like property and the argument for psh functions (see e.g. [24]). Set $U = \{u < v\}$, and define $T \geq 0$ by

$$\omega_u^n - \omega_v^n = (\omega_u - \omega_v) \wedge T.$$

Fact 3.3. *For some positive currents $T_1, T_2, ...T_{n-1}$ we have*

$$dT = d\omega \wedge T_1, \quad dT_1 = d\omega \wedge T_2, ..., \quad dT_{n-2} = d\omega \wedge T_{n-1}.$$

Proof. It is enough to observe that

$$d[\omega_u^p \wedge \omega_v^q] = pd\omega \wedge \omega_u^{p-1} \wedge \omega_v^q + qd\omega \wedge \omega_u^p \wedge \omega_v^{q-1}.$$

☐

By comparing the forms of the same bidegree one easily sees also the following identity.

Fact 3.4. *$d(u-v) \wedge d^c\omega \wedge S = -d^c(u-v) \wedge d\omega$ for S positive.*

Let us denote by A the boundary term $\int_{\partial U} d^c(u-v) \wedge T \geq 0$ (non negativity of A follows from the proof of the comparison principle in [3]). We thus have, applying the above facts,

$$\begin{aligned}&\int_U \omega_u^n - \omega_v^n = \int_{\partial U} d^c(u-v) \wedge T - \int_U d^c(u-v) \wedge d\omega \wedge T_1 \\ &= A + \int_U d^c(u-v) \wedge d\omega \wedge T_1 = A + \int_U d[(u-v) \wedge d^c\omega \wedge T_1] \\ &- \int_U (u-v) dd^c\omega \wedge T_1 - \int_U (u-v) d^c\omega \wedge dT_1\end{aligned}$$

(by Stokes and $dd^c\omega \geq 0$)

$$\geq A - \int_U (u-v) d^c\omega \wedge d\omega \wedge T_2.$$

The last integral is nonpositive if $d^c\omega \wedge d\omega \wedge T_2 \geq 0$ which follows from our hypothesis. ☐

The *balanced* metrics, studied extensively (see [27] and references therein for more details), are defined by $d(\omega^{n-1})=0$. They need not satisfy the condition (3.1), except in the case $n=2$. Instead, the reasoning from the first proposition of this section gives us the following comparison principle for the Laplacian in this case:

Let ω be a balanced metric and let ϕ, $\psi\in PSH(X,\omega)\cap L^\infty(X)$. Then

$$\int_{\{u<v\}}\omega_v\wedge\omega^{n-1}\leq\int_{\{u<v\}}\omega_u\wedge\omega^{n-1}.$$

Now we present a weaker form of comparison principle with "error terms" which will be useful in obtaining a priori estimates:

Theorem 3.5. *Let ω be a Hermitian metric on a complex compact manifold X and let $u,v\in PSH(X,\omega)\cap L^\infty(X)$. Then there exists a polynomial P_n of degree n, with zeroth coefficient equal to 0, depending only on the dimension, such that*

$$\int_{\{u<v\}}\omega_v^n\leq\int_{\{u<v\}}\omega_u^n+P_n(BM)\sum_{k=0}^{n}\int_{\{u<v\}}{\omega_u}^k\wedge\omega^{n-k},$$

where B is defined by (2.1)) and $M=sup_{\{u<v\}}(v-u)$.

Proof. Note that

$$\begin{aligned}\int_{\{u<v\}}\omega_v^n&=\int_{\{u<v\}}\omega\wedge{\omega_v}^{n-1}+\int_{\{u<v\}}dd^c\,v\wedge{\omega_v}^{n-1}\\&\leq\int_{\{u<v\}}\omega\wedge{\omega_v}^{n-1}+\int_{\{u<v\}}dd^c\,u\wedge{\omega_v}^{n-1}+\int_{\{u<v\}}d^c(v-u)\wedge d({\omega_v}^{n-1}),\end{aligned}$$

where we have used Lemma 2.6. Again by (2.1) we have

$$dd^c({\omega_v}^{n-1})\leq B[\omega^2\wedge{\omega_v}^{n-2}+\omega^3\wedge{\omega_v}^{n-3}].$$

Thus by the Stokes theorem

$$\begin{aligned}\int_{\{u<v\}}\omega_v^n&\leq\int_{\{u<v\}}\omega_u\wedge{\omega_v}^{n-1}-\int_{\{u<v\}}d(v-u)\wedge d^c({\omega_v}^{n-1})\\&\leq\int_{\{u<v\}}\omega_u\wedge{\omega_v}^{n-1}+\int_{\{u<v\}}(v-u)\wedge dd^c({\omega_v}^{n-1})\\&\leq\int_{\{u<v\}}\omega_u\wedge{\omega_v}^{n-1}+sup_{\{u<v\}}(v-u)B\int_{\{u<v\}}(\omega^2\wedge{\omega_v}^{n-2}+\omega^3\wedge{\omega_v}^{n-3}).\end{aligned}$$

Repeating the above procedure of replacing ω_v by ω and ω_u in the end one obtains the statement. □

In complex dimension 2 the next proposition yields better inequality. Recall first a classical notion in Hermitian geometry, the Gauduchon metric (see [15]). Let ϕ be the unique function such that $inf_X\phi=0$ and $e^\phi\omega$ is a Gauduchon metric i.e.

$$dd^c(e^{(n-1)\phi}\omega^{n-1})=0. \tag{3.2}$$

Proposition 3.6. *Let ω be a Hermitian metric on a complex compact manifold X of dimension 2 and let u, $v\in PSH(X,\omega)\cap L^\infty(X)$. Let also ϕ be defined by the Gauduchon condition with respect to ω. Then the following comparison principle holds*

$$\int_{\{u<v\}}2(e^\phi-1)\,\omega_v\wedge\omega+\int_{\{u<v\}}{\omega_v}^2\leq\int_{\{u<v\}}2(e^\phi-1)\,\omega_u\wedge\omega+\int_{\{u<v\}}{\omega_u}^2.$$

Proof. Note that

$$\int_{\{u<v\}} (e^{\phi} - 1)\, \omega_v \wedge \omega + \int_{\{u<v\}} {\omega_v}^2 = \int_{\{u<v\}} (e^{\phi}\, \omega + dd^c\, v) \wedge \omega_v$$
$$\leq \int_{\{u<v\}} (e^{\phi}\, \omega + dd^c\, v) \wedge \omega_u$$

(because $e^{\phi}\, \omega + dd^c\, \psi$ is also dd^c- closed).

The last term equals

$$\int_{\{u<v\}} (e^{\phi} - 1)\, \omega \wedge \omega_u + \int_{\{u<v\}} \omega_v \wedge (e^{\phi}\, \omega + dd^c\, u) - \int_{\{u<v\}} (e^{\phi} - 1)\, \omega \wedge \omega_v$$
$$\leq \int_{\{u<v\}} (e^{\phi} - 1)\, \omega \wedge \omega_u + \int_{\{u<v\}} \omega_u \wedge (e^{\phi}\, \omega + dd^c\, u) - \int_{\{u<v\}} (e^{\phi} - 1)\, \omega \wedge \omega_v$$
$$= \int_{\{u<v\}} 2(e^{\phi} - 1)\, \omega \wedge \omega_u + \int_{\{u<v\}} {\omega_u}^2 - \int_{\{u<v\}} (e^{\phi} - 1)\, \omega \wedge \omega_v\, .$$

□

We end this section proving the comparison principle for the Laplace operator.

Proposition 3.7. *Let ω be a Hermitian metric and let $u,\ v \in PSH(X, \omega) \cap L^{\infty}(X)$. Then there exists a constant $C \geq 1$ dependent only on ω such that*

$$\int_{\{u<v\}} \omega_v \wedge \omega^{n-1} \leq C \int_{\{u<v\}} \omega_u \wedge \omega^{n-1}\, .$$

Proof. Note that $\int_{\{u<v\}} \omega_v \wedge \omega^{n-1} \leq \int_{\{u<v\}} \omega_v \wedge e^{(n-1)\phi}\, \omega^{n-1}$, with v defined by (3.2). Since $\int_X \omega_v \wedge e^{(n-1)\phi}\, \omega^{n-1}$ is independent of v (due to the Gauduchon metric assumption) we can repeat the proof of the first proposition in this section to conclude that

$$\int_{\{u<v\}} \omega_v \wedge \omega^{n-1} \leq \int_{\{u<v\}} \omega_u \wedge e^{(n-1)\phi}\, \omega^{n-1} \leq e^{(n-1) sup_X \phi} \int_{\{u<v\}} \omega_u \wedge \omega^{n-1}\, .$$

□

While the constant C is unsatisfactory for many applications, note that the argument for obtaining L^{∞} estimates for equations with L^p- right hand side from [23] is virtually unaffected by that. So, as a by-product of the above comparison principle we obtain another proof of the L^{∞} estimates for the Herimitan Laplacian equation which has the advantage that is independent of any Green type argument and hence independent from cumbersome curvature estimates (strictly speaking, those are hidden in the Gauduchon function).

4. L^{∞} ESTIMATES

In this section we shall prove uniform a priori estimates for the Monge-Ampère equations with L^p right hand side ($p > 1$) essentially repeating the proof from [23]. Note that whenever the comparison principle holds we have all the needed ingredients and the proof is the same as in [23]. The crux of the matter is however that one can prove those a priori estimates using only the weaker form of the comparison principle (Theorem 3.5).

We begin with an auxiliary proposition:

Proposition 4.1. *Let* $A > 1, t$ *and* ϵ *be positive constants satisfying* $At + \epsilon < \frac{1}{2B}$, *with* B *defined by (2.1). Let* $u, v < 0$ *be bounded* ω*-psh functions such that* $1 - A < v < 0$, $sup_X(u - v) = 0$, *and* $inf_X(u - v) = -S$. *Then*

$$t^n\ cap_\omega(\{\,u < v - S + \epsilon\,\}) \leq C_n \sum_{k=0}^{n} \int_{\{\,u<(1-t)v-S+\epsilon+t\,\}} {\omega_u}^k \wedge \omega^{n-k},$$

with a constant C_n *dependent only on* n.

Proof. Choose any function $w \in PSH(X, \omega)$, $0 \leq w \leq 1$. Note that we have the set inclusions

$$\{\,u < v - S + \epsilon\,\} \subset \{\,u < (1-t)v - S + \epsilon + tw\,\} =: U \subset \{\,u < (1-t)v - S + \epsilon + t\,\}.$$

Then

$$t^n \int_{\{\,u<v-S+\epsilon\,\}} \omega_w^n \leq \int_{\{\,u<v-S+\epsilon\,\}} \omega_{tw}^n \leq \int_{\{\,u<(1-t)v-S+\epsilon+tw\,\}} \omega_{tw}^n,$$

where we have used that $t \leq \frac{1}{2} \leq 1$. Since by our assumptions $M = sup_U((1-t)v + t - S + \epsilon - u) < \frac{1}{2B}$ we conclude from Theorem 3.5 the latter quantity can be estimated by

$$\int_U \omega_u^n + \frac{C_n}{2} \sum_{k=0}^{n} \int_U {\omega_u}^k \wedge \omega^{n-k} \leq C_n \sum_{k=0}^{n} \int_{\{\,u<(1-t)v-S+\epsilon+t\,\}} {\omega_u}^k \wedge \omega^{n-k},$$

and the result follows. □

Below we state a crucial lemma which allows to control the mixed Monge-Ampère measures appearing in the above proposition if $v = const$.

Lemma 4.2. *Let* $u, t,\ \epsilon$ *be as above (take now* $A = 1$*). Then for every* $k \in 1, \cdots, n-1$ *we have the estimate*

$$\int_{\{\,u<-S+\epsilon+t\,\}} {\omega_u}^k \wedge \omega^{n-k} \leq C \int_{\{\,u<-S+\epsilon+t\,\}} (\omega_u^n + \omega^n)$$

for some C *dependent only on* n.

Proof. Set $a_k := \int_{\{\,u<-S+\epsilon+t\,\}} {\omega_u}^k \wedge \omega^{n-k}$. An application of Lemma 2.6 yields

$$\begin{aligned} a_k &\leq a_{k+1} - \int_{\{\,u<-S+\epsilon+t\,\}} d^c(u + S - \epsilon - t) \wedge d({\omega_u}^k \wedge \omega^{n-k-1}) \\ &\leq a_{k+1} + \int_{\{\,u<-S+\epsilon+t\,\}} d(u + S - \epsilon - t) \wedge d^c({\omega_u}^k \wedge \omega^{n-k-1}) \\ &= a_{k+1} + \int_{\{\,u<-S+\epsilon+t\,\}} (\epsilon + t - S - u) \wedge dd^c({\omega_u}^k \wedge \omega^{n-k-1}). \end{aligned}$$

The last term is bounded by $(\epsilon + t)B(a_k + a_{k-1} + a_{k-2}) \leq \frac{1}{2}(a_k + a_{k-1} + a_{k-2})$. Thus we obtain

$$a_k \leq 2a_{k+1} + a_{k-1} + a_{k-2}. \tag{4.1}$$

To finish the proof just observe that any sequence $\{\,a_j\,\}_{j=0}^n$ of non negative numbers satisfying (4.1) also satisfies

$$a_j \leq C(a_n + a_0)$$

with some C dependent only on n. □

Now coupling Theorem 3.5 with Lemma 4.2 and following the lines of Lemma 2.2 in [23] we obtain the following result:

Theorem 4.3. *Let u be an ω-psh function with $sup_X u = 0$, while $inf_X u = -S$. Suppose for some positive number α and an increasing function $h : \mathbb{R}_+ \to (1, \infty)$ satisfying*

$$\int_1^\infty (yh^{1/n}(y))^{-1} dy < \infty$$

the following inequality holds

$$\int_K \omega_u^n \leq F(cap_\omega(K)), \quad \text{with } F(t) = \frac{\alpha t}{h(t^{-1/n})}, \ \alpha > 0, \tag{4.2}$$

for any compact set K Then for $D < \frac{1}{2B}$ we have

$$D \leq \kappa(cap_\omega(\{ u < -S + D \})),$$

where

$$\kappa(s) := c(n)\alpha^{1/n}(1 + C)[\int_{s^{-1/n}}^\infty y^{-1}h^{-1/n}(y)dy + h^{-1/n}(s^{-1/n})].$$

As in the Kähler case Theorem 4.3 together with universal capacity estimates for sublevel sets (Proposition 2.5) yield a priori estimates for $u \in PSH(X, \omega)$ having Monge-Ampère measure dominated by capacity.

5. Weak solutions

Here we prove the existence of continuous solutions of the complex Monge-Ampère equation for nonnegative right hand side which belongs to some Orlicz spaces (including L^p, $p > 1$) working under the assumption (3.1).

Lemma 5.1. *Assume that ω satisfies (3.1) and that $u_j \in PSH(X, \omega) \cap C(\bar{X})$ is a uniformly bounded sequence converging weakly to $u \in PSH(\omega)$. Suppose further that for h as in Theorem 4.3*

$${\omega_{u_j}}^n = f_j \ \omega^n,$$

with f_j satisfying

$$\int_K f_j \, \omega^n \leq F(cap_\omega(K)), \quad \text{with } F(t) = \frac{\alpha t}{h(t^{-1/n})}, \ \alpha > 0, \tag{5.1}$$

and such that all f_j belong to the Orlicz space $L^\psi(X)$, where $\frac{\psi(x)}{x}$ is increasing to ∞ as x goes to ∞. Then $u_j \to u$ uniformly in X.

Proof. Assume $0 < u < A - 1$. Choose $\delta > 0$ and $\eta = \min(\delta, 1/(4A))$ so that the set $E_j(\delta + 2A\,\eta) = \{u_j + \delta + 2A\,\eta \leq u\}$ is nonempty, whereas the set $E_j(\delta + 3A\,\eta) = \{u_j + \delta + 3A\,\eta \leq u\}$ is empty. Denote by $a_j(\delta + A\,\eta) = cap_\omega(E_j(\delta + A\,\eta))$ the capacity of the set $E_j(\delta + A\,\eta)$. Let v_j be a $PSH(X, \omega)$ function satisfying $-1 < v_j < 0$ and

$$\int_{E_j(\delta + A\,\eta)} (dd^c v_j + \omega)^n \geq a_j(\delta + A\,\eta)/2.$$

Observe that for $V = \{u_j \leq \eta\, v_j + (1 - \eta)u - \delta\}$ the following inclusions are true

$$E_j(\delta + A\,\eta) \subset V \subset E_j(\delta).$$

Applying the comparison principle we get

$$a_j(\delta + A\,\eta)\,\eta^n \leq \int_{E_j(\delta + A\,\eta)} (dd^c(\eta\, v_j + (1 - \eta)u) + \omega)^n \leq \int_V {\omega_{u_j}}^n$$
$$\leq \int_{E_j(\delta)} f_j \, \omega^n \, .$$

Therefore, using the notation $u_+ := \max(u,0)$ we have for any $M > 0$

$$\begin{aligned} a_j(\delta + A\,\eta)\,\eta^n\,\delta &\le \int_X (u-u_j)_+ f_j\,\omega^n \\ &= \int_{\{f_j>M\}} (u-u_j)_+ f_j\,\omega^n + \int_{\{f_j\le M\}} (u-u_j)_+ f_j\,\omega^n \\ &\le \max_X (u-u_j)_+ \int_{\{f_j>M\}} f_j\,\omega^n + M\int_X (u-u_j)_+\,\omega^n \\ &\le \max_X (u-u_j)_+ \frac{M}{\psi(M)} \int_X \psi f_j\,\omega^n + M\int_\Omega (u-u_j)_+\,\omega^n\,. \end{aligned}$$

For the last inequality we use the assumption that $\frac{\psi(x)}{x}$ is increasing. Fix $\epsilon > 0$. By the assumptions and the L^∞ estimates above the quantities

$$\max_X (u-u_j)_+ \int_X \psi f_j\,\omega^n$$

are uniformly bounded. Using the assumptions on ψ we can make $\frac{M}{\psi(M)}$ arbitrarily small by taking M big enough. We choose M so that the first term on the right hand side of the last estimate is less than $\epsilon/2$ for any j. Since $u_j \to u$ in $L^1(\Omega)$, by the psh-like property, the other term is less than $\epsilon/2$ for $j > j_0$. Therefore

$$a_j(\delta + A\,\eta) \le \epsilon\,\eta^{-n}\,\delta^{-1} \quad \text{for} \quad j > j_0.$$

Since $E_j(\delta + 2A\,\eta)$ is nonempty, applying Theorem 4.3 we obtain

$$\eta \le \kappa(a_j(\delta + A\,\eta)) \le \kappa(\epsilon\,\eta^{-n}\,\delta^{-1}), \quad j > j_0.$$

By the assumption on h we have $\lim_{s\to 0}\kappa(s) = 0$, so the last inequality yields a contradiction if we take ϵ small enough. Therefore, using the psh-property and the Hartogs lemma we conclude that u_j tend to u uniformly. □

Theorem 5.2. *With ω satisfying (3.1) take f such that*

$$\int_X f\,\omega^n = \int_X \omega^n, \quad \int_K f\,\omega^n \le F(cap_\omega(K)), \quad \text{with } F(t) = \frac{\alpha t}{h(t^{-1/n})}, \ \alpha > 0, \tag{5.2}$$

for h as in Theorem 4.3, and such that f belongs to the Orlicz space $L^\psi(X)$, where $\frac{\psi(x)}{x}$ is increasing to ∞ as x goes to ∞. Then there exists continuous function $u \in PSH(X,\omega)$ that fulfils the equation

$$\omega_u{}^n = f\ \omega^n\,.$$

Proof. It follows from [28] [17] that the theorem is true for smooth positive f. Thus we approximate f by smooth positive functions f_j in L^ψ, and obtain solutions u_j. A priori estimates from Section 4 yield uniform bounds for u_j. Passing to a subsequence we can therefore assume that $u_j \to u$ in $L^1(X)$. The last lemma says that the convergence of functions is uniform. The statement now follows by monotone convergence theorem (see Preliminaries). □

6. A REMARK ON THE UNIQUENESS

In [28] the authors proved that if u, v are smooth ω-psh functions and their Monge-Ampère measures satisfy $\omega_u^n = e^{F+b}\,\omega^n$, $\omega_v^n = e^{F+c}\,\omega^n$ for some smooth function F and some constants b and c then in fact $b = c$ and u and v differ by a constant. This corresponds to the uniqueness of potentials in the Calabi conjecture from the Kähler case.

Below we give an alternative proof of the fact that $b = c$ which is in the spirit of pluripotential theory. Suppose, to the contrary, that

$$\omega_u^n = e^{F+b}\,\omega^n,\ \omega_v^n = e^{F+c}\,\omega^n$$

for some u, v; and without loss of generality assume that $c > b$.

Consider the Hermitian metric $\omega + dd^c\,u$. Since by the assumptions above it is smooth and strictly positive one finds a unique Gauduchon function ϕ_u, such that

$$inf_X\phi_u = 0,\ dd^c(e^{(n-1)\phi_u}(\omega + dd^c\,u)^{n-1}) = 0.$$

Then one can apply the comparison principle for the Laplacian with respect to $e^{\phi_u}(\omega + dd^c\,u)$ which yields

$$\int_{\{u<v\}} e^{(n-1)\phi_u}(\omega + dd^c\,u)^{n-1}\wedge\omega_v \leq \int_{\{u<v\}} e^{(n-1)\phi_u}\,\omega_u^n\,.$$

Exchanging now v with $v + C$ (which does not affect the reasoning above) for big enough C one obtains

$$\int_X e^{(n-1)\phi_u}(\omega + dd^c\,u)^{n-1}\wedge\omega_v \leq \int_X e^{(n-1)\phi_u}\,\omega_u^n\,. \tag{6.1}$$

However the left hand side can be estimated from below using (pointwise) the inequality for mixed Monge-Ampère measures (see [9] for a discussion on the topic):

$$\int_X e^{(n-1)\phi_u}(\omega + dd^c\,u)^{n-1}\wedge\omega_v \geq \int_X e^{(n-1)\phi_u+\frac{(c-b)}{n}}\,\omega_u^n\,.$$

Coupling the above estimates one obtains

$$0 < e^{\frac{(c-b)}{n}} - 1 \leq 0,$$

a contradiction.

Applying the Calabi argument directly, as in [27] one obtains that $v - u$ would be a subharmonic function with respect to some elliptic operator (dependent on v, u). Thus it might be possible to conclude the result directly in this manner. Note however that Calabi's argument depends heavily on the smoothness of u and v and even in the Kähler case extending uniqueness to non smooth solutions is a highly nontrivial matter (we refer to [10] for a discussion and to [1] and [10] for the proofs of uniqueness of bounded and general functions respectively). In our approach the smoothness is needed only to produce Gauduchon function, which perhaps will be easier to generalize in the non-smooth setting.

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Sławomir Dinew: Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland `slawomir.dinew@im.uj.edu.pl`
Current adress: Erwin Schrödinger Institute for mathematical physics, Boltzmanngasse 8, A2010 Vienna, Austria

Sławomir Kołodziej: Institute of Mathematics, Jagiellonian University, ul. Łojasiewicza 6, 30-348 Kraków, Poland `slawomir.kolodziej@im.uj.edu.pl`