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Boltzmanngasse 9
A-1090 Wien, Austria

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Bernard Helffer
Yuri A. Kordyukov

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SEMICLASSICAL SPECTRAL ASYMPTOTICS FOR A TWO-DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATOR: THE CASE OF DISCRETE WELLS

BERNARD HELFFER AND YURI A. KORDYUKOV

Dedicated to Misha Shubin on the occasion of his 65th birthday

ABSTRACT. We consider a magnetic Schrödinger operator H^h, depending on the semiclassical parameter $h > 0$, on a two-dimensional Riemannian manifold. We assume that there is no electric field. We suppose that the minimal value b_0 of the magnetic field b is strictly positive, and there exists a unique minimum point of b, which is non-degenerate. The main result of the paper is a complete asymptotic expansion for the low-lying eigenvalues of the operator H^h in the semiclassical limit. We also apply these results to prove the existence of an arbitrary large number of spectral gaps in the semiclassical limit in the corresponding periodic setting.

1. Preliminaries and main results

Let M be a compact oriented manifold of dimension $n \geq 2$ (possibly with boundary). Let g be a Riemannian metric and $\mathbf{B}$ a real-valued closed 2-form on M. Assume that $\mathbf{B}$ is exact and choose a real-valued 1-form $\mathbf{A}$ on M such that $d\mathbf{A} = \mathbf{B}$. Thus, one has a natural mapping

$$u \mapsto ih\, du + \mathbf{A}u$$

from $C_c^\infty(M)$ to the space $\Omega_c^1(M)$ of smooth, compactly supported one-forms on M. The Riemannian metric allows to define scalar products in these spaces and consider the adjoint operator

$$(ih\, d + \mathbf{A})^* : \Omega_c^1(M) \to C_c^\infty(M).$$

A Schrödinger operator with magnetic potential $\mathbf{A}$ is defined by the formula

$$H^h = (ih\, d + \mathbf{A})^*(ih\, d + \mathbf{A}).$$

Here $h > 0$ is a semiclassical parameter. If M has non-empty boundary, we will assume that the operator H^h satisfies the Dirichlet boundary conditions.

We are interested in semiclassical asymptotics of the low-lying eigenvalues of the operator H^h. This problem was studied in [3, 8, 10, 11, 12, 13, 14, 23, 25, 26, 27, 29, 30, 31] (see [4, 9] for surveys).

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In this paper, we study the problem in a particular situation. We come back to the case considered in [11]. We suppose that M is two-dimensional. Then we can write $\mathbf{B} = b dx_g$, where $b \in C^\infty(M)$ and dx_g is the Riemannian volume form. Let

$$b_0 = \min_{x\in M} b(x).$$

We furthermore assume that:

(1) $b_0 > 0$;
(2) there exist a unique point x_0, which belongs to the interior of M, $k \in \mathbb{N}$ and $C > 0$ such that for all x in some neighborhood of x_0 the estimates hold:
$$C^{-1}\, d(x, x_0)^2 \leq b(x) - b_0 \leq C\, d(x, x_0)^2\,.$$

Denote

$$a = \mathrm{Tr}\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right)^{1/2}, \quad d = \det\left(\frac{1}{2}\mathrm{Hess}\, b(x_0)\right).$$

Denote by $\lambda_0(H^h) \leq \lambda_1(H^h) \leq \lambda_2(H^h) \leq \dots$ the eigenvalues of the operator H^h in $L^2(M)$.

Theorem 1.1. *Under current assumptions, for any natural j, there exist $C_j > 0$ and $h_j > 0$ such that, for any $h \in (0, h_j]$,*

$$hb_0 + h^2\left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] - C_j h^{19/8} \leq \lambda_j(H^h) \leq hb_0 + h^2\left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] + C_j h^{5/2}.$$

In particular, we have lower and upper bounds for the groundstate energy $\lambda_0(H^h)$:

$$hb_0 + h^2\frac{a^2}{2b_0} - C_0 h^{19/8} \leq \lambda_0(H^h) \leq hb_0 + h^2\frac{a^2}{2b_0} + C_0 h^{5/2}, \quad h \in (0, h_0].$$

and the asymptotics of the splitting between the groundstate energy and the first excited state :

$$\lambda_1(H^h) - \lambda_0(H^h) \sim h^2\frac{2d^{\frac{1}{2}}}{b_0}\,.$$

The previous statement can be completed in the following way.

Theorem 1.2. *Under current assumptions, for any natural j, there exists a sequence $(\alpha_{j,\ell})_{\ell\in\mathbb{N}}$, and for any N, there exist $C_{j,N} > 0$ and $h_{j,N} > 0$ such that, for any $h \in (0, h_{j,N}]$,*

$$|\lambda_j(H^h) - h\sum_{\ell=0}^{N}\alpha_{j,\ell}h^{\frac{\ell}{2}}| \leq C_{j,N}h^{\frac{N+3}{2}}\,, \tag{1.1}$$

with $\alpha_{j,0} = b_0$, $\alpha_{j,1} = 0$, $\alpha_{j,2} = \frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}$.

This theorem improves the result of [11] which only gives a two-terms asymptotics for the ground state energy in the flat case.

The scheme of the proof is to first prove the weak version, given by Theorem 1.1, with $N = 2$, permitting to determine j_0 disjoint intervals in which the first j_0 eigenvalues are localized, for h small enough, and then to determine a complete expansion of each eigenvalue lying in a given interval.

The paper is organized as follows. In Section 2 we construct approximate eigenfunctions of the operator H^h with any order of precision. This allows us to prove accurate upper bounds for the jth eigenvalue of H^h. In Section 3 we prove a lower bound for the jth eigenvalue of H^h and complete the proofs of Theorems 1.1 and 1.2. In Section 4 we consider the case when the magnetic field is periodic. We combine the construction of approximate eigenfunctions given in Section 2 with the results of [6] to prove the existence of arbitrary large number of gaps in the spectrum of the periodic operator H^h in the semiclassical limit.

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2. Upper bounds

2.1. **Approximate eigenfunctions: main result.** The purpose of this section is to prove the following accurate upper bound for the eigenvalues of the operator H^h.

Theorem 2.1. *Under current assumptions, for any natural j and k, there exists a sequence $(\mu_{j,k,\ell})_{\ell\in\mathbb{N}}$ with*

$$\mu_{j,k,0} = (2k+1)b_0, \quad \mu_{j,k,1} = 0,$$

and

$$\mu_{j,k,2} = (2j+1)(2k+1)\frac{d^{1/2}}{b_0} + (2k^2+2k+1)\frac{t}{2b_0} + \frac{1}{2}(k^2+k)R(x_0)\,,$$

where R is the scalar curvature, and

$$t = \operatorname{Tr}\left(\frac{1}{2}\operatorname{Hess} b(x_0)\right),$$

and for any N, there exist $\phi^h_{jkN} \in C^\infty(M)$, $C_{jk,N} > 0$ and $h_{jk,N} > 0$ such that

$$(\phi^h_{j_1k_1N}, \phi^h_{j_2k_2N}) = \delta_{j_1j_2}\delta_{k_1k_2} + \mathcal{O}_{j_1,j_2,k_1,k_2}(h) \tag{2.1}$$

and, for any $h \in (0, h_{jk,N}]$,

$$\|H^h\phi^h_{jkN} - \mu^h_{jkN}\phi^h_{jkN}\| \le C_{jkN}h^{\frac{N+3}{2}}\|\phi^h_{jkN}\|, \tag{2.2}$$

where

$$\mu^h_{jkN} = h\sum_{\ell=0}^{N}\mu_{j,k,\ell}h^{\frac{\ell}{2}}.$$

Since the operator H^h is self-adjoint, using Spectral Theorem, we immediately deduce the existence of eigenvalues near the points μ^h_{jk}.

Corollary 2.2. *For any natural j, k and N, there exist $C_{jk,N} > 0$ and $h_{jk,N} > 0$ such that, for any $h \in (0, h_{jk,N})$,*

$$\operatorname{dist}(\mu^h_{jkN}, \operatorname{Spec}(H^h)) \le C_{jk,N}h^{\frac{N+3}{2}}.$$

Remark 2.3. The low-lying eigenvalues of the operator H^h, as $h \to 0$, are obtained by taking $k = 0$ in Theorem 2.1. Therefore, as an immediate consequence of Theorem 2.1, we deduce that, for any natural j and N, there exists $h_{j,N} > 0$ such that, for any $h \in (0, h_{j,N}]$, we have

$$\lambda_j(H^h) \le \mu^h_{j0N} + C_{j0,N}h^{\frac{N+3}{2}}.$$

In particular, this implies the upper bound in Theorem 1.1.

Remark 2.4. Our interest in the case of arbitrary k in Theorem 2.1 is motivated by its importance for proving the existence of gaps in the spectrum of the operator H^h in the semiclassical limit. This will be discussed in Section 4.

Proof of Theorem 2.1. The proof is long, so we will split it in different steps in the next subsections. □

2.2. **Expanding operators in fractional powers of h.** The approximate eigenfunctions $\phi^h_{jk} \in C^\infty(M)$, which we are going to construct, will be supported in a small neighborhood of x_0. So, in a neighborhood of x_0, we will consider some special local coordinate system with coordinates (x, y) such that x_0 corresponds to $(0, 0)$. We will only apply our operator on functions which are a product of cut-off functions with functions of the form of linear combinations of terms like $h^\nu w(h^{-1/2}x, h^{-1/2}y)$ with w in $\mathcal{S}(\mathbb{R}^2)$. These functions are consequently $O(h^\infty)$ outside a fixed neighborhood of $(0, 0)$. We will start by doing the computations formally in the sense that everything is determined modulo $O(h^\infty)$, and any smooth function will be replaced by its Taylor's expansion. It is then easy to construct non formal approximate eigenfunctions.

First, we recall that in local coordinates $X = (X^1, X^2) = (x, y)$ on M the 1-form $\mathbf{A}$ is written as

$$\mathbf{A} = A_1(X)\, dX_1 + A_2(X)\, dX_2,$$

the matrix of the Riemannian metric g as

$$g(X) = (g_{j\ell}(X))_{1\leq j,\ell\leq 2}$$

and its inverse as

$$g(X)^{-1} = (g^{j\ell}(X))_{1\leq j,\ell\leq 2}.$$

Denote $|g(X)| = \det(g(X))$. Then the magnetic field $\mathbf{B}$ is given by

$$\mathbf{B} = B\, dx \wedge dy, \quad B = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y},$$

and

$$B = b\sqrt{|g|}.$$

Finally, the operator H^h has the form

$$H^h = \frac{1}{\sqrt{|g(X)|}} \sum_{1\leq \alpha,\beta\leq 2} \nabla^h_\alpha \left(\sqrt{|g(X)|}\, g^{\alpha\beta}(X) \nabla^h_\beta\right),$$

where

$$\nabla^h_\alpha = ih\frac{\partial}{\partial X^\alpha} + A_\alpha(X), \quad \alpha = 1, 2,$$

or, equivalently,

$$H^h = \sum_{1\leq \alpha,\beta\leq 2} g^{\alpha\beta}(X)\nabla^h_\alpha \nabla^h_\beta + ih \sum_{1\leq\alpha\leq 2} \Gamma^\alpha \nabla^h_\alpha,$$

where, for $\alpha = 1, 2$,

$$\Gamma^\alpha = \frac{1}{\sqrt{|g(X)|}} \sum_{1\leq\beta\leq 2} \frac{\partial}{\partial X^\beta}\left(\sqrt{|g(X)|}\, g^{\beta\alpha}(X)\right). \tag{2.3}$$

We will consider normal local coordinates near x_0 such that x_0 corresponds to $(0, 0)$ and, in a neighborhood of x_0,

$$b(X) = b_0 + \alpha_1 x^2 + \beta_1 y^2 + O(|X|^3).$$

Thus, we have

$$a = (\alpha_1)^{1/2} + (\beta_1)^{1/2}, \quad d = \alpha_1\beta_1, \quad t = \alpha_1 + \beta_1.$$

By well-known properties of normal coordinates we have

$$g_{11}(X) = 1 + O(|X|^2), \quad g_{12}(X) = O(|X|^2), \quad g_{22}(X) = 1 + O(|X|^2).$$

Moreover (see, for instance, [1, Proposition 1.28]), we have

$$g_{ij}(X) = \delta_{ij} - \frac{1}{3}\sum_{kl} R_{ikjl}(x_0)X^k X^l + O(|X|^3),$$

where R_{ijkl} is the Riemann curvature tensor. Therefore, Taylor's expansion of $g^{\alpha\beta}$ has the form

$$g^{\alpha\beta}(X) = \delta^{\alpha\beta} + \sum_{k=2}^{\infty} g^{\alpha\beta}_{(k)}(X), \tag{2.4}$$

where

$$g^{\alpha\beta}_{(2)}(X) = \frac{1}{3}\sum_{kl} R_{\alpha k\beta l}(x_0)X^k X^l.$$

In the two-dimensional case, due to its symmetries, the Riemann curvature tensor is determined by a single component

$$R_{1212} = -R_{2112} = R_{2121} = -R_{1221}.$$

The other components equal zero. We have

$$2R_{1212} = R(g_{11}g_{22} - g_{12}^2),$$

where R is the scalar curvature. So we have

$$R_{1212}(x_0) = \frac{1}{2}R(x_0).$$

Thus we have

$$g^{11}_{(2)}(X) = \frac{1}{6}R(x_0)y^2, \quad g^{12}_{(2)}(X) = -\frac{1}{6}R(x_0)xy, \quad g^{22}_{(2)}(X) = \frac{1}{6}R(x_0)x^2. \tag{2.5}$$

We also have

$$\sqrt{|g(X)|} = 1 - \frac{1}{12}R(x_0)x^2 - \frac{1}{12}R(x_0)y^2 + O(|X|^3). \tag{2.6}$$

Let us write Taylor's expansion of Γ^α in the form

$$\Gamma^\alpha(X) = \sum_{k=0}^{\infty} \Gamma^\alpha_{(k)}(X), \quad \alpha = 1, 2. \tag{2.7}$$

Using (2.5) and (2.6), one can show that

$$\Gamma^\alpha_{(0)}(X) = 0, \quad \alpha = 1, 2,$$

and

$$\Gamma^1_{(1)}(X) = -\frac{1}{3}R(x_0)x\,, \quad \Gamma^2_{(1)}(X) = -\frac{1}{3}R(x_0)y\,. \tag{2.8}$$

If we write $\mathbf{B} = B(x, y)\,dx\,dy$ then

$$B(X) = b(X)\sqrt{|g(X)|} = b_0 + \alpha x^2 + \beta y^2 + O(|X|^3),$$

where

$$\alpha_1 = \alpha + \frac{1}{12}b_0R(x_0) > 0, \quad \beta_1 = \beta + \frac{1}{12}b_0R(x_0) > 0.$$

We can also choose a gauge A such that

$$A_1(X) = 0 \text{ and } A_2(X) = b_0 x + \frac{\alpha}{3}x^3 + \beta x y^2 + O(|X|^4).$$

We expand A_2 into the Taylor series:

$$A_2(X) = b_0 x + \sum_{j=3}^{\infty} S_j(x,y),$$

with

$$S_j(x,y) = \sum_{\ell=0}^{j} S_{j\ell} x^\ell y^{j-\ell}.$$

In particular, we have

$$S_3(x,y) = \frac{\alpha}{3}x^3 + \beta x y^2.$$

Next we move the operator H^h into the Hilbert space $L^2(\mathbb{R}^n)$ equipped with the Euclidean inner product, considering the operator

$$\begin{aligned} \hat{H}^h &= |g(X)|^{1/4} H_h |g(X)|^{-1/4} \\ &= \sum_{1\leq \alpha,\beta\leq 2} g^{\alpha\beta}(X) \hat{\nabla}^h_\alpha \hat{\nabla}^h_\beta + ih \sum_{1\leq\alpha\leq 2} \Gamma^\alpha \hat{\nabla}^h_\alpha, \end{aligned}$$

where, for $\alpha = 1, 2$,

$$\hat{\nabla}^h_\alpha = |g(X)|^{1/4} \nabla^h_\alpha |g(X)|^{-1/4} = \nabla^h_\alpha + \frac{1}{12} ihR(x_0)X^\alpha + O(h|X|^2)). \tag{2.9}$$

Now we use the scaling $x = h^{1/2}x_1, y = h^{1/2}y_1$ and expand the resulting operator $\hat{H}^h(x_1, y_1, D_{x_1}, D_{y_1})$ into a formal power series of $h^{1/2}$. By (2.9), we have

$$\begin{aligned} \hat{\nabla}^h_1 &= h^{1/2}(-D_{x_1} + \frac{1}{12} ihR(x_0)x_1 + O(h^{3/2})), \\ \hat{\nabla}^h_2 &= h^{1/2}(-D_{y_1} + b_0 x_1 + \frac{1}{12} ihR(x_0)y_1 + hS_3(x_1,y_1) + O(h^{3/2})). \end{aligned}$$

From (2.4) and (2.7), we get

$$\begin{aligned} g^{\alpha\beta}(x_1,y_1) &= \delta^{\alpha\beta} + hg^{\alpha\beta}_{(2)}(x_1,y_1) + \sum_{k=3}^{\infty} h^{k/2} g^{\alpha\beta}_{(k)}(x_1,y_1), \\ \Gamma^\alpha(x_1,y_1) &= h^{1/2}\Gamma^\alpha_{(1)}(x_1,y_1) + \sum_{k=2}^{\infty} h^{k/2}\Gamma^\alpha_{(k)}(x_1,y_1), \quad \alpha = 1, 2. \end{aligned}$$

Using these expansions, one can check that the operator $\hat{H}^h$ has the form

$$\hat{H}^h(x_1, y_1, D_{x_1}, D_{y_1}) = hQ^h(x_1, y_1, D_{x_1}, D_{y_1}),$$

with

$$Q^h(x_1, y_1, D_{x_1}, D_{y_1}) = \sum_{k=0}^{\infty} h^{k/2} Q_k(x_1, y_1, D_{x_1}, D_{y_1}),$$

where

$$\begin{aligned} &Q_0(x_1, y_1, D_{x_1}, D_{y_1}) = D^2_{x_1} + (D_{y_1} - b_0 x_1)^2, \\ &Q_1(x_1, y_1, D_{x_1}, D_{y_1}) = 0, \end{aligned}$$

and

$$\begin{aligned}
Q_2(x_1, y_1, D_{x_1}, D_{y_1}) = &- \frac{1}{12} iR(x_0)(x_1 D_{x_1} + D_{x_1} x_1) \\
&- \frac{1}{12} iR(x_0)(y_1(D_{y_1} - b_0 x_1) + (D_{y_1} - b_0 x_1) y_1) \\
&- ((D_{y_1} - b_0 x_1) S_3(x_1, y_1) + S_3(x_1, y_1)(D_{y_1} - b_0 x_1)) \\
&+ g^{11}_{(2)}(x_1, y_1) D^2_{x_1} \\
&+ g^{12}_{(2)}(x_1, y_1)\, D_{x_1}\, (D_{y_1} - b_0 x_1) \\
&+ g^{12}_{(2)}(x_1, y_1)\, (D_{y_1} - b_0 x_1)\, D_{x_1} \\
&+ g^{22}_{(2)}(x_1, y_1)(D_{y_1} - b_0 x_1)^2 \\
&- i\Gamma^1_{(1)}(x_1, y_1) D_{x_1} \\
&- i\Gamma^2_{(1)}(x_1, y_1)(D_{y_1} - b_0 x_1).
\end{aligned}$$

Using the Fourier transform in y_1, we can write the operator Q^h as

$$Q^h(x_1, -D_\xi, D_{x_1}, \xi) = \sum_{k=0}^{\infty} h^{k/2} Q_k(x_1, -D_\xi, D_{x_1}, \xi),$$

where

$$\begin{aligned}
Q_0(x_1, -D_\xi, D_{x_1}, \xi) &= D^2_{x_1} + (\xi - b_0 x_1)^2, \\
Q_1(x_1, -D_\xi, D_{x_1}, \xi) &= 0,
\end{aligned}$$

and

$$\begin{aligned}
Q_2(x_1, -D_\xi, D_{x_1}, \xi) = &- \frac{1}{12} iR(x_0)(2x_1 D_{x_1} - i) \\
&+ \frac{1}{12} iR(x_0)(2(\xi - b_0 x_1) D_\xi - i) \\
&- ((\xi - b_0 x_1) S_3(x_1, -D_\xi) + S_3(x_1, -D_\xi)(\xi - b_0 x_1)) \\
&+ g^{11}_{(2)}(x_1, -D_\xi) D^2_{x_1} \\
&+ g^{12}_{(2)}(x_1, -D_\xi)\, D_{x_1}\, (\xi - b_0 x_1) \\
&+ g^{12}_{(2)}(x_1, -D_\xi)\, (\xi - b_0 x_1)\, D_{x_1} \\
&+ g^{22}_{(2)}(x_1, -D_\xi)(\xi - b_0 x_1)^2 \\
&- i\Gamma^1_{(1)}(x_1, -D_\xi) D_{x_1} \\
&- i\Gamma^2_{(1)}(x_1, -D_\xi)(\xi - b_0 x_1).
\end{aligned}$$

A further translation $x_2 = x_1 - \frac{\xi}{b_0}$ gives

$$\hat{H}^h = hT^h(x_2, \xi, D_{x_2}, D_\xi),$$

where

$$T^h(x_2, \xi, D_{x_2}, D_\xi) = Q^h\left(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}, D_{x_2}, \xi\right).$$

We have (denoting $w = (x_2, \xi)$)

$$T^h(w, D_w) = T_0(x_2, D_{x_2}) + hT_2(w, D_w) + \sum_{j=3}^{\infty} h^{j/2} T_j(w, D_w)$$

with

$$T_0(x_2, D_{x_2}) = D_{x_2}^2 + b_0^2 x_2^2,$$

and

$$\begin{aligned}
T_2(w, D_w) = & -\frac{1}{12} iR(x_0)(2\left(x_2 + \frac{\xi}{b_0}\right) D_{x_2} - i) \\
& + \frac{1}{12} iR(x_0)(2b_0 x_2 \left(-D_\xi + \frac{1}{b_0} D_{x_2}\right) + i) \\
& + (b_0 x_2 \hat{S}_3(x_2, \xi, D_{x_2}, D_\xi) + \hat{S}_3(x_2, \xi, D_{x_2}, D_\xi) b_0 x_2) \\
& + g^{11}_{(2)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}) D_{x_2}^2 \\
& - g^{12}_{(2)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2})\, D_{x_2}\, (b_0 x_2) \\
& - g^{12}_{(2)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2})\, b_0 x_2\, D_{x_2} \\
& + g^{22}_{(2)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}) b_0^2 x_2^2 \\
& - i\Gamma^1_{(1)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}) D_{x_2} \\
& + i\Gamma^2_{(1)}(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}) b_0 x_2,
\end{aligned}$$

where

$$\hat{S}_3(x_2, \xi, D_{x_2}, D_\xi) = S_3\left(x_2 + \frac{\xi}{b_0}, -D_\xi + \frac{1}{b_0} D_{x_2}\right).$$

The operator $\hat{S}_3$ has the following form:

$$\begin{aligned}
\hat{S}_3(x_2, \xi, D_{x_2}, D_\xi) = & \, x_2 L(\xi, D_\xi) + M_0(\xi, D_\xi) + M_1(x_2, D_{x_2}) \\
& + M_2(x_2, D_{x_2}, D_\xi) + M_3(x_2, D_{x_2}, D_\xi) + M_4(x_2, \xi, D_{x_2}),
\end{aligned}$$

where

$$\begin{aligned}
L(\xi, D_\xi) &= \frac{\alpha}{b_0^2} \xi^2 + \beta D_\xi^2, \\
M_0(\xi, D_\xi) &= \frac{\alpha}{3b_0^3} \xi^3 + \frac{\beta}{b_0} \xi D_\xi^2, \\
M_1(x_2, D_{x_2}) &= \frac{\alpha}{3} x_2^3 + \frac{\beta}{b_0^2} x_2 D_{x_2}^2, \\
M_2(x_2, D_{x_2}, D_\xi) &= -2\frac{\beta}{b_0^2} D_{x_2} \xi D_\xi, \\
M_3(x_2, D_{x_2}, D_\xi) &= -2\frac{\beta}{b_0} x_2 D_{x_2} D_\xi, \\
M_4(x_2, \xi, D_{x_2}) &= \frac{\alpha}{b_0} x_2^2 \xi + \frac{\beta}{b_0^3} \xi D_{x_2}^2.
\end{aligned}$$

So we get

$$\begin{aligned}
T_2(w, D_w) = & -\frac{1}{6} iR(x_0) \frac{\xi}{b_0} D_{x_2} - \frac{1}{6} iR(x_0) b_0 x_2 D_\xi \\
& + 2b_0 x_2^2 L(\xi, D_\xi) + b_0 \left(x_2 M(w, D_w) + M(w, D_w) x_2\right))
\end{aligned}$$

$$+ g^{11}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)D^2_{x_2}$$
$$- g^{12}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)\Big(D_{x_2}(b_0x_2)+b_0x_2D_{x_2}\Big)$$
$$+ g^{22}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)b_0^2x_2^2$$
$$- i\Gamma^1_{(1)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)D_{x_2}$$
$$+ i\Gamma^2_{(1)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)b_0x_2,$$

where

$$M(w,D_w)=\sum_{\ell=0}^{3}M_\ell(w,D_w).$$

We have

$$\mathrm{Sp}(T_0(x_2,D_{x_2}))=\{\mu_k=(2k+1)b_0 : k\in\mathbb{N}\}.$$

The eigenfunction of $T_0(x_2,D_{x_2})$ associated to the eigenvalue μ_k is

$$\psi_k(x_2)=\pi^{-1/4}b_0^{1/2}H_k(b_0^{1/2}x_2)e^{-b_0x_2^2/2},$$

where H_k is the Hermite polynomial:

$$H_k(x)=(-1)^ke^{x^2}\frac{d^k}{dx^k}(e^{-x^2}).$$

The norm of ψ_k in $L^2(\mathbb{R},dx)$ equals the norm of H_k in $L^2(\mathbb{R},e^{-x^2}\,dx)$, which is given by

$$\|H_k\|=\sqrt{2^kk!\sqrt{k}}.$$

2.3. **Construction of approximate eigenfunctions.** First, we construct a formal eigenfunction u^h of the operator $T^h(w,D_w)$ admitting an asymptotic expansion in the form of a formal asymptotic series in powers of $h^{1/2}$

$$u^h=\sum_{\ell=0}^{\infty}u^{(\ell)}h^{\ell/2},\quad u^{(\ell)}\in\mathcal{S}(\mathbb{R}^2),$$

with the corresponding formal eigenvalue

$$\lambda^h=\sum_{\ell=0}^{\infty}\lambda^{(\ell)}h^{\ell/2},$$

such that

$$T^h(w,D_w)u^h-\lambda^hu^h=0$$

in the sense of asymptotic series in powers of $h^{1/2}$.

The first terms. Looking at the coefficient of h^0, we obtain:

$$T_0(x_2,D_{x_2})u^{(0)}=\lambda^{(0)}u^{(0)}\,.$$

Thus, we have

$$\lambda^{(0)}=\lambda^{(0)}_k=(2k+1)b_0,\quad u^{(0)}(x_2,\xi)=\frac{1}{\|H_k\|}\psi_k(x_2)\chi_0(\xi),\quad k\in\mathbb{N}, \tag{2.10}$$

where χ_0 is some function, which will be determined later, and c is some constant.

Looking at the coefficient of $h^{1/2}$, we obtain:

$$T_0(x_2, D_{x_2})u^{(1)} = \lambda^{(0)}u^{(1)} + \lambda^{(1)}u^{(0)}.$$

The orthogonality condition implies that

$$\lambda^{(1)} = 0.$$

Under this condition, we get

$$u^{(1)}(x_2, \xi) = \frac{1}{\|H_k\|}\psi_k(x_2)\chi_1(\xi), \tag{2.11}$$

where χ_1 is some function, which will be determined later.

Next, the cancelation of the coefficient of h^1 gives:

$$(T_0(x_2, D_{x_2}) - \lambda^{(0)})u^{(2)} = \lambda^{(2)}u^{(0)} - T_2(w, D_w)u^{(0)}. \tag{2.12}$$

The orthogonality condition implies that

$$\lambda^{(2)}\chi_0(\xi) - \frac{1}{\|H_k\|}\int T_2(w, D_w)u^{(0)}\psi_k(x_2)dx_2 = 0.$$

Lemma 2.5. *For any function u of the form $u(x_2, \xi) = \frac{1}{\|H_k\|}\psi_k(x_2)\chi(\xi)$, we have*

$$\frac{1}{\|H_k\|}\int T_2(w, D_w)u(w)\psi_k(x_2)dx_2 = \mathcal{H}_k\chi(\xi),$$

where $\mathcal{H}_k$ is the harmonic oscillator:

$$\begin{aligned}\mathcal{H}_k =&(2k+1)\beta_1 D_\xi^2 + (2k+1)\alpha_1\frac{1}{b_0^2}\xi^2\\ &+ \frac{1}{2b_0}(2k^2+2k+1)\left(\alpha_1 + \beta_1 + \frac{1}{2}b_0 R(x_0)\right) - \frac{1}{4}R(x_0).\end{aligned}$$

Proof. We have

$$\begin{aligned}D_{x_2}\psi_k &= -ib_0^{1/2}\left(2k\psi_{k-1} - b_0^{1/2}x_2\psi_k\right),\\ D_{x_2}^2\psi_k &= b_0(2k+1-b_0x_2^2)\psi_k,\end{aligned}$$

and

$$D_{x_2}^3\psi_k = 2ib_0^2x_2\psi_k + b_0(2k+1-b_0x_2^2)D_{x_2}\psi_k.$$

We also have

$$\begin{aligned}2xH_k =&H_{k+1} + 2kH_{k-1},\\ 4x^2H_k =&H_{k+2} + (4k+2)H_k + 4k(k-1)H_{k-2},\\ 8x^3H_k =&H_{k+3} + (6k+6)H_{k+1} + 12k^2H_{k-1} + 8k(k-1)(k-2)H_{k-3}\\ 16x^4H_k(x) =&H_{k+4}(x) + (8k+12)H_{k+2}(x) + 12(2k^2+2k+1)H_k(x)\\ &+ 16(2k^2-3k+1)kH_{k-2}(x) + 16k(k-1)(k-2)(k-3)H_{k-4}(x),\end{aligned}$$

that implies that

$$\begin{aligned}\frac{1}{\|H_k\|^2}\langle x_2\psi_{k-1}, \psi_k\rangle =&\frac{1}{2b_0^{1/2}},\\ \frac{1}{\|H_k\|^2}\langle x_2^3\psi_{k-1}, \psi_k\rangle =&\frac{3}{4b_0^{3/2}}k,\\ \frac{1}{\|H_k\|^2}\langle x_2^2\psi_k, \psi_k\rangle =&\frac{1}{2b_0}(2k+1),\end{aligned}$$

$$\frac{1}{\|H_k\|^2}\langle x_2^4\psi_k,\psi_k\rangle = \frac{3}{4b_0^2}(2k^2+2k+1).$$

Next, we have

$$\begin{aligned}
\frac{1}{\|H_k\|^2}\langle x_2 D_{x_2}\psi_k,\psi_k\rangle =& \frac{-ib_0^{1/2}}{\|H_k\|^2}\langle 2kx_2\psi_{k-1} - b_0^{1/2}x_2^2\psi_k,\psi_k\rangle\\
=&\frac{1}{2}i,\\
\frac{1}{\|H_k\|^2}\langle D_{x_2}^2\psi_k,\psi_k\rangle =& \frac{b_0}{\|H_k\|^2}\langle (2k+1-b_0x_2^2)\psi_k,\psi_k\rangle\\
=&\frac{1}{2}(2k+1)b_0,\\
\frac{1}{\|H_k\|^2}\langle x_2^2 D_{x_2}^2\psi_k,\psi_k\rangle =& \frac{b_0}{\|H_k\|^2}\langle ((2k+1)x_2^2 - b_0x_4^2)\psi_k,\psi_k\rangle\\
=&\frac{1}{4}(2k^2+2k-1),\\
\frac{1}{\|H_k\|^2}\langle D_{x_2}^4\psi_k,\psi_k\rangle =& \frac{b_0^2}{\|H_k\|^2}\langle (2k+1-b_0x_2^2)^2\psi_k,\psi_k\rangle\\
=&\frac{3}{4}(2k^2+2k+1)b_0^2.
\end{aligned}$$

First, remark that

$$\delta_0\lambda = \frac{1}{\|H_k\|}\int\left(-\frac{1}{6}iR(x_0)\frac{\xi}{b_0}D_{x_2} - \frac{1}{6}iR(x_0)b_0x_2D_\xi\right)u(w)\psi_k(x_2)dx_2 = 0,$$

since the operator $-\frac{1}{6}iR(x_0)\frac{\xi}{b_0}D_{x_2} - \frac{1}{6}iR(x_0)b_0x_2D_\xi$ is odd in the x_2 variable.

Next, we have

$$\begin{aligned}
\delta_1\lambda =& \frac{2b_0}{\|H_k\|^2}\langle x_2^2\psi_k,\psi_k\rangle L(\xi,D_\xi)\chi(\xi)\\
=&(2k+1)\left(\frac{\alpha}{b_0^2}\xi^2+\beta D_\xi^2\right)\chi(\xi).
\end{aligned}$$

Now we consider

$$\delta_2\lambda = \frac{1}{\|H_k\|}\int b_0\left(x_2M(w,D_w)+M(w,D_w)x_2\right)u(w)\psi_k(x_2)dx_2.$$

The operators $M_1(w,D_w)$ and $M_2(w,D_w)$ are odd in the x_2 variable, so we obtain

$$\begin{aligned}
\delta_2\lambda =& \frac{1}{\|H_k\|}\int b_0\left(x_2M_1(w,D_w)+M_1(w,D_w)x_2\right)u(w)\psi_k(x_2)dx_2\\
&+\frac{1}{\|H_k\|}\int b_0\left(x_2M_2(w,D_w)+M_2(w,D_w)x_2\right)u(w)\psi_k(x_2)dx_2\\
=&\frac{b_0}{\|H_k\|^2}\langle\left(\frac{2\alpha}{3}x_2^4+\frac{\beta}{b_0^2}x_2^2D_{x_2}^2+\frac{\beta}{b_0^2}x_2D_{x_2}^2x_2\right)\psi_k,\psi_k\rangle\chi(\xi)\\
&-\frac{b_0}{\|H_k\|^2}\langle\left(2\frac{\beta}{b_0^2}x_2D_{x_2}+2\frac{\beta}{b_0^2}D_{x_2}x_2\right)\psi_k,\psi_k\rangle\xi D_\xi\chi(\xi)\\
=&\frac{2b_0}{\|H_k\|^2}\langle\left(\frac{\alpha}{3}x_2^4+\frac{\beta}{b_0^2}x_2^2D_{x_2}^2-i\frac{\beta}{b_0^2}x_2D_{x_2}\right)\psi_k,\psi_k\rangle\chi(\xi)
\end{aligned}$$

$$+\frac{2b_0}{\|H_k\|^2}\langle\left(-\frac{2\beta}{b_0^2}x_2D_{x_2}+i\frac{\beta}{b_0^2}\right)\psi_k,\psi_k\rangle\xi D_\xi\chi(\xi).$$

Thus we arrive at

$$\delta_2\lambda=\frac{\alpha+\beta}{2b_0}(2k^2+2k+1)\chi(\xi).$$

Next, we consider

$$\delta_3\lambda=\frac{1}{\|H_k\|}\int g^{11}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)D^2_{x_2}u(w)\psi_k(x_2)dx_2.$$

By (2.5), we have

$$g^{11}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)=\frac{1}{6}R(x_0)(-D_\xi+\frac{1}{b_0}D_{x_2})^2.$$

Therefore,

$$\delta_3\lambda=\frac{1}{6\|H_k\|}R(x_0)\int(-D_\xi+\frac{1}{b_0}D_{x_2})^2D^2_{x_2}u(w)\psi_k(x_2)dx_2.$$

It suffices to consider the terms, which are even with respect to x_2:

$$\delta_3\lambda=\frac{1}{6\|H_k\|}R(x_0)\int(D^2_\xi+\frac{1}{b_0^2}D^2_{x_2})D^2_{x_2}u(w)\psi_k(x_2)dx_2.$$

Thus, we obtain that

$$\begin{aligned}\delta_3\lambda=&\frac{1}{6\|H_k\|^2}R(x_0)\langle D^2_{x_2}\psi_k,\psi_k\rangle D^2_\xi\chi(\xi)\\&+\frac{1}{6b_0^2\|H_k\|^2}R(x_0)\langle D^4_{x_2}\psi_k,\psi_k\rangle\chi(\xi)\\=&\frac{1}{12}(2k+1)b_0R(x_0)D^2_\xi\chi(\xi)\\&+\frac{1}{8}(2k^2+2k+1)R(x_0)\chi(\xi).\end{aligned}$$

Next, we consider

$$\delta_4\lambda=-\frac{1}{\|H_k\|}\int g^{12}_{(2)}\left(x_2+\frac{\xi}{b_0},-D_\xi+\frac{1}{b_0}D_{x_2}\right)\left(D_{x_2}(b_0x_2)+b_0x_2D_{x_2}\right)u(w)\psi_k(x_2)dx_2.$$

By (2.5), we have

$$\delta_4\lambda=\frac{1}{6\|H_k\|}R(x_0)\int\left(x_2+\frac{\xi}{b_0}\right)\left(-D_\xi+\frac{1}{b_0}D_{x_2}\right)(2b_0x_2D_{x_2}-ib_0)\,u(w)\psi_k(x_2)dx_2.$$

It suffices to consider the terms, which are even with respect to x_2:

$$\begin{aligned}\delta_4\lambda=&\frac{1}{6\|H_k\|}R(x_0)\int(x_2D_{x_2}-\xi D_\xi)\Big(2x_2D_{x_2}-i\Big)u(w)\psi_k(x_2)dx_2\\=&\frac{1}{3\|H_k\|^2}R(x_0)\langle x_2^2D^2_{x_2}\psi_k,\psi_k\rangle\chi(\xi)\\&-\frac{1}{2\|H_k\|^2}iR(x_0)\langle x_2D_{x_2}\psi_k,\psi_k\rangle\chi(\xi)\\&-\frac{1}{6\|H_k\|^2}R(x_0)\langle(2x_2D_{x_2}-i)\psi_k,\psi_k\rangle\xi D_\xi\chi(\xi)\\=&\left(\frac{1}{12}(2k^2+2k+1)R(x_0)+\frac{1}{12}R(x_0)\right)\chi(\xi).\end{aligned}$$

Consider

$$\delta_5\lambda = \frac{1}{\|H_k\|}\int g^{22}_{(2)}\left(x_2+\frac{\xi}{b_0}, -D_\xi+\frac{1}{b_0}D_{x_2}\right) b_0^2 x_2^2 u(w)\psi_k(x_2)dx_2.$$

By (2.5), we have

$$\delta_5\lambda = \frac{1}{6\|H_k\|}R(x_0)\int (x_2+\frac{\xi}{b_0})^2 b_0^2 x_2^2 u(w)\psi_k(x_2)dx_2.$$

It suffices to consider the terms, which are even with respect to x_2:

$$\begin{aligned}\delta_5\lambda =&\frac{1}{6\|H_k\|}R(x_0)\int (x_2^2+\frac{\xi^2}{b_0^2}) b_0^2 x_2^2 u(w)\psi_k(x_2)dx_2\\ =&\frac{1}{6\|H_k\|^2}b_0^2R(x_0)\langle x_2^4\psi_k,\psi_k\rangle\chi(\xi)+\frac{1}{6\|H_k\|^2}R(x_0)\langle x_2^2\psi_k,\psi_k\rangle\xi^2\chi(\xi)\\ =&\frac{1}{8}(2k^2+2k+1)R(x_0)\chi(\xi)+\frac{1}{12b_0}(2k+1)R(x_0)\xi^2\chi(\xi).\end{aligned}$$

Next, consider

$$\delta_6\lambda = -\frac{i}{\|H_k\|}\int \Gamma^1_{(1)}\left(x_2+\frac{\xi}{b_0}, -D_\xi+\frac{1}{b_0}D_{x_2}\right) D_{x_2}u(w)\psi_k(x_2)dx_2.$$

By (2.8), we have

$$\delta_6\lambda = \frac{i}{3\|H_k\|}R(x_0)\int\left(x_2+\frac{\xi}{b_0}\right)D_{x_2}u(w)\psi_k(x_2)dx_2.$$

It suffices to consider the terms, which are even with respect to x_2:

$$\begin{aligned}\delta_6\lambda =&\frac{i}{3\|H_k\|^2}R(x_0)\langle x_2D_{x_2}\psi_k,\psi_k\rangle\chi(\xi)\\ =&-\frac{1}{6}R(x_0)\chi(\xi).\end{aligned}$$

Finally, we take

$$\delta_7\lambda = \frac{i}{\|H_k\|}\int \Gamma^2_{(1)}\left(x_2+\frac{\xi}{b_0}, -D_\xi+\frac{1}{b_0}D_{x_2}\right) b_0x_2u(w)\psi_k(x_2)dx_2.$$

By (2.8), we have

$$\delta_7\lambda = \frac{i}{3\|H_k\|}R(x_0)\int\left(-D_\xi+\frac{1}{b_0}D_{x_2}\right)b_0x_2u(w)\psi_k(x_2)dx_2.$$

It suffices to consider the terms, which are even with respect to x_2:

$$\begin{aligned}\delta_7\lambda =&\frac{i}{3\|H_k\|^2}R(x_0)\langle (x_2D_{x_2}-i)\psi_k(w),\psi_k(w)\rangle\chi(\xi)\\ =&-\frac{1}{6}R(x_0)\chi(\xi).\end{aligned}$$

We conclude

$$\begin{aligned}&\frac{1}{\|H_k\|}\int T_2(w,D_w)u(w)\psi_k(x_2)dx_2\\ =&\delta_0\lambda+\delta_1\lambda+\delta_2\lambda+\delta_3\lambda+\delta_4\lambda+\delta_5\lambda+\delta_6\lambda+\delta_7\lambda\\ =&\frac{1}{2b_0}(2k^2+2k+1)\left(\alpha+\beta+\frac{2}{3}b_0R(x_0)\right)\chi(\xi)\end{aligned}$$

$$
\begin{aligned}
&+(2k+1)\left(\beta+\frac{1}{12}b_0R(x_0)\right)D_\xi^2\chi(\xi)\\
&+(2k+1)\left(\alpha+\frac{1}{12}b_0R(x_0)\right)\frac{1}{b_0^2}\xi^2\chi(\xi)\\
&-\frac{1}{4}R(x_0)\chi(\xi)\\
=&\frac{1}{2b_0}(2k^2+2k+1)\left(\alpha_1+\beta_1+\frac{1}{2}b_0R(x_0)\right)\chi(\xi)\\
&+(2k+1)\beta_1D_\xi^2\chi(\xi)+(2k+1)\alpha_1\frac{1}{b_0^2}\xi^2\chi(\xi)-\frac{1}{4}R(x_0)\chi(\xi).
\end{aligned}
$$

□

Thus, we obtain that

$$\lambda^{(2)}=\lambda^{(2)}_{jk}=\nu_{jk},\quad j,k\in\mathbb{N},$$

where ν_{jk} is an eigenvalue of the harmonic oscillator $\mathcal{H}_k$:

$$
\begin{aligned}
\nu_{jk}=&(2j+1)(2k+1)(\alpha_1\beta_1)^{1/2}\frac{1}{b_0}\\
&+\frac{1}{2b_0}(2k^2+2k+1)(\alpha_1+\beta_1)+\frac{1}{2}(k^2+k)R(x_0),\quad j,k\in\mathbb{N},
\end{aligned}
$$

and

$$\chi_0(\xi)=\Psi_{jk}(\xi),$$

where Ψ_{jk} is the normalized eigenfunction of $\mathcal{H}_k$ associated to the eigenvalue ν_{jk}.

Moreover, we conclude that $u^{(2)}$ is a solution of (2.12), which can be written as

$$u^{(2)}=\phi^{(2)}(x_2,\xi)+\psi_k(x_2)\chi_2(\xi),$$

where $\phi^{(2)}$ is a solution of (2.12), satisfying the condition

$$\int\phi^{(2)}(x_2,\xi)\psi_k(x_2)\,dx_2=0,$$

and χ_2 will be determined later.

Now the cancelation of the coefficient of $h^{3/2}$ gives:

$$
\begin{aligned}
&(T_0(x_2,D_{x_2})-\lambda^{(0)})u^{(3)}\\
&\qquad=\lambda^{(3)}u^{(0)}-T_3(w,D_w)u^{(0)}+\lambda^{(2)}u^{(1)}-T_2(w,D_w)u^{(1)}.
\end{aligned}
\tag{2.13}
$$

The orthogonality condition for (2.13) is written as

$$\lambda^{(3)}\chi_0-\frac{1}{\|H_k\|}\int T_3(w,D_w)u^{(0)}\psi_k(x_2)\,dx_2+\lambda^{(2)}\chi_1-\mathcal{H}_k\chi_1=0.\tag{2.14}$$

Under this assumption, we obtain that $u^{(3)}$ is a solution of (2.13), which can be written as

$$u^{(3)}=\phi^{(3)}(x_2,\xi)+\frac{1}{\|H_k\|}\psi_k(x_2)\chi_3(\xi),$$

where $\phi^{(3)}$ is a solution of (2.13), satisfying the condition

$$\int\phi^{(3)}(x_2,\xi)\psi_k(x_2)\,dx_2=0,$$

and χ_3 will be determined later.

The equation (2.14) has a solution if and only if

$$\lambda^{(3)} = \frac{1}{\|H_k\|}\iint T_3(w,D_w)u^{(0)}(w)\psi_k(x_2)\Psi_{jk}(\xi)\,dx_2\,d\xi,$$

which allow us to find $\lambda^{(3)}$. Under this condition, there exists a unique solution χ_1 of (2.14), orthogonal to χ_0

The iteration procedure. Suppose that the coefficients of $h^{\ell/2}$ equal zero for $\ell = 0,\dots,n-1$, $n > 3$. Then we know the coefficients $\lambda^{(\ell)}$ for $\ell = 0,\dots,n-1$. We also know that $u^{(\ell)}$ for $\ell = 0,\dots,n-1$ can be written as

$$u^{(\ell)} = \phi^{(\ell)}(x_2,\xi) + \frac{1}{\|H_k\|}\psi_k(x_2)\chi_\ell(\xi),$$

where $\phi^{(\ell)}, \ell = 0,\dots,n-1$, are some known functions in $\mathcal{S}(\mathbb{R}^2)$, satisfying the condition

$$\int \phi^{(\ell)}(x_2,\xi)\psi_k(x_2)\,dx_2 = 0,$$

and $\chi_\ell \in \mathbb{S}(\mathbb{R})$ are known for $\ell = 0,\dots,n-3$, $\chi_\ell \perp \chi_0$.

The cancelation of the coefficient of $h^{n/2}$ gives:

$$(T_0(x_2,D_{x_2}) - \lambda^{(0)})u^{(n)} = \lambda^{(n)}u^{(0)} - T_n(w,D_w)u^{(0)} \\ + \sum_{\ell=3}^{n-1}(\lambda^{(\ell)}u^{(n-\ell)} - T_\ell(w,D_w)u^{(n-\ell)}) + \lambda^{(2)}u^{(n-2)} - T_2(w,D_w)u^{(n-2)}. \tag{2.15}$$

The orthogonality condition for (2.15) is written as

$$\lambda^{(n)}\chi_0 - \frac{1}{\|H_k\|}\int T_n(w,D_w)u^{(0)}\psi_k(x_2)\,dx_2 \\ + \sum_{\ell=3}^{n-1}(\lambda^{(\ell)}\chi_{n-\ell} - \frac{1}{\|H_k\|}\int T_\ell(w,D_w)u^{(n-\ell)}\psi_k(x_2)\,dx_2) \\ - \frac{1}{\|H_k\|}\int T_\ell(w,D_w)\phi^{(n-\ell)}\psi_k(x_2)\,dx_2 + \lambda^{(2)}\chi_{n-2} - \mathcal{H}_k\chi_{n-2} = 0. \tag{2.16}$$

Under this assumption, we obtain that $u^{(n)}$ is a solution of (2.15), which can be written as

$$u^{(n)} = \phi^{(n)}(x_2,\xi) + \frac{1}{\|H_k\|}\psi_k(x_2)\chi_n(\xi),$$

where $\phi^{(n)}$ is a solution of (2.15), satisfying the condition

$$\int \phi^{(n)}(x_2,\xi)\psi_k(x_2)\,dx_2 = 0,$$

and χ_n will be determined later.

The orthogonality condition for (2.16) allows us to find $\lambda^{(n)}$. Under this condition, there exists a unique solution χ_{n-2} of (2.16), orthogonal to χ_0.

Thus, for any $j \in \mathbb{N}$ and $k \in \mathbb{N}$, we have constructed an approximate eigenfunction u^h_{jk} of the operator $T^h(w,D_w)$ admitting an asymptotic expansion in the form of a formal asymptotic series in powers of $h^{1/2}$

$$u^h_{jk} = \sum_{\ell=0}^{\infty} u^{(\ell)}_{jk}h^{\ell/2}, \quad u^{(\ell)}_{jk} \in \mathcal{S}(\mathbb{R}^2),$$

such that

$$u_{jk}^{(0)}(x_2,\xi)=\frac{1}{\|H_k\|}\psi_k(x_2)\Psi_{jk}(\xi). \tag{2.17}$$

with the corresponding approximate eigenvalue

$$\lambda_{jk}^h=\sum_{\ell=0}^{\infty}\lambda_{jk}^{(\ell)}h^{\ell/2}.$$

For any $N\in\mathbb{N}$, consider

$$u_{jk(N)}^h=\sum_{\ell=0}^{N}u_{jk}^{(\ell)}h^{\ell/2},\quad \lambda_{jk(N)}^h=\sum_{\ell=0}^{N}\lambda_{jk}^{(\ell)}h^{\ell/2}.$$

Then we have

$$T^h(w,D_w)u_{jk(N)}^h-\lambda_{jk(N)}^h u_{jk(N)}^h=O(h^{\frac{N+1}{2}}).$$

The constructed functions $u_{jk(N)}^h$ have sufficient decay properties. Therefore, by changing back to the original coordinates and multiplying by a fixed cut-off function, we obtain the desired functions ϕ_{jkN}^h, which satisfy (2.2) with $\mu_{j,k,\ell}=\lambda_{jk}^{(\ell)}$.

The system $\{u_{jk}^{(0)}\}$ is an orthonormal system. Since each change of variables, which we use, is unitary, this implies the condition (2.1).

3. Lower bounds

In this section, we will prove the lower bound in Theorem 1.1. First, we recall a general lower bound due to Montgomery [27]. Suppose that U is a domain in M. Then, for any $u\in C_c^\infty(U)$, the following estimate holds:

$$\|(ih\,d+\mathbf{A})u\|_U^2\geq\left|\int_U b|u|^2dx_g\right|. \tag{3.1}$$

This fact is an immediate consequence of a Weitzenböck-Bochner type identity.

From (3.1), it follows that we can restrict our considerations by any sufficiently small neighborhood Ω of x_0. Denote by H_D^h the Dirichlet realization of the operator H^h in $L^2(\Omega,dx_g)$.

The estimate (3.1) implies that

$$\tau hH_D^h+(1-\tau h)hb\leq H_D^h,\quad 0<\tau<h^{-1}.$$

Taking $\tau=h^{-1/2}$, we obtain

$$h^{1/2}(H_D^h-hb+h^{1/2}b)\leq H_D^h,\quad 0<h<1.$$

Consider the Dirichlet realization P_D^h of the operator $H^h-hb+h^{1/2}(b-b_0)$ in $L^2(\Omega,dx_g)$. Then we have

$$hb_0+h^{1/2}\lambda_j(P_D^h)\leq\lambda_j(H_D^h). \tag{3.2}$$

Therefore, the desired lower bound for $\lambda_j(H_D^h)$ is an immediate consequence of the following theorem.

Theorem 3.1. *For any $j\in\mathbb{N}$, there exist $C_j>0$ and $h_j>0$ such that*

$$\lambda_j(P_D^h)\geq h^{3/2}\left[\frac{2d^{1/2}}{b_0}j+\frac{a^2}{2b_0}\right]-C_jh^{15/8},\quad h\in(0,h_j].$$

To prove Theorem 3.1, we will follow the lines of the proof of [11, Theorem 7.4]. First, we observe that the upper bound in Theorem 1.1 and (3.2) imply an upper bound for $\lambda_j(P^h_D)$:

$$\lambda_j(P^h_D) \leq h^{3/2}\left[\frac{2d^{1/2}}{b_0}j+\frac{a^2}{2b_0}\right]+C_jh^2. \tag{3.3}$$

For any eigenvalue $\lambda_j(P^h_D)$, denote by $u_h^{(j)}$ an associated eigenfunction. By a straightforward repetition of the arguments of [11], we can easily show the following analogue of Lemmas 7.10 and 7.11 in [11][1].

Lemma 3.2. *For any $j\in\mathbb{N}$ and any real $k\geq 0$, we have*

$$\||X|^k u_h^{(j)}\|_{L^2(\Omega,dx_g)} \leq C_{k,j}(h^{k/2}+h^{(3k+1)/8})\|u_h^{(j)}\|_{L^2(\Omega,dx_g)}.$$

For any $j\in\mathbb{N}$, any $\alpha=1,2$ and any $k\geq 0$, we have

$$\||X|^k\nabla^h_\alpha u_h^{(j)}\|_{L^2(\Omega,dx_g)} \leq C_{k,j}(h^{(k+1)/2}+h^{(3k+5)/8})\|u_h^{(j)}\|_{L^2(\Omega,dx_g)}.$$

Take normal local coordinates near x_0 such that x_0 corresponds to $(0,0)$ and, in a neighborhood of x_0,

$$b(X)=b_0+\alpha_1x^2+\beta_1y^2+O(|X|^3).$$

So we have

$$g_{11}(X)=1+O(|X|^2),\quad g_{12}(X)=O(|X|^2),\quad g_{22}(X)=1+O(|X|^2).$$

We can take a magnetic potential A such that

$$A_1(X)=-\frac{1}{2}b_0y+O(|X|^3),\quad A_2(X)=\frac{1}{2}b_0x+O(|X|^3).$$

Let us introduce

$$b_2(X)=\frac{1}{2}X\cdot\operatorname{Hess}b(0)\cdot X.$$

Thus, we have

$$b(X)=b_0+b_2(X)+O(|X|^3).$$

We have

$$P^h_D=\sum_{1\leq\alpha,\beta\leq 2}g^{\alpha\beta}(X)\nabla^h_\alpha\nabla^h_\beta+ih\sum_{1\leq\alpha\leq 2}\Gamma^\alpha\nabla^h_\alpha-hb(X)+h^{1/2}(b(X)-b_0),$$

so its quadratic form is given by

$$\begin{aligned}(P^h_Du,u)=&\int_\Omega\sum_{1\leq\alpha,\beta\leq 2}g^{\alpha\beta}(X)\nabla^h_\alpha u(X)\overline{\nabla^h_\beta u(X)}\sqrt{g(X)}dX\\ &-h\int_\Omega b(X)|u(X)|^2\sqrt{g(X)}dX+h^{1/2}\int_\Omega(b(X)-b_0)|u(X)|^2\sqrt{g(X)}dX.\end{aligned}$$

Note that

$$P^h_D\geq 0.$$

Now we move the operator P^h_D into the Hilbert space $L^2(\Omega,dX)$, using the unitary change of variables $v=|g(X)|^{1/4}u$. For the corresponding operator

$$\hat{P}^h_D=|g(X)|^{1/4}P^h_D|g(X)|^{-1/4}$$

[1]There are a few inaccuracies in [11], concerning Lemma 7.11. For the erratum, see http://www.math.u-psud.fr/ helffer/erratum164II.pdf

in $L^2(\Omega, dX)$, we obtain

$$(\hat{P}^h_D v, v) = \int_\Omega \sum_{1\leq \alpha,\beta\leq 2} g^{\alpha\beta}(X) \left(\nabla^h_\alpha - \frac{1}{4} ih|g(X)|^{-1}\frac{\partial}{\partial X_\alpha}|g(X)|\right) v(X)\times$$

$$\times \overline{\left(\nabla^h_\beta - \frac{1}{4} ih|g(X)|^{-1}\frac{\partial}{\partial X_\alpha}|g(X)|\right) v(X)} dX$$

$$- h\int_\Omega b(X)|v(X)|^2 dX + h^{1/2}\int_\Omega (b(X)-b_0)|v(X)|^2 dX.$$

Put

$$q(v) = \int_\Omega \sum_{1\leq \alpha\leq 2} \left|\left(\nabla^h_\alpha - \frac{1}{4} ih|g(X)|^{-1}\frac{\partial}{\partial X_\alpha}|g(X)|\right) v(X)\right|^2 dX.$$

Then we have

$$|(\hat{P}^h_D v, v) - q(v)| \tag{3.4}$$

$$\leq \int_\Omega \sum_{1\leq \alpha\leq 2} |X|^2 \left|\left(\nabla^h_\alpha - \frac{1}{4} ih|g(X)|^{-1}\frac{\partial}{\partial X_\alpha}|g(X)|\right) v(X)\right|^2 dX$$

$$+ C_1 h\int_\Omega |v(X)|^2 dX + C_2 h^{1/2}\int_\Omega |X|^2|v(X)|^2 dX.$$

Consider the Dirichlet realization $P^h_{flat,D}$ of the operator

$$P^h_{flat} = \left(ih\frac{\partial}{\partial x} - \frac{1}{2}b_0 y\right)^2 + \left(ih\frac{\partial}{\partial y} + \frac{1}{2}b_0 x\right)^2 - hb_0 + h^{1/2}b_2(X)$$

in the space $L^2(\Omega, dX)$. So its quadratic form is given by

$$(P^h_{flat} v, v) = q^{flat}(v) - hb_0\int_\Omega |v(X)|^2 dX + h^{1/2}\int_\Omega b_2(X)|v(X)|^2 dX,$$

where

$$q^{flat}(v) = \int_\Omega \left|\left(ih\frac{\partial}{\partial x} - \frac{1}{2}b_0 y\right) v(X)\right|^2 dX + \int_\Omega \left|\left(ih\frac{\partial}{\partial y} + \frac{1}{2}b_0 x\right) v(X)\right|^2 dX.$$

So we have

$$|(\hat{P}^h_D v, v) - (P^h_{flat} v, v)| \leq |q(v) - q^{flat}(v)| \tag{3.5}$$

$$+ \sum_\alpha \int_\Omega |X|^2|\nabla^h_\alpha v(X)|^2 dX + h\int_\Omega |X|^2|v(X)|^2 dX + h^{1/2}\int_\Omega |X|^3|v(X)|^2 dX.$$

Finally, we have

$$|q(v) - q^{flat}(v)| \tag{3.6}$$

$$\leq C(q(v))^{1/2}\left[h\left(\int_\Omega |X|^2|v(X)|^2 dX\right)^{1/2} + \left(\int_\Omega |X|^6|v(X)|^2 dX\right)^{1/2}\right].$$

For a fixed $j\in\mathbb{N}$, consider the subspace $V^{h,j}$ of $L^2(\Omega, dx_g)$, generated by all eigenfunctions of the operator P^h_D associated to the eigenvalue $\lambda_\ell(P^h_D)$ with $\ell = 0, 1, \dots, j$. Thus, $V^{h,j}$ is a $(j+1)$-dimensional space such that

$$(P^h_D u_h, u_h) \leq \lambda_j(P^h_D)\|u_h\|^2_{L^2(\Omega, dx_g)}, \quad u_h\in V^{h,j}. \tag{3.7}$$

Moreover, by Lemma 3.2, for any real $k \geq 0$, there exists $C_k > 0$ such that, for any $u_h \in V^{h,j}$,

$$\| |X|^k u_h \|_{L^2(\Omega, dx_g)} \leq C_k (h^{k/2} + h^{(3k+1)/8}) \|u_h\|_{L^2(\Omega, dx_g)}, \tag{3.8}$$

for any $\alpha = 1, 2$ and any $k \geq 0$,

$$\| |X|^k \nabla^h_\alpha u_h \|_{L^2(\Omega, dx_g)} \leq C_k (h^{(k+1)/2} + h^{(3k+5)/8}) \|u_h\|_{L^2(\Omega, dx_g)}. \tag{3.9}$$

By (3.8) and (3.9), for any real $k \geq 0$, there exists $C_k > 0$ such that, for any $v_h \in L^2(\Omega, dX)$ of the form $v_h = |g(X)|^{1/4} u_h$ with $u_h \in V^{h,j}$,

$$\| |X|^k v_h \|_{L^2(\Omega, dX)} \leq C_k (h^{k/2} + h^{(3k+1)/8}) \|v_h\|_{L^2(\Omega, dX)}, \tag{3.10}$$

for any $\alpha = 1, 2$ and for any $k \geq 0$.

$$\begin{aligned} &\left(\int_\Omega |X|^{2k} \left| \left(ih\frac{\partial}{\partial x} - \frac{1}{2} b_0 y \right) v_h(X) \right|^2 dX \right)^{1/2} \\ &\quad + \left(\int_\Omega |X|^{2k} \left| \left(ih\frac{\partial}{\partial y} + \frac{1}{2} b_0 x \right) v_h(X) \right|^2 dX \right)^{1/2} \\ &\qquad \leq C_k (h^{(k+1)/2} + h^{(3k+5)/8}) \|v_h\|_{L^2(\Omega, dX)}. \end{aligned} \tag{3.11}$$

The estimates (3.10) and (3.11) allow us show for any $v_h \in L^2(\Omega, dX)$ of the form $v_h = |g(X)|^{1/4} u_h$ with $u_h \in V^{h,j}$, first, using (3.7), (3.3) and (3.4), that,

$$q(v_h) \leq C h^{3/2} \|v_h\|^2_{L^2(\Omega, dX)},$$

next, using (3.6), that

$$|q(v_h) - q^{flat}(v_h)| \leq C h^2 \|v_h\|^2_{L^2(\Omega, dX)},$$

and finally, using (3.5), that

$$(P^h_{flat} v_h, v_h) \leq (\lambda_j(P^h_D) + C_j h^{15/8}) \|v_h\|^2_{L^2(\Omega, dX)}. \tag{3.12}$$

Let χ be a function from $C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp} \chi \subset \Omega$ and $\chi \equiv 1$ in a neighborhood of zero. By (3.10) and (3.11), it follows that, for any $k \in \mathbb{N}$ there exists $C_k > 0$ such that, for any $v_h \in L^2(\Omega, dX)$ of the form $v_h = |g(X)|^{1/4} u_h$ with $u_h \in V^{h,j}$,

$$\begin{aligned} &\|(1-\chi) v_h\|_{L^2(\Omega, dX)} + \left\|\frac{\partial \chi}{\partial x} v_h\right\|_{L^2(\Omega, dX)} + \left\|\frac{\partial \chi}{\partial y} v_h\right\|_{L^2(\Omega, dX)} \\ &\qquad \leq C_k h^k \|v_h\|_{L^2(\Omega, dX)}. \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &\left(\int_\Omega (1-\chi(X)) \left| \left(ih\frac{\partial}{\partial x} - \frac{1}{2} b_0 y \right) v_h(X) \right|^2 dX \right)^{1/2} \\ &\quad + \left(\int_\Omega (1-\chi(X)) \left| \left(ih\frac{\partial}{\partial y} + \frac{1}{2} b_0 x \right) v_h(X) \right|^2 dX \right)^{1/2} \\ &\qquad \leq C_k h^k \|v_h\|_{L^2(\Omega, dX)}. \end{aligned} \tag{3.14}$$

Using (3.13) and (3.14), it is easy to check that, for any $k>0$, there exists $C_k>0$ such that

$$|(P^h_{flat}(\chi v_h),\chi v_h)-(P^h_{flat}v_h,v_h)|\leq C_kh^k\|v_h\|^2. \tag{3.15}$$

Consider the self-adjoint realization of the operator P^h_{flat} in $L^2(\mathbb{R}^2,dX)$. We will the same notation P^h_{flat} for this operator. Consider the $(j+1)$-dimensional subspace $W^{h,j}$ of $C^\infty_c(\mathbb{R}^2)$, which consists of all functions $w_h\in C^\infty_c(\mathbb{R}^2)$ of the form $w_h=\chi|g(X)|^{1/4}u_h$ with $u_h\in V^{h,j}$. By (3.12) and (3.15), it follows that, for any $w_h\in W^{h,j}$,

$$(P^h_{flat}w_h,w_h)\leq(\lambda_j(P^h_D)+C_jh^{15/8})\|w_h\|^2_{L^2(\Omega,dX)}.$$

By the mini-max principle, this immediately implies that, for any $j>0$, there exists $C_j>0$ such that, for j-th eigenvalue $\lambda_j(P^h_{flat})$ of P^h_{flat}, we have

$$\lambda_j(P^h_{flat})\leq\lambda_j(P^h_D)+C_jh^{15/8}. \tag{3.16}$$

It remains to recall that the eigenvalues of the Schrödinger operator with constant magnetic field and positive quadratic potential in $\mathbb{R}^n$ can be computed explicitly. More precisely (see, for instance, [26, Theorem 2.2]), the eigenvalues of the operator

$$H_{b,K}=\left(i\frac{\partial}{\partial x}-\frac{1}{2}by\right)^2+\left(i\frac{\partial}{\partial y}+\frac{1}{2}bx\right)^2+\sum_{ij}K_{ij}X_iX_j$$

are given by

$$\lambda_{n_1n_2}=(2n_1+1)s_1+(2n_2+1)s_2,\quad n_1,n_2\in\mathbb{N},$$

where

$$s_1=\frac{1}{\sqrt{2}}\left[t_K+b^2-[(t_K+b^2)^2-4d_K]^{1/2}\right]^{1/2},$$

$$s_2=\frac{1}{\sqrt{2}}\left[t_K+b^2+[(t_K+b^2)^2-4d_K]^{1/2}\right]^{1/2},$$

and

$$t_K=\operatorname{Tr}K,\quad d_K=\det K.$$

Applying this formula to the operator P^h_{flat}, we obtain that its eigenvalues have the form:

$$\lambda_{n_1n_2}=(2n_1+1)s_1+(2n_2+1)s_2-hb_0,\quad n_1,n_2\in\mathbb{N},$$

where

$$\begin{aligned}s_1&=\frac{h}{\sqrt{2}}\left[h^{1/2}t+b_0^2-[(h^{1/2}t+b_0^2)^2-4h^{1/2}d]^{1/2}\right]^{1/2}\\&=d^{1/2}b_0^{-1}h^{3/2}+O(h^2),\end{aligned}$$

and

$$\begin{aligned}s_2&=\frac{h}{\sqrt{2}}\left[h^{1/2}t+b_0^2+[(h^{1/2}t+b_0^2)^2-4h^{1/2}d]^{1/2}\right]^{1/2}\\&=hb_0+\frac{1}{2}tb_0^{-1}h^{3/2}+O(h^2).\end{aligned}$$

Thus, we obtain

$$\lambda_{n_1n_2}=2n_2hb_0+\left[n_1\frac{2d^{1/2}}{b_0}+n_2\frac{t}{b_0}+\frac{a^2}{2b_0}\right]h^{3/2}+O(h^2),\quad n_1,n_2\in\mathbb{N}.$$

For jth eigenvalue $\lambda_j(P^h_{flat})$ of P^h_{flat}, we obtain

$$\lambda_j(P^h_{flat}) = \left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] h^{3/2} + O(h^2). \tag{3.17}$$

Combining (3.16) and (3.17), we immediately complete the proof of Theorem 3.1.

Proof of Theorem 1.2. Fix $j \in \mathbb{N}$. By Theorem 1.1, there exist $C > 0$ and $h_0 > 0$ such that, for any $h \in (0, h_0]$,

$$I_j \bigcap \operatorname{Spec}(H^h) = \{\lambda_j(H^h)\},$$

where

$$I_j = \left(hb_0 + h^2 \left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] - Ch^{19/8}, hb_0 + h^2 \left[\frac{2d^{1/2}}{b_0}j + \frac{a^2}{2b_0}\right] + Ch^{5/2}\right).$$

On the other hand, by Corollary 2.2, for any natural N, there exist $C' > 0$ and $h_0' > 0$ such that, for any $h \in (0, h_0']$,

$$\operatorname{dist}(\mu^h_{j0N}, \operatorname{Spec}(H^h)) \leq C' h^{\frac{N+3}{2}}.$$

Without loss of generality, we can assume that, for any $h \in (0, \min(h_0, h_0')]$,

$$(\mu^h_{j0N} - C' h^{\frac{N+3}{2}}, \mu^h_{j0N} + C' h^{\frac{N+3}{2}}) \cap I_\ell = \emptyset, \forall \ell \neq j.$$

Hence, for any $h \in (0, \min(h_0, h_0')]$, $\lambda_j(H^h)$ is the point of $\operatorname{Spec}(H^h)$, closest to μ^h_{j0N}. It follows that

$$|\lambda_j(H^h) - \mu^h_{j0N}| \leq C' h^{\frac{N+3}{2}}, \quad h \in (0, \min(h_0, h_0')]\,,$$

that proves (1.1) with $\alpha_{j,\ell} = \mu_{j,0,\ell}$. ☐

4. Periodic case and spectral gaps

In this Section, we apply the results of Section 2 to the problem of existence of gaps in the spectrum of a periodic magnetic Schrödinger operator. Some related results on spectral gaps for periodic magnetic Schrödinger operators can be found in [2, 5, 15, 16, 17, 18, 19, 20, 21, 22, 24, 28] (see also the references therein).

Let M be a two-dimensional noncompact oriented manifold of dimension $n \geq 2$ equipped with a properly discontinuous action of a finitely generated, discrete group Γ such that M/Γ is compact. Suppose that $H^1(M, \mathbb{R}) = 0$, i.e. any closed 1-form on M is exact. Let g be a Γ-invariant Riemannian metric and $\mathbf{B}$ a real-valued Γ-invariant closed 2-form on M. Assume that $\mathbf{B}$ is exact and choose a real-valued 1-form $\mathbf{A}$ on M such that $d\mathbf{A} = \mathbf{B}$. Write $\mathbf{B} = b\,dx_g$, where $b \in C^\infty(M)$ and dx_g is the Riemannian volume form. Let

$$b_0 = \min_{x \in M} b(x).$$

Assume that there exist a (connected) fundamental domain $\mathcal{F}$ and a constant $\epsilon_0 > 0$ such that

$$b(x) \geq b_0 + \epsilon_0, \quad x \in \partial\mathcal{F}.$$

We will consider the magnetic Schrödinger operator H^h as an unbounded self-adjoint operator in the Hilbert space $L^2(M)$. Using the results of [6], one can immediately derive from Theorem 2.1 the following result on existence of gaps in the spectrum of H^h in the semiclassical limit.

We will use the above notation

$$t = \operatorname{Tr}\left(\frac{1}{2}\operatorname{Hess} b(x_0)\right), \quad d = \det\left(\frac{1}{2}\operatorname{Hess} b(x_0)\right).$$

For any $k \in \mathbb{N}$, put

$$c_k = (2k+1)\frac{d^{1/2}}{b_0} + (2k^2+2k+1)\frac{t}{2b_0} + \frac{1}{2}(k^2+k)R(x_0).$$

Theorem 4.1. *Assume that $b_0 > 0$ and there exist $x_0 \in \mathcal{F}$ and $C > 0$ such that for all x in some neighborhood of x_0 the estimates hold:*

$$C^{-1}\, d(x, x_0)^2 \le b(x) - b_0 \le C\, d(x, x_0)^2 .$$

Then, for any natural k and N, there exist $C_{k,N} > c_k$ and $h_{k,N} > 0$ such that the spectrum of H^h in the interval

$$[(2k+1)hb_0 + h^2 c_k, (2k+1)hb_0 + h^2 C_{k,N}]$$

has at least N gaps for any $h \in (0, h_{k,N}]$.

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Département de Mathématiques, Bâtiment 425, Univ Paris-Sud et CNRS, F-91405 Orsay Cédex, France

E-mail address: Bernard.Helffer@math.u-psud.fr

Institute of Mathematics, Russian Academy of Sciences, 112 Chernyshevsky str. 450008 Ufa, Russia

E-mail address: yurikor@matem.anrb.ru