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ON DIFFERENT EXTREMAL BASES FOR $\mathbb{C}$-CONVEX DOMAINS

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ABSTRACT. We discuss some extremal bases for $\mathbb{C}$-convex domains.

1. INTRODUCTION

In order to estimate the Bergman kernel (on the diagonal) and the Carathéodory, Bergman, and Kobayashi metric on convex (or linearly convex) domains special coordinates near the boundary were introduced by J.-H. Chen [Che 89], in his Ph. D. dissertation, and by J. D. McNeal [McN 92, McN 94]. A lot of further studies were based on this orthonormal basis introduced in that papers, see e.g. [MS 94, MS 97, Gau 97, McN 01]. We will call this basis a *maximal basis*; a detailed construction, which also justifies the name, will be described later. On the other side, for the same purpose a *minimal basis* was introduced, for example, in the papers [Hef 02, Con 02, NP 03, Hef 04, DF 06, NPZ 09]. For a general notion of extremal basis see [CD 08]. Looking more carefully at the work based on the maximal basis it can be seen that from the very beginning a property of that basis is deeply used that was never proved. In this note we will present an example showing that exactly this statement is not true for the maximal basis. But it is true for the minimal one. This property may be phrased by saying that certain vectors connected with this basis are orthogonal to certain complex tangent planes; details will be given later. Starting with this observation it should be asked whether the

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estimates for invariant metrics given with help of the maximal basis remain to be true. In this note we show that the answer is positive.

Description of the two extremal bases for a domain $D \subset \mathbb{C}^n$ containing no complex lines.

(a) *Maximal basis.* Fix a point $q \in D$. Then choose a boundary point $p_1 \in \partial D$ such that $m_1 := \|p_1 - q\| = d_D(q)$, where $d_D(q)$ denotes the Euclidean boundary distance of the point q. Put $a_1 := (p_1 - q)/\|p_1 - q\|$. Note that the point p_1 is not in general uniquely determined. Denote by H_1 the affine hyperplane through q which is orthogonal to the vector a_1, i.e. $H_1 = q + \operatorname{span}\{a_1\}^\perp$. Put $D_2 := D \cap H_1$. Choose a unit vector $a_2 \in \operatorname{span}\{a_1\}^\perp$ and a boundary point $p_2 \in \partial D \cap H_1$ such that $m_2 := \operatorname{dist}(q, a_2, \partial D) = \sup\{d_D(q; a)\}$, where the supremum is taken over all unit vectors a in $\operatorname{span}\{a_1\}^\perp$, and $p_2 = q + m_2 a_2$. Here $d_D(q; a) := \sup\{r > 0 : q + r\mathbb{D}a \subset D\}$ denotes the boundary distance of q in direction of a and $\mathbb{D}$ is the open unit disc in $\mathbb{C}$. In the next step put $H_2 := q + \operatorname{span}\{a_1, a_2\}^\perp$; H_2 is the affine $(n-2)$-dimensional plane through q orthogonal to $\operatorname{span}\{a_1, a_2\}$. Define $D_3 := H_2 \cap D$ and continue this procedure, which finally leads to an orthonormal basis $a_1, \dots, a_n$, which is called a *maximal basis* of D at q, to a sequence of positive numbers $m_2 \geq \cdots \geq m_n$, and to boundary points $p_1, \dots, p_n$ with $p_j = q + m_j a_j$ for all j's. Obviously, in this general context this basis depend on q and it is not in general uniquely determined.

(b) *Minimal basis*: Let q, $e_1 := a_1$, $s_1 := m_1$, $\widetilde{p}_1 := p_1$, where the a_1, p_1, and m_1 are taken from the former construction. Let H_1 be also as above. Define a new boundary point $\widetilde{p}_2 = q + s_2 e_2$, where $e_2 \in \operatorname{span}\{e_1\}^\perp$ is an unit vector and $s_2 = \operatorname{dist}(q, \partial_{H_1}(D \cap H_1))$. Note that in this construction, opposite to the above one, $\widetilde{p}_2$ is chosen to be a nearest boundary point of $\partial D \cap H_1$ to q. Then put $H_2 := q + \operatorname{span}\{e_1, e_2\}^\perp$; H_2 is the affine $(n-2)$-dimensional plane through q orthogonal to $\operatorname{span}\{e_1, e_2\}$. Define $D_3 := H_2 \cap D$ and continue now this procedure for H_3 by always taking the nearest boundary point. This finally leads to an orthonormal basis $e_1, \dots, e_n$, which is called the *minimal basis* of D at q, to a sequence of positive numbers $s_1 \leq s_2 \leq \cdots \leq s_n$, and to boundary points $\widetilde{p}_1, \dots, \widetilde{p}_n$ of D with $\widetilde{p}_j = q + p_j e_j$, $1 \leq j \leq n$. As above, this basis depend on q and it is, in general, not uniquely determined.

Assume now that, in addition, D is convex and $\mathcal{C}^\infty$-smooth near a boundary point p_1 (of finite type). Let r be its boundary function. Then the property indicated in the introduction can be described as follows: Fix $q \in D$ on the inner normal at p_1, sufficiently near to p_1, and take the coordinate system given by the maximal basis at q, i.e.

take $q = 0$ and write any point z of $\mathbb{C}^n$ as $z = \sum_{j=1}^n w_j a_j$. Then it is claimed (see, for example, [Che 89, Proposition 2.2 (ii)] and [McN 92, Proposition 3.1 (i)]) that

$$\frac{\partial r(p_k)}{\partial w_j} = 0, \quad j = k+1, \ldots, n. \quad (*)$$

In the original coordinate system this property reads as

$$\sum_{s=1}^n \frac{\partial r(p_k)}{\partial z_s} a_{j,s}, \quad j = k+1, \ldots, n.$$

Therefore, an equivalent form to state (*) is to say that the vectors a_j, $j = k+1, \ldots, n$, belong to the complex tangent space $T^{\mathbb{C}}_{p_k}(\partial D)$ or that $T^{\mathbb{C}}_{p_k}(\partial D) \cap \operatorname{span}\{a_1, \ldots, a_k\}^{\perp} = \operatorname{span}\{a_{k+1}, \ldots, a_n\}$. We should point out that exactly the property (*) is the basis of the arguments in those papers dealing with maximal bases (minimal bases have this crucial property). But, as the following example will show, (*) is not true near the boundary of a domain in $\mathbb{C}^3$. Nevertheless, in section 3 it will be proved that the estimates obtained in terms of the maximal basis remain to be true.

2. An Example

Let β_1 and β_2 be real numbers with $0 < \beta_2 < \beta_1 < 1$. Define

$$D := \{z \in \mathbb{C}^2 \times \mathbb{C} : \rho(z) + |z_3|^2 < 1\},$$

where $\rho(z) = x_1^2 + \beta_1 y_1^2 + x_2^2 + \beta_2 y_2^2$. Note that D is a strictly (pseudo)convex domain with real-analytic boundary. Fix $q = (0, 0, \delta)$, $0 < \delta < 1$. Then following the construction of the maximal basis of D at q leads to $m_1 = 1 - \delta$ and $a_1 = p_1 = (0, 0, 1)$. In the next step the construction gives the domain

$$D_\delta := \{z \in \mathbb{C}^2 : \rho(z) < 1 - \delta^2\}.$$

Note that D_δ is up to a dilatation D_0. So it suffices to study D_0. Put

$$\mathcal{T} := \{b \in \mathbb{C}^2 : \frac{\partial \rho(b)}{\partial z_1}(-\bar{b}_2) + \frac{\partial \rho(b)}{\partial z_2}(\bar{b}_1) = 0\}.$$

Lemma 2.1. $\mathcal{T} = \{b \in \mathbb{C}^2 : b_1 = 0$ *or* $b_2 = 0$ *or* $\operatorname{Im} b_1 = \operatorname{Im} b_2 = 0\}$.

Proof. Simple calculations show that $b \in \mathcal{T}$ if and only if

$$(\beta_1 - \beta_2) \operatorname{Im} b_1 \operatorname{Im} b_2 = 0$$
$$(1 - \beta_1) \operatorname{Im} b_1 \operatorname{Re} b_2 = (1 - \beta_2) \operatorname{Im} b_2 \operatorname{Re} b_1,$$

from which the statement of the lemma follows. □

The next result shows that the property (*) is, in general, not true for a maximal basis. Let $p_2 \in \partial D_0$ be such that

$$\frac{d_{D_0}(0;p_2)}{||p_2||} = m_2 = \sup_{a\in\mathbb{C}^2, ||a||=1} d_{D_0}(0;a).$$

Proposition 2.2. $p_2 \not\in \mathcal{T}$.

Proof. Let $b \in \mathcal{T}$ be a unit vector. Observe that $\rho(re^{i\alpha}b) < 1$ for all $\alpha \in \mathbb{R}$ if and only if $r^2R(b) < 1$, where $R(b) := \max\{\rho(e^{i\alpha}b) : \alpha \in \mathbb{R}\}$. Therefore, $d_{D_0}(0;b) = 1/\sqrt{R(b)}$. Write $b = (e^{i\varphi_1}\cos\Theta, e^{i\varphi_2}\sin\Theta)$, where $0 \le \Theta < 2\pi$ and $0 \le \varphi_1, \varphi_2 \le \pi/2$. According to Lemma 2.1, there are three possibilities for b:

- $\Theta = 0$ or $\Theta = \pi$: $\rho(e^{i\alpha}b) = \cos^2(\alpha + \varphi_1) + \beta_1 \sin^2(\alpha + \varphi_1)$.
- $\Theta = \pi/2$ or $\Theta = 3\pi/2$: $\rho(e^{i\alpha}b) = \cos^2(\alpha + \varphi_2) + \beta_2 \sin^2(\alpha + \varphi_2)$.
- $\varphi_1 = \varphi_2 = 0$: $\rho(e^{i\alpha}b) = \cos^2\alpha + \sin^2\alpha(\beta_1\cos^2\Theta + \beta_2\sin^2\Theta)$.

Hence $R(b) = 1$ in all the three cases.

On the other hand, there are unit vectors $b^* \in \mathbb{C}^2$ with $R(b^*) < 1$ which implies that $p_2 \not\in \mathcal{T}$. To define b^*, take $\Theta := \pi/4$ $\varphi_1 := 0$ and $\varphi_2 := \pi/2$. Then $2\rho(e^{i\alpha}b^*) = 1+\beta_2+(\beta_1-\beta_2)\sin^2\alpha$. Since $\beta_1 < \beta_2 < 1$, it follows that $R(b^*) = \frac{1+\beta_2}{2} < 1$. ☐

3. Estimates and localization

Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain containing no complex lines, i.e. any non-empty intersection with a complex line is biholomorphic to $\mathbb{D}$ (cf. [APS 04, Hör 94]). For $z \in D$ denote by $e_1(z), \dots, e_n(z)$ a minimal basis at z and by $a_1(z), \dots, a_n(z)$ a reordered maximal basis at z which means that the new $a_1(z)$ is the old one, but then $a_2(z) = a_n$, $a_3(z) = a_{n-1}$, etc. Let $s_1(z) \le \dots \le s_n(z)$ and $m_1(z) \le \dots \le m_n(z)$ be the respective numbers (recall that $s_1(z) = m_1(z) = d_D(z)$). Set $s_D(z) := \prod_{j=1}^n s_j(z)$ and $m_D(z) := \prod_{j=1}^n m_j(z)$. Moreover, denote by $K_D(z)$ and $F_D(z;X)$ the Bergman kernel and any of the invariant metrics of D, respectively. For $X \in \mathbb{C}^n$, set

$$E_D(z;X) := \sum_{j=1}^n \frac{|\langle X, e_j(z)\rangle|}{s_j(z)}, \quad A_D(z;X) := \sum_{j=1}^n \frac{|\langle X, a_j(z)\rangle|}{m_j(z)}.$$

We shall write $f(z) \lesssim g(z)$ if $f(z) \le cg(z)$ for some constant $c > 0$ depending only on n; $f(z) \sim g(z)$ means that $f(z) \lesssim g(z) \lesssim f(z)$. By [NPZ 09], we know that

$$K_D(z) \sim 1/s_D^2(z), \quad F_D(z;X) \sim E_D(z;X) \sim 1/d_D(z;X)$$

(for weaker versions of these results, see [NP 03, Blu 05, Lie 05]). For short, sometimes we shall omit the arguments z and X. It follows by

[NPZ 09, Lemma 15] that

$$K_D \lesssim 1/m_D^2, \quad F_D \lesssim A_D.$$

In particular,

$$1/d_D(z;X) \sim E_D(z;X) \lesssim A_D(z;X)$$

The main consequence of the (wrong) property $(*)$ for maximal bases of a smooth convex bounded domain of finite type is the fact that

$$A_D(z;X) \sim_D 1/d_D(z;X),$$

where the constant in $\sim_D$ depends on D. Using this fact, it is shown in [Che 89, McN 94, McN 01] that

$$K_D \sim_D 1/m_D^2, \quad F_D \sim_D A_D$$

The following two propositions imply that fortunately these estimates remain to be true.

The first one is contained in [Hef 04] for the case of a smooth convex bounded domain of finite type. The proof there invokes the estimate $1/d_D(z;X) \sim_D A_D(z;X)$ but, in fact, it uses only the trivial part of this estimate: $1/d_D(z;X) \lesssim_D A_D(z;X)$.

Proposition 3.1. *Let $D \subset \mathbb{C}^n$ be a $\mathbb{C}$-convex domain containing no complex lines. Then $m_j(z) \sim s_j(z)$, $j = 1, \dots, n$, $z \in D$.*

Proof. Fix $z \in D$ and put $m_j = m_j(z)$, $s_j = s_j(z)$ First, we shall prove that $m_j \lesssim s_j$. Since $E_D \lesssim A_D$, it is enough show that if $E_D \le cA_D$, then $m_j \le c's_j$, where $c' = n!c$.

Expanding the determinant of the matrix of the unitary transformation between the bases, it follows that $\prod_{j=1}^n |\langle a_j, e_{\sigma(j)}\rangle| \ge 1/n!$ for some permutation σ of $\{1, \dots, n\}$ In particular, $|\langle a_j, e_{\sigma(j)}\rangle| \ge 1/n!$. Then $E_D(z;a_j) \le cA(z;a_j)$ implies that $m_j \le c's_{\sigma(j)}$.

Assume now that $c's_k < m_k$ for some k. Then

$$c's_k < m_k \le m_j \le c's_{\sigma(j)}, \quad j \ge k.$$

This shows that $\sigma(j) > k$ for any $j \ge k$, which is a contradiction, since σ is a permutation.

The above arguments show that $\widetilde{s}_j \sim s_j$, where $\widetilde{s}_j$ are the respective numbers for another minimal basis at z. So we may assume that $e_1 = a_1$. We know that $m_1 = p_1$. It remains to prove that $m_k \gtrsim s_k$ for $k \ge 2$. Choose a unit vector a'_k in $\operatorname{span}(e_k, \dots, e_n)$ orthogonal to $a_{k+1}, \dots a_n$ ($a'_n = e_n$ if $k = n$). Then a'_k is also orthogonal to $a_1 = e_1$. Hence

$m_k \geq d_D(z; a'_k)$ (by construction of maximal basis). On the other hand, since a'_k is orthogonal to $e_1, \dots, e_{k-1}$, then

$$\frac{1}{d_D(z; a'_k)} \sim E_D(z; a'_k) = \sum_{j=k}^{n} \frac{\langle |a'_k, e_j\rangle|}{s_j} \lesssim \frac{1}{s_k}.$$

So $m_k \geq d_D(z; a'_k) \gtrsim s_k$. □

Proposition 3.2. *Let D be as in Proposition 3.1. Then $A_D \sim E_D$.*

Proof. Using the inequality $E_D \lesssim A_D$ and Proposition 3.1, it is enough to show that for any k,

$$\frac{|\langle X, a_k\rangle|}{s_k} \lesssim E_D.$$

Set $b_{jk} = \langle a_j, e_k\rangle$. Since

$$\frac{1}{s_j} \sim \frac{1}{d_D(z; a_j)} \sim E_D(z; a_j) \geq \frac{|b_{jk}|}{s_k},$$

it follows that $|b_{jk}| \lesssim s_k/s_j$. The unitary matrix $B = (b_{jk})$ transforms the basis $e_1, \dots, e_n$ to the basis $a_1, \dots, a_n$. For the inverse matrix $C = (c_{jk})$ we have

$$|c_{jk}| \leq \sum_{\sigma} |b_{1\sigma(1)} \dots b_{k-1,\sigma(k-1)} b_{k+1,\sigma(k+1)} \dots b_{n,\sigma(n)}|$$

$$\lesssim \sum_{\sigma} \frac{s_{\sigma(1)}}{s_1} \dots \frac{s_{\sigma(k-1)}}{s_{k-1}} \frac{s_{\sigma(k+1)}}{s_{k+1}} \dots \frac{s_{\sigma(n)}}{s_n} = \sum_{\sigma} \frac{s_k}{s_j} = (n-1)! \frac{s_k}{s_j},$$

where σ runs over all permutations of $\{1, \dots, j-1, j+1, \dots, n\}$.

It follows that

$$\frac{|\langle X, a_k\rangle|}{s_k} \leq \sum_{j=1}^{n} |\langle X, e_j\rangle| \frac{|c_{jk}|}{s_k} \lesssim E_D.$$

□

Remark. Replace the construction of a maximal basis by the following: choose "minimal" discs on steps $1, \dots, k$ and "maximal" discs on steps $k+1, \dots, n-1$ (the n-th choice is unique); $k = n-1$ provides a minimal basis, $k = 1$ – a maximal basis, and $k = 0$ – a basis with no "minimal" discs. Note that Propositions 3.1 and 3.2 remain true with A_D expressed in the new basis. (This construction has an obvious real analog).

Let a be a boundary point of a bounded domains $D \subset \mathbb{C}^n$. It is easy to see that for any neighborhood U of a one has that $s_D \sim_* s_{D\cap U}$, and $E_D \sim_* E_{D\cap U}$ near a, where the constant in $\sim_*$ depends on D and

U (the same holds for m_D and A_D). Assume that D is $\mathcal{C}^{2n}$-smooth and (weakly) linearly convex near a (cf. [APS 04, Hör 94] for this and other notions of convexity). Then Proposition 3.3 below and the localization principle for the Kobayashi metric κ_D (cf. [JP 93]) imply that $\kappa_D \sim_D E_D$ near a (the constant in $\sim_D$ depends on D). If, in addition, D is pseudoconvex, then the same principle for K_D and the Bergman metric b_D (cf. [JP 93]) implies that $K_D \sim_D 1/s_D^2$ and $b_D \sim_D E_D$ near a. Assume that a is a $\mathcal{C}^\infty$-smooth finite type point (but D not necessarily bounded). Then a is a local holomorphic peak point (see [DF 03]), and strong localization principles (cf. [Nik 02]) imply that $\kappa_D \sim E_D(= E_{D\cap U})$ and (if D is pseudoconvex) $K_D \sim 1/s_D^2(= 1/s_{D\cap U}^2)$, $b_D \sim E_D$ near a.

Proposition 3.3. *Let a be a $\mathcal{C}^k$-smooth boundary point ($2 \le k \le \infty$) of a domain $D \subset \mathbb{C}^n$ with the following property: for any $b \in \partial D$ near a there are a neighborhood U_b such that $D \cap U_b \cap T_b^{\mathbb{C}}(\partial D) = \varnothing$. Then there is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain $G \subset D$ and a neighborhood U of a such that $D \cap U = G \cap U$.*

Proof. We may assume that $a = 0$. Denote by $H_f(z; X)$ the Hessian of a C^2-smooth function f. Set $B_s := \mathbb{B}_n(0, s)$ $(s > 0)$ and

$$r(z) := \begin{cases} -d_D(z), & z \in D \\ d_D(z), & z \notin D. \end{cases}$$

It follows by the differential inequality for r^2 in the proof of [APS 04, Proposition 2.5.18 (ii)⇒(iii)] that there is an $\varepsilon > 0$ such r is a $\mathcal{C}^k$-smooth defining function of D in $B_{3\varepsilon}$ and $H_r(z; X) \ge 0$ if $\langle \partial r(z), \overline{X} \rangle = 0$ and $z \in D \cap B_{2\varepsilon}$. Then the proof of [DF 77, Lemma 1] implies that there is a $c > 0$ such that $H_r(z; X) \ge c|X| \cdot |\langle \partial r(z), \overline{X} \rangle|$, $z \in D \cap B_{2\varepsilon}$. We may assume that $2\varepsilon c \le 1$ and $D \cap B_\varepsilon$ is connected. Choose now a smooth function χ such that $\chi(x) = 0$ if $x \le \varepsilon^2$ and $\chi'(x), \chi''(x) > 0$ if $x > \varepsilon^2$. Set $\theta(z) = \chi(|z|^2)$. We may find a $C \ge 1/2$ such that

$$B_{2\varepsilon} \Supset G' := \{z \in B_{2\varepsilon} : 0 > \rho(z) = r(z) + C\theta(z)\} \subset D.$$

Further, the inequalities $2c\varepsilon \le 1$ and $|\langle \partial\theta(z), \overline{X} \rangle| \le \chi'(|z|^2)|z| \cdot |X|$ give $\chi'(|z|^2)|X| > c|\langle \partial\theta(z), \overline{X} \rangle|$ if $z \in B_{2\varepsilon} \setminus \overline{B_\varepsilon}$ and $X \ne 0$. This together with

$$H_r(z; X) \ge -c|X| \cdot |\langle \partial r(z), \overline{X} \rangle|, \quad z \in G',$$

$$H_\rho(z; X) = H_r(z; X) + 4C\chi''(|z|^2)\mathrm{Re}^2\langle z, X \rangle + 2C\chi'(|z|^2)|X|^2,$$

$2C \ge 1$, and the triangle inequality show that

$$H_\rho(z; X) \ge -c|X| \cdot |\langle \partial\rho(z), \overline{X} \rangle|, \quad z \in \overline{G'}.$$

Moreover, the last inequality is strict if $z \in \overline{G'} \setminus \overline{B_\varepsilon}$ and $X \neq 0$. This implies that $\partial\rho \neq 0$ on $\partial G' \setminus \overline{B_\varepsilon}$; (otherwise, ρ will attain local minima at some point of this set which is impossible). So $\partial\rho \neq 0$ on $\partial G'$.

Let G be the connected component of G' containing $D \cap B_\varepsilon$. Then [APS 04, Proposition 2.5.18] (see also [Hör 94, Proposition 4.6.4]) implies that G is a $\mathcal{C}^k$-smooth $\mathbb{C}$-convex domain. $\square$

REFERENCES

[APS 04] M. Andersson, M. Passare, R. Sigurdsson, *Complex convexity and analytic functionals*, Birkhäuser, Basel–Boston–Berlin, 2004.

[Blu 05] S. Blumberg, *Das Randverhalten des Bergman-Kerns und der Bergman-Metrik auf lineal konvexen Gebieten endlichen Typs*, Dissertation, BUGH Wuppertal, 2005; http://elpub.bib.uni-wuppertal.de.

[CD 08] P. Charpantier, Y. Dupan, *Extremal basis, geometrically separated domains and applications*, arXiv:0810.1889.

[Che 89] J.-H. Chen, *Estimates of the Invariant Metrics on Convex Domains*, Ph. D. dissertation, Purdue University., 1989.

[Con 02] M. Conrad, *Nicht-isotrope Abschätzungen für lineal konvexe Gebiete endlichen Types*, Dissertation, BUGH Wuppertal, 2002; http://elpub.bib.uni-wuppertal.de.

[DF 77] K. Diederich, J. E. Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. 39 (1977), 129–141.

[DF 03] K. Diederich, J. E. Fornaess, *Lineally convex domains of finite type: holomorphic support functions*, manuscripta math. 112 (2003), 403–431.

[DF 06] K. Diederich, J. E. Fornaess, *Hölder Estimaztes on Lineally Convex Domains of Finite Type*, Michigan Math. J. 54 (2006), 341–352.

[Gau 97] M. Gaussier, *Characterization of convex domains with noncompact automorphismus group*, Michigan Math. J. 44 (1997), 375–388.

[Hef 02] T. Hefer, *Hölder and L^p estimates for $\overline{\partial}$ on convex domains of finite type depending on Catlin's multitype*, Math. Z. 242 (2002), 367–398.

[Hef 04] T. Hefer, *Extremal bases and Hölder estimates for $\overline{\partial}$ on convex domains of finite type*, Michigan Math. J. 52 (2004), 573–602.

[Hör 94] L. Hörmander, *Notions of convexity*, Birkhäuser, Basel–Boston–Berlin, 1994.

[JP 93] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis*, de Gruyter, Berlin–New–York, 1993.

[Lie 05] M. Lieder, *Das Randverhalten der Kobayashi und Carathéodory-Metrik auf lineal konvexen Gebieten endlichen Typs*, Dissertation, BUGH Wuppertal, 2005; http://elpub.bib.uni-wuppertal.de.

[McN 92] J. D. McNeal, *Convex domains of finite type*, J. Funct. Anal. 108 (1992), 361–373.

[McN 94] J. D. McNeal, *Estimates on the Bergman Kernels of Convex domains*, Adv. Math. 109 (1994), 108–139.

[MS 94] J. D. McNeal, E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. 73 (1994), 177–199.

[MS 97] J. D. McNeal, E. M. Stein, *The Szegö projection on convex domains*, Math. Z. 224 (1997), 519–553.

[McN 01] J. D. McNeal, *Invariant metric estimates for* $\overline{\partial}$ *on some pseudoconvex domains*, Ark. Mat. 39 (2001), 121–136.

[Nik 02] N. Nikolov, *Localization of invariant metrics*, Arch. Math. 79 (2002), 67–73.

[NP 03] N. Nikolov, P. Pflug, *Estimates for the Bergman kernel and metric on convex domains in* $\mathbb{C}^n$, Ann. Polon. Math.81 (2003), 73–78.

[NPZ 09] N. Nikolov, P. Pflug, W. Zwonek, *Estimates for invariant metrics on* $\mathbb{C}$*-convex domains*, Trans. Amer. Math. Soc. (to appear); arXiv:0812.2793.

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