ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Boltzmanngasse 9
A-1090 Wien, Austria

Compactness for the $\overline{\partial}$–Neumann problem – a Functional Analysis Approach.

Friedrich Haslinger

Vienna, Preprint ESI 2208 (2009) December 18, 2009

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available online at http://www.esi.ac.at

COMPACTNESS FOR THE $\overline{\partial}$ - NEUMANN PROBLEM - A FUNCTIONAL ANALYSIS APPROACH.

FRIEDRICH HASLINGER

Abstract.
We discuss compactness of the $\overline{\partial}$-Neumann operator in the setting of weighted L^2-spaces on $\mathbb{C}^n$. For this purpose we use a description of relatively compact subsets of L^2- spaces. We also point out how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain and mention an abstract functional analysis characterization of compactness of the $\overline{\partial}$-Neumann operator.

1. Introduction.

In this paper we continue the investigations of [HaHe] concerning existence and compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces over $\mathbb{C}^n$. Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-\varphi} \, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Let

$$\langle f, g \rangle_\varphi = \int_{\mathbb{C}^n} f \, \overline{g} e^{-\varphi} \, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda$$

the norm in $L^2(\mathbb{C}^n, \varphi)$.
We consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \underset{\overline{\partial}^*_\varphi}{\overset{\overline{\partial}}{\rightleftarrows}} L^2_{(0,2)}(\mathbb{C}^n, \varphi),$$

where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j d\overline{z}_j \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = - \sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j.$$

2000 *Mathematics Subject Classification.* Primary 32W05; Secondary 32A36, 35J10.
Key words and phrases. $\overline{\partial}$-Neumann problem, Sobolev spaces, compactness.
Partially supported by the FWF-grant P19147.

The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\,\overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

where the symbol $\Box_\varphi$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}_0^\infty$, i.e., the space of smooth functions with compact support.
$\Box_\varphi$ is a selfadjoint and positive operator, which means that

$$\langle \Box_\varphi f, f\rangle_\varphi \geq 0 \ , \ \text{for } f \in dom(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$Q_\varphi(f,g) = \langle \overline{\partial} f, \overline{\partial} g\rangle_\varphi + \langle \overline{\partial}^*_\varphi f, \overline{\partial}^*_\varphi g\rangle_\varphi, \tag{1.1}$$

for $f, g \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. The weighted $\overline{\partial}$-Neumann operator N_φ is - if it exists - the bounded inverse of $\Box_\varphi$.

We indicate that $f \in dom(\overline{\partial}^*_\varphi)$ if and only if

$$\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} f_j \right) \in L^2(\mathbb{C}^n, \varphi)$$

and that forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ are dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2_\varphi + \|\overline{\partial}^*_\varphi f\|^2_\varphi)^{\frac{1}{2}}$ (see [GaHa]).

Now we suppose that the lowest eigenvalue μ_φ of the Levi - matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

of φ satisfies

$$\liminf_{|z|\to\infty} \mu_\varphi(z) > 0, \quad (*)$$

and mention the Kohn-Morrey formula:

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi}\, d\lambda \tag{1.2}$$

from which we get

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi}\, d\lambda \leq \|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi, \tag{1.3}$$

hence for a plurisubharmonic weight function φ satisfying (*), there is a $C > 0$ such that

$$\|u\|^2_\varphi \leq C(\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi)$$

for each $(0,1)$-form $u \in$dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$.
For the proof see [FS], [GaHa] or [Str].
Now it follows that there exists a uniquely determined $(0,1)$-form
$N_\varphi u \in$dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$ such that

$$\langle u, v\rangle_\varphi = Q_\varphi(N_\varphi u, v) = \langle \overline{\partial} N_\varphi u, \overline{\partial} v\rangle_\varphi + \langle \overline{\partial}^*_\varphi N_\varphi u, \overline{\partial}^*_\varphi v\rangle_\varphi,$$

and that

$$\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi u\|_\varphi^2 \le C_1 \|u\|_\varphi^2 \tag{1.4}$$

which means that

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \operatorname{dom}(\overline{\partial}) \cap \operatorname{dom}(\overline{\partial}_\varphi^*)$$

is continuous in the graph topology, as well as

$$\|N_\varphi u\|_\varphi^2 \le C_2(\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}_\varphi^* N_\varphi u\|_\varphi^2) \le C_3 \|u\|_\varphi^2,$$

where $C_1, C_2, C_3 > 0$ are constants. Hence we get that N_φ is a continuous linear operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself (see also [ChSh]).

We will give a new proof of the main result in [HaHe] using a direct approach, see [B], Corollaire IV.26, where two conditions are given which imply that a subset of an L^2- space is relatively compact. The first of these conditions will correspond to Gårding's inequality (see for instance [F] , [GaHa],) and the second condition corresponds to our assumption on the lowest eigenvalue of the Levi matrix M_φ.

We indicate how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ and finally mention an abstract necessary and sufficient condition for the $\overline{\partial}$-Neumann operator to be compact.

2. Weighted Sobolev spaces

Now we define an appropriate Sobolev space and prove compactness of the corresponding embedding, for related settings see [BDH], [Jo], [KM] .

Definition 2.1. *Let*

$$\mathcal{W}^{Q_\varphi} = \{u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi) \ : \ \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 < \infty\}$$

with norm

$$\|u\|_{Q_\varphi} = (\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2)^{1/2}.$$

Remark: $\mathcal{W}^{Q_\varphi}$ coincides with the form domain $dom(\overline{\partial}) \cap dom(\overline{\partial}_\varphi^*)$ of Q_φ (see [Ga], [GaHa]).

Proposition 2.2. *Suppose that the weight function φ is plurisubharmonic and that the lowest eigenvalue μ_φ of the Levi - matrix M_φ satisfies*

$$\lim_{|z|\to\infty} \mu_\varphi(z) = +\infty . \quad (^{**})$$

Then the embedding

$$j_\varphi : \mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

is compact.

Proof. For $u \in \mathcal{W}^{Q_\varphi}$ we have by 1.3

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 \ge \langle M_\varphi u, u\rangle_\varphi.$$

This implies

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 \ge \int_{\mathbb{C}^n} \mu_\varphi(z)\, |u(z)|^2 \, e^{-\varphi(z)} \, d\lambda(z). \tag{2.1}$$

We show that the unit ball in $\mathcal{W}^{Q_\varphi}$ is relatively compact in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. For this purpose we use the following lemma, see for instance [B] Corollaire IV.26.

Lemma 2.3. *Let $\mathcal{A}$ be a bounded subset of $L^2(\mathbb{C}^n, \varphi)$. Suppose that*
(i) for each $\epsilon > 0$ and for each $R > 0$ there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_{L^2(\mathbb{B}_R,\varphi)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in \mathcal{A}$, where $\tau_h f(z) = f(z+h)$ and $\mathbb{B}_R = \{z \in \mathbb{C}^n : |z| < R\}$;
(ii) for each $\epsilon > 0$ there exists $R > 0$ such that

$$\|f\|_{L^2(\mathbb{C}^n \setminus \mathbb{B}_R,\varphi)} < \epsilon$$

for each $f \in \mathcal{A}$.
Then $\mathcal{A}$ is relatively compact in $L^2(\mathbb{C}^n, \varphi)$.

Remark 2.4. *Conditions (i) and (ii) are also necessary for $\mathcal{A}$ to be relatively compact in $L^2(\mathbb{C}^n, \varphi)$ (see* [B]*).*

First we show that condition (i) of Lemma 2.3 is satisfied in our situation. Let $u = \sum_{j=1}^n u_j \, dz_j$ be a $(0,1)$-form with coefficients in $\mathcal{C}_0^\infty$. For each u_j and for $t \in \mathbb{R}$ and $h = (h_1, \dots, h_n) \in \mathbb{C}^n$ let

$$v_j(t) := u_j(z + th).$$

Note that

$$|v_j'(t)| \leq |h| \left[\sum_{k=1}^n \left(\left| \frac{\partial u_j}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial u_j}{\partial y_k}(z+th) \right|^2 \right) \right]^{1/2},$$

where $z_k = x_k + iy_k$, for $k = 1, \dots, n$. By the fact that

$$u_j(z+h) - u_j(z) = v_j(1) - v_j(0) = \int_0^1 v_j'(t) \, dt$$

we can now estimate for $|h| < R$

$$\int_{\mathbb{B}_R} |\tau_h u_j(z) - u_j(z)|^2 e^{-\varphi(z)} \, d\lambda(z) = \int_{\mathbb{B}_R} |\tau_h(\chi_R u_j)(z) - \chi_R u_j(z)|^2 e^{-\varphi(z)} \, d\lambda(z)$$

$$\leq |h|^2 \int_{\mathbb{B}_R} \left[\int_0^1 \sum_{k=1}^n \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z+th) \right|^2 \right) dt \right] e^{-\varphi(z)} \, d\lambda(z)$$

$$\leq C_{R,\varphi} |h|^2 \int_{\mathbb{B}_{3R}} \sum_{k=1}^n \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z) \right|^2 \right) e^{-\varphi(z)} \, d\lambda(z)$$

for $j = 1, \dots, n$ where χ_R is a $\mathcal{C}^\infty$ cutoff function which is identically 1 on $\mathbb{B}_{2R}$ and zero outside $\mathbb{B}_{3R}$ and by Gårding's inequality for $\mathbb{B}_{3R}$ (see [ChSh], [F], [GaHa])

$$\begin{aligned} \|\chi_R u\|^2_{\varphi,1} &\leq C'_{\varphi,R} \left(\|\overline{\partial}(\chi_R u)\|^2_\varphi + \|\overline{\partial}^*_\varphi(\chi_R u)\|^2_\varphi + \|\chi_R u\|^2_\varphi \right) \\ &\leq C''_{\varphi,R} \left(\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi \right) \end{aligned}$$

we can control the last integral by the norm $\|u\|^2_{Q_\varphi}$. Since we started from the unit ball in $\mathcal{W}^{Q_\varphi}$ we get that condition (i) of Lemma 2.3 is satisfied.

Condition (ii) of Lemma 2.3 is satisfied for the unit ball of $\mathcal{W}^{Q_\varphi}$ since we have

$$\int_{\mathbb{C}^n\setminus\mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\,d\lambda(z) \le \int_{\mathbb{C}^n\setminus\mathbb{B}_R} \frac{\mu_\varphi(z)|u(z)|^2}{\inf\{\mu_\varphi(z) : |z|\ge R\}} e^{-\varphi(z)} d\lambda(z).$$

So formula (2.1) together with assumption (**) shows that

$$\int_{\mathbb{C}^n\setminus\mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\,d\lambda(z) \le \frac{\|u\|^2_{Q_\varphi}}{\inf\{\mu_\varphi(z)\,:\,|z|\ge R\}} < \epsilon,$$

if R is big enough.

□

We are now able to give a short proof of the main result in [HaHe] or [GaHa]

Proposition 2.5. *Let φ be a plurisubharmonic $\mathcal{C}^2$- weight function. If the lowest eigenvalue $\mu_\varphi(z)$ of the Levi - matrix M_φ satisfies (**), then N_φ is compact.*

Proof. By Proposition 2.2, the embedding $\mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ is compact. The inverse N_φ of $\Box_\varphi$ is continuous as an operator from $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ into $\mathcal{W}^{Q_\varphi}$, this follows from 1.4. Therefore we have that N_φ is compact as an operator from $L^2_{(0,1)}(\mathbb{C}^n,\varphi)$ into itself.

□

Now notice that

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n,\varphi) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n,\varphi)$$

can be written in the form

$$N_\varphi = j_\varphi \circ j_\varphi^*\ ,$$

where

$$j_\varphi^* : L^2_{(0,1)}(\mathbb{C}^n,\varphi) \longrightarrow \mathcal{W}^{Q_\varphi}$$

is the adjoint operator to j_φ (see [Str]).
This means that N_φ is compact if and only if j_φ is compact and summarizing the above results we get the following

Proposition 2.6. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function . The $\overline{\partial}$-Neumann operator*

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n,\varphi) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n,\varphi)$$

is compact if and only if for each $\epsilon > 0$ there exists $R > 0$ such that

$$\|u\|_{L^2_{(0,1)}(\mathbb{C}^n\setminus\mathbb{B}_R,\varphi)} < \epsilon$$

for each $u \in \mathcal{W}^{Q_\varphi}$ with

$$\|\overline{\partial}u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \le 1.$$

3. Smoothly bounded pseudoconvex domains and properties (P) and ($\tilde{\text{P}}$)

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Ω satisfies property (P), if or each $M > 0$ there exists a a neighborhood U of $\partial\Omega$ and a plurisubharmonic function $\varphi_M \in \mathcal{C}^2(U)$ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t}_k \geq M \|t\|^2,$$

for all $p \in \partial\Omega$ and for all $t \in \mathbb{C}^n$.

Ω satisfies property ($\tilde{\text{P}}$) if the following holds: there is a constant C such that for all $M > 0$ there exists a $\mathcal{C}^2$ function φ_M in a neighborhood U (depending on M) of $\partial\Omega$ with
(i) $\left|\sum_{j=1}^{n} \frac{\partial \varphi_M}{\partial z_j}(z) t_j\right|^2 \leq C \sum_{j=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k$
and
(ii) $\sum_{j=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \geq M \|t\|^2$,
for all $z \in U$ and for all $t \in \mathbb{C}^n$.

In [C] Catlin showed that condition (P) implies compactness of the $\overline{\partial}$- operator N on $L^2_{(0,1)}(\Omega)$ and McNeal ([McN]) showed that property ($\tilde{\text{P}}$) also implies compactness of the $\overline{\partial}$- operator N on $L^2_{(0,1)}(\Omega)$. It is not difficult to show that property (P) implies property ($\tilde{\text{P}}$), see for instance [Str].
We can now use a similar approach as in section 2 to prove Catlin's result. For this purpose we use the following version of lemma 2.3

Lemma 3.1. *Let $\mathcal{A}$ be a bounded subset of $L^2(\Omega)$. Suppose that*
(i) for each $\epsilon > 0$ and for each $\omega \subset\subset \Omega$ there exists $\delta > 0, \delta < dist(\omega, \Omega^c)$ such that

$$\|\tau_h f - f\|_{L^2(\omega)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in \mathcal{A}$,
(ii) for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\|f\|_{L^2(\Omega \setminus \omega)} < \epsilon$$

for each $f \in \mathcal{A}$.
Then $\mathcal{A}$ is relatively compact in $L^2(\Omega)$.

Remark 3.2. *Conditions (i) and (ii) are also necessary for $\mathcal{A}$ to be relatively compact in $L^2(\Omega)$.*

In order to show that the unit ball in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ satisfies condition (i) of 3.1 we remark that Gårding's inequality holds for $\omega \subset\subset \Omega$ (see section 2). To verify condition (ii) we use property (P) and the following version of the Kohn-Morrey formula

$$\int_\Omega \sum_{j,k=1}^{n} \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi_M} \, d\lambda \leq \|\overline{\partial} u\|^2_{\varphi_M} + \|\overline{\partial}^*_{\varphi_M} u\|^2_{\varphi_M}, \tag{3.1}$$

here we used that Ω is pseudoconvex, which means that the boundary terms in the Kohn-Morrey formula can be neglected. Now we point out that the weighted $\overline{\partial}$-complex is equivalent to the unweighted one and that the expression $\sum_{j=1}^n \frac{\partial \varphi_M}{\partial z_j} u_j$ which appears in $\overline{\partial}^*_{\varphi_M} u$, can be controlled by the complex Hessian $\sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k$, which follows from the fact that property (P) implies property ($\tilde{\text{P}}$) (see [Str]). Of course we also use that the weight φ_M is bounded on $\Omega \subset\subset \mathbb{C}^n$. In this way the same reasoning as in section 2 shows that property (P) implies condition (ii) of lemma 3.1. Therefore condition (P) gives that the unit ball of $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ is relatively compact in $L^2_{(0,1)}(\Omega)$ and hence that the $\overline{\partial}$-Neumann operator is compact.

Now let

$$j : dom(\overline{\partial}) \cap dom(\overline{\partial}^*) \hookrightarrow L^2_{(0,1)}(\Omega)$$

denote the embedding. It follows from [Str] that

$$N = j \circ j^*.$$

Hence N is compact if and only if j is compact, where $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ is endowed with the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$.

Proposition 3.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Let $\mathcal{B}$ denote the unit ball of $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$. The $\overline{\partial}$- Neumann operator N is compact if and only if $\mathcal{B}$ as a subset of $L^2_{(0,1)}(\Omega)$ satisfies the following condition:*
for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\|f\|_{L^2_{(0,1)}(\Omega \setminus \omega)} < \epsilon$$

for each $f \in \mathcal{B}$.

This follows from the above remarks about the embedding j and the fact that the two conditions in 3.1 are also necessary for a bounded set in L^2 to be relatively compact. For a localized version of the above result see [Sa].

Acknowledgement. The author is grateful to the Erwin Schrödinger Institute (ESI), Vienna, where most of this work was carried out and to Mehmet Celik, Bernard Helffer and Emil Straube for valuable discussions during the ESI - program on the $\overline{\partial}$-Neumann operator.

References

[BDH] P. Bolley, M. Dauge and B. Helffer, *Conditions suffisantes pour l'injection compacte d'espace de Sobolev à poids,* Séminaire équation aux dérivées partielles (France), vol.1, Université de Nantes (1989), 1–14.

[B] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Masson , Paris, 1983.

[C] D.W. Catlin, *Global regularity of the $\overline{\partial}$-Neumann operator,* Proc. Symp. Pure Math. **41** (1984), 39-49.

[ChSh] So-Chin Chen and Mei-Chi Shaw, *Partial differential equations in several complex variables*, Studies in Advanced Mathematics, Vol. 19, Amer. Math. Soc. 2001.

[F] G.B. Folland, *Introduction to partial differential equations*, Princeton University Press, Princeton, 1995.

[FS] S. Fu and E.J. Straube, *Compactness in the $\overline{\partial}-$Neumann problem*, Complex Analysis and Geometry (J. McNeal, ed.), Ohio State Math. Res. Inst. Publ. **9** (2001), 141–160.

[Ga] K. Gansberger, Compactness of the $\overline{\partial}$-Neumann operator, Dissertation, University of Vienna, 2009.

[GaHa] K. Gansberger and F. Haslinger, *Compactness estimates for the $\overline{\partial}$- Neumann problem in weighted L^2- spaces*, Proceedings of the conference on Complex Analysis 2008 in honour of Linda Rothschild, Fribourg 2008, to appear.

[HaHe] F. Haslinger and B. Helffer, *Compactness of the solution operator to $\overline{\partial}$ in weighted L^2 - spaces*, J. of Functional Analysis, **243** (2007), 679-697.

[Jo] J. Johnsen, *On the spectral properties of Witten Laplacians, their range projections and Brascamp-Lieb's inequality*, Integral Equations Operator Theory **36** (3), 2000, 288–324.

[KM] J.-M. Kneib and F. Mignot, *Equation de Schmoluchowski généralisée,* Ann. Math. Pura Appl. (IV) **167** (1994), 257–298.

[McN] J.D. McNeal , *A sufficient condition for compactness of the $\overline{\partial}$-Neumann operator*, J. of Functional Analysis, **195** (2002), 190-205.

[Sa] S. Sahutoglu , *Compactness of the $\overline{\partial}$- Neumann problem and Stein neighborhood bases*, Dissertation Texas A & M University, 2006.

[Str] E. Straube, *The L^2-Sobolev theory of the $\overline{\partial}$-Neumann problem,* ESI Lectures in Mathematics and Physics, EMS (to appear).

F. Haslinger: Institut für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: friedrich.haslinger@univie.ac.at