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Local causal structures, Hadamard states and the principle of local covariance in quantum field theory.

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Abstract. In the framework of the algebraic formulation, we discuss and analyse some new features of the local structure of a real scalar quantum field theory in a strongly causal spacetime. Particularly we use the properties of the exponential map to set up a local version of a bulk-to-boundary correspondence. The bulk is a suitable subset of a geodesic neighbourhood of any but fixed point p of the underlying background, while the boundary is a part of the future light cone having p as its own tip. In this regime, we provide a novel notion for the extended $*$-algebra of Wick polynomials on the said cone and, on the one hand, we prove that it contains the information of the bulk counterpart via an injective $*$-homomorphism while, on the other hand, we associate to it a distinguished state whose pull-back in the bulk is of Hadamard form. The main advantage of this point of view arises if one uses the universal properties of the exponential map and of the light cone in order to show that, for any two given backgrounds M and M' and for any two subsets of geodesic neighbourhoods of two arbitrary points, it is possible to engineer the above procedure such that the boundary extended algebras are related via a restriction homomorphism. This allows for the pull-back of boundary states in both spacetimes and, thus, to set up a machinery which permits the comparison of expectation values of local field observables in M and M'.

MSC: 81T20, 81T05

Contents

1 Introduction

In the framework of quantum field theory over curved backgrounds, we witnessed a considerable series of leaps forward due to a novel use of advanced mathematical techniques combined with new physical insights leading to an improved understanding of the underlying foundations of the theory. It is far from our intention to give a recollection of all of them, but we would like at least to draw the attention at least to some of them. On the one hand, in [BFV03], the principle of general local covariance was formulated leading to the realization of a quantum field theory as a covariant functor between the category of globally hyperbolic (four-dimensional) Lorentzian manifolds with isometric embeddings as morphisms and the category of C*-algebras with invertible endomorphisms as morphisms and to the new interpretation of local fields as natural transformations from compactly supported smooth function to suitable operators. On the other hand, the presence of a non trivial background comes with the grievous problem of the a priori absence of a sufficiently large symmetry group to identify a natural ground state as in Minkowski spacetime where Poincaré invariance enables this.

Nonetheless, it is still possible to identify a class of physically relevant states as those fulfilling the so-called Hadamard condition. This guarantees that the ultraviolet behaviour of the chosen state mimics that of the Minkowski vacuum at short distances and that the quantum fluctuations of observables such as the smeared components of the stress-energy tensor are bounded. From a practical point of view, the original characterization of the Hadamard form was realized by means of the local structure of the integral kernel of the two-point function of the chosen quasi-free state in a suitably small neighbourhood of a background point. Unfortunately, such criterion is rather difficult to check in a concrete example and a real step forward has been achieved in [Rad96a, Rad96b] in which is is proven and fully characterised the connection between the Hadamard condition and the microlocal properties of the two-point function.

This result has prompted a series of interesting developments in the analysis of physically relevant states in a curved background, but we focus mainly on a few recent progresses (*c.f.*, [DMP06, DMP07, DMP08]) where it has been shown that, either in asymptotically flat or cosmological spacetimes, it is possible to exploit the conformal structure of the manifold to identify a preferred null submanifold of codimension one, the conformal boundary. On the latter it is possible to suitably encode the information of the bulk algebra of observables and to identify a state fulfilling suitable uniqueness properties and whose pull-back in the bulk satisfies the Hadamard condition, being at the same time invariant under all spacetime isometries.

The main problem in the above construction is the need to find a rigid and global geometric structure which acts as an auxiliary background out of which the bulk state is constructed. Hence the local applicability of a similar scheme seems rather limited; yet one of the main goals of this paper is to show that such a procedure can indeed be set up at a local level and for all spacetimes of physical interest. Particularly this statement is demonstrated on the basis of a careful use of some rather well-known geometrical object.

To be more precise, the taken point of view will be the following: if one considers an arbitrary but fixed point p in a strongly causal four-dimensional spacetime, it is always possible to single out a geodesic neighbourhood where the exponential map is a local diffeomorphism. Within this set we can also always select a second point q such that double cone $\mathscr{D} \equiv \mathscr{D}(p,q) \doteq I^+(p) \cap I^-(q)$ is a globally hyperbolic spacetime. This line of reasoning has a twofold advantage: on the one hand, one can single out a local natural null submanifold of codimension one $\mathscr{C}_p^+$ as the portion of $J^+(p)$ contained in the closure of $\mathscr{D}$, while, on the other hand, we are free to repeat the very same construction for a second point p' with associated double cone $\mathscr{D}'$ in another spacetime M'. Since the exponential map is invertible and the tangent spaces T_pM and $T_{p'}M$ are isomorphic, it turns out that it is possible to engineer all the geometric data in such a way that the two boundaries $\mathscr{C}_p^+$ and $\mathscr{C}_{p'}^+$ can be related by a suitable restriction map, the only freedom being the choice of a frame at p and at p'.

These two advantages can be used to draw some important conclusions on the structure of local quantum field theories. More precisely, we shall focus on a real scalar field theory in $\mathscr{D}$ and the associated Borchers-Uhlmann and extended algebras of observables. Particularly we shall show that it is possible to construct a scalar field theory also on $\mathscr{C}_p$ and, as a novel result, that also, in the boundary, there exists a natural notion of extended algebra which is here made precise. Besides the check of the mathematical consistency of our definition, we strengthen our proposal showing that there exists an injective $*$-homomorphism Π between the bulk and the boundary counterparts. The relevance of this result is emphasized by the identification of a natural state on the boundary independent from the choice of the frame at p, whose pull-back in $\mathscr{D}$ via Π turns out still to be invariant under a choice of the frame (hence, physically speaking, it is the same for all inertial observers at p) and of Hadamard form. This results provides a potential candidate for a local vacuum in this large class of backgrounds we are considering.

Yet we have still not made profitable use of the second advantage outlined before. As a matter of fact, we can now consider two arbitrary strongly causal spacetimes M and M' as well as two points therein so that the relevant portions of the two boundaries, say $\mathscr{C}_p$ and $\mathscr{C}_{p'}$ associated with the double cones, can be related by a suitable restriction map. The construction of the boundary field theory shows that such a map becomes an injective homomorphism between the boundary extended algebras, hence allowing for the construction of a local Hadamard state in two different backgrounds starting from the same building block on the boundary.

Therefore we are now in a position to have a reference state with respect to which we can compare the expectation values of the same field observables in two different spacetimes. Particularly, if one of these is (a portion of) Minkowski spacetime, it is clear that the result of the comparison will be related to the geometric data of the second background which, than, can now be assessed with a crystal clear procedure. Furthermore, we shall show that this method admits an interpretation in the language of category theories, and, thus, it will be manifest that how our proposal is not in contrast with the principle of general local covariance, but actually it can be seen as a generalization. As a matter of fact it reduces to the latter whenever isometric embeddings are involved and, furthermore, in this case the fields recover the interpretation as natural transformation as in [BFV03], *i.e.*, they transform in a covariant manner under local isometries.

To reinforce the above procedure we also provide an explicit example of this "comparison" strategy considering a massless real scalar field minimally coupled to scalar curvature both in Minkowski and in a Friedman-Robertson-Walker spacetime with flat spatial sections. We explicitly show how the difference of the expectation values of the regularized squared scalar fields in these two spacetimes can be expanded in a power series of a suitable local coordinate system (null-advanced) yielding, at first order, a contribution dependent on the structure of the so-called scale factor of the curved background.

Since we have already extensively discussed the plan of action, we only briefly sketch the synopsis of the paper. In section 2 we shall analyse all the geometric structure needed. Although most of the material, devoted to the construction of frames and of the exponential map, is rather well-known in the literature, we try nonetheless to recollect it here to provide guidance through the construction of the main geometric objects required, the boundary in particular. In section 3 we shall instead tackle the problem of constructing a quantum scalar field theory on a null cone; particularly, in section 3.1 and 3.2 we discuss the structure of the bulk and boundary algebras therein while, in section 3.3, we identify the distinguished boundary state. The novel construction of the extended algebra on the boundary is presented in section 3.4 and all these results are connected to the bulk counterpart in section 3.5. Eventually, in section 4, we discuss, by means of the language of categories, the scheme which leads to the possibility to compare field theories on different spacetimes. The concrete example mentioned above is in section 4.2. Section 5 summarizes the paper and sets out a few presented conclusions.

2 Frames and Cones

As outlined in the introduction, the keyword of this paper is "comparison", *i.e.* our ultimate goal will be to correlate quantum field theories in different backgrounds both at the level of algebras and of states and, moreover, to try in the process also to extract information on the local geometry. To this avail one needs a crystal clear control both of the underlying background and of its properties. Therefore, we cannot consider arbitrary manifolds, but we need to focus only on those which are of physical relevance, insofar as they can carry a full-fledged quantum field theory.

If we keep in mind this perspective, we shall henceforth call *spacetime* a four-dimensional, Hausdorff, connected smooth manifold M endowed with a Lorentzian metric whose signature is $(-,+,+,+)$. Then, consequently, M is also second countable and paracompact [Ge68, Ge70]. One could of course add the requirement of global hyperbolicity of M as customarily done (see for example [BGP07] or [BFV03]) in order to have a well-defined Cauchy problem for a classical field theory whose dynamics is ruled by an hyperbolic partial differential equation. Nonetheless, we shall refrain from adding such an hypothesis from the very beginning, because we wish to emphasize only those data which are indispensable to obtain the sought result. Actually we shall see that the assumption of global hyperbolicity of M is not strictly necessary in our analysis, though it becomes desirable whenever one wishes to address global issues.

The next natural step is the identification of further local geometric structures which could serve as a useful tool in the comparison of two different field theories on two different spacetimes, M and M'. To this avail, one should keep in mind two important remarks: on the one hand we already know that, whenever M and M' can be either isometrically, or conformally, embedded into each other, it is possible to compare real scalar field theories on the two spacetimes [BFV03, Pi09]. On the other hand, one has to admit that this scenario is a rather special one and that one should try to weaken the mentioned hypothesis in order to ultimately account for potentially interesting physical scenarios, such as, for example, M being Minkowski spacetime and M' the de Sitter one in the case of theories not conformally invariant.

One should also bear in mind that, if two spacetimes are related by a diffeomorphism, it is always possible to transplant smooth compactly supported functions, interpreted, as local field configurations on one manifold into smooth ones on another, hence leading to the natural idea that these are the natural transformations one should use to compare theories in different backgrounds. Unfortunately this is not the case, since the diffeomorphisms usually do not preserve the geometric structures at the heart of the quantum or even of the classical field theories. A typical example of such a problem arises in connection with the equations of motion of a dynamical system whenever these are constructed out of the spacetime metric. The action of a generic diffeomorphism preserves their form only in some special cases where they are related to isometries. Hence we are forced to find an alternative way if our goal is to compare field theories on different backgrounds.

A potential proposal to solve the above query stems from an apparently unrelated series of developments in quantum field theory over curved backgrounds. More precisely, in the last few years, it has become clear that important information on the structure and on the existence of well-behaved states of physical relevance (*i.e.*, Hadamard states) could be extracted on a wide class of spacetimes relating a field theory living therein to a second one constructed on the conformal boundary [DMP06, DMP07]. The latter turns out to be a null differentiable submanifold of codimension one of (a suitable extension of) the full spacetime and the main reason for its usefulness lies in universality and intrinsicality which allow for the encoding therein of the data of different field theories constructed on different spacetimes.

Unfortunately, one cannot slavishly adopt this point of view for our purposes since the conformal boundary structure ultimately is a global feature shared only by a certain class of manifolds, while we would like to consider a scenario as general as possible scenario. Nonetheless these remarks prompted our search for a different local and, to a certain extent, universal structure which, as a guiding tool to compare quantum field theories on different spacetimes, would make available a procedure similar to the bulk-to-boundary correspondence outlined in [DMP06, DMP07].

In the remaining part of this section, we propose a natural potential solution of this task that arises both from the study of the local causal structures in a neighbourhood of any point of the spacetime and from a careful use of the properties of the exponential map.

2.1 Frames and the exponential map

The aim of this subsection is to introduce the basic geometric tools to be used. Most of the concepts are certainly well-known in the literature and the reader might refer either to [Hus96] for a full-fledged analysis of those related to bundles and their properties or to [KoNo96, ON83] for a discussion focused on the differential geometric aspects. Nonetheless it is worthwhile to recapitulate part of them since they will play a pivotal role in this paper and we can, at the same time, fix the notation.

Consider an arbitrary four-dimensional differentiable manifold M. To any point $p \in M$, we can associate

- a *linear frame* F_p of the tangent space, *i.e.*, a non-singular linear mapping $e : \mathbb{R}^4 \to T_pM$, or, equivalently, an assignment of an ordered basis $e_1, ..., e_4$ of T_pM.

It is straightforward to infer that the set of all such linear frames FM at an arbitrary but fixed $p \in M$ naturally comes with a right and free action of the group $GL(4, \mathbb{R})$ which is tantamount to the possible changes of basis in $\mathbb{R}^4$, *i.e.*, $(A, e) \mapsto eA$ where eA denotes the ordered basis $A^i_j e_i$ for all $A \in GL(4, R)$. Thus FM can be endowed with the following additional structure:

- Given a four-dimensional differentiable manifold M, a **frame bundle** is the principal bundle $FM = FM[GL(4, \mathbb{R}), \pi', M]$ built from the disjoint union $\bigsqcup_p F_pM$, where F_pM is identified with the typical fibre $GL(4, \mathbb{R})$ and $\pi' : FM \to M$ is the projection map. Furthermore, the tangent bundle TM can be constructed as the associated bundle $TM = FM \times_{GL(4,\mathbb{R})} \mathbb{R}^4$.

Although a rather well-known concept, we emphasize that the structure introduced last guarantees that the typical fibre of the tangent bundle at any point p is $\mathbb{R}^4$ regardless of the chosen manifold, a fact we shall use in the forthcoming discussion. Nonetheless, the above data are still not sufficient for our purposes since we need another ingredient which allows to probe the local geometry in a neighbourhood of p in comparison to $\mathbb{R}^4$, namely the exponential map. Following [KoNo96, ON83], recall that

- for any $p \in M$, if D_p is the set of all vectors v in T_pM such that the geodesic $\gamma_v : [0,1] \to M$ admits v as tangent vector in 0, then the exponential map at p is $\exp_p : D_p \to M$ with $\exp_p(v) = \gamma_v(1)$.

- for any point $p \in M$ there always exists a neighbourhood $\widetilde{\mathcal{O}}$ of the 0-vector in T_pM such that the exponential map is a diffeomorphism onto an open subset $\mathcal{O} \subset M$. Furthermore, whenever $\widetilde{\mathcal{O}}$ is star-shaped, $\mathcal{O}$ is called a *normal neighbourhood*, and the inverse map therein will be indicated as $\exp_p^{-1} : \mathcal{O} \to \widetilde{\mathcal{O}}$.

Already this information suggests that a possible way to compare local quantum field theories on different spacetimes might arise by taking advantage both of the universality of the fibre of the tangent bundle at any point in a differentiable manifold and of the existence of open sets where the exponential map is a diffeomorphism. Nonetheless, since we wish to implement a bulk-to-boundary procedure, we also need to single out a preferred structure of codimension 1. For this purpose one has to go one step further and to recall that, ultimately, we are interested in spacetimes. Thus all the manifolds are endowed with a smooth Lorentzian metric. This yields further important geometric properties:

- Since a linear frame at a point $p \in M$ can be seen as the assignment of an ordered basis of $\mathbb{R}^4$, one can endow this latter vector space with the standard Minkowski metric η, which, by construction, is invariant under the Lorentz group $SO(3,1)$. In this case it is possible to consider the frame bundle and reduce its structure group to $SO(3,1)$. Hence, one can start from the definition of frame bundle to introduce $F'[SO(3,1), \pi', M]$ which is also referred to as the *bundle of orthonormal frames over M*. Furthermore, if the spacetime is oriented and time-oriented, we can further reduce the group to $SO_0(3,1)$, the component of $SO(3,1)$ connected to the identity.

- Since our construction relies strongly on the local diffeomorphism property of the exponential map, a key aspect of the spacetimes we consider arises from the realization that every point in a Lorentzian manifold admits a normal neighbourhood (see proposition 7 and also definition 5 in §5 of [ON83]).

- There is always a choice of coordinates, called normal coordinates, such that, in these coordinates, the pull-back of the metric g under the inverse of the exponential map equals η (the Minkowksi metric in standard coordinates) on the counter image of the point p.

- Since we shall ultimately need to single out a sort of preferred codimension 1 structure, it is rather important that, in a Lorentzian manifold, the so-called Gauss lemma holds true (lemma 1 in §5 of [ON83]). Particularly this yields that, taken any $p \in M$, if we consider the null cone $\widetilde{C} \subset T_p(M)$ having p as its own tip, than the subset $\widetilde{C} \cap \widetilde{\mathcal{O}}$ is mapped into a local null cone in $\mathcal{O} \subset M$ which consists of initial segments of all null geodesics starting at p.

Therefore we are now in position to outline the building block of the geometric construction we shall consider. Let us consider two spacetimes (M, g) and (M', g') and two generic points $p \in M$ and $p' \in M'$, together with their normal neighbourhoods $\mathcal{O}_p$ and $\mathcal{O}_{p'}$. If we equip each tangent space with an orthonormal basis via a frame, $e_p : \mathbb{R}^4 \to T_p(M)$ and $e_{p'} : \mathbb{R}^4 \to T_{p'}(M')$, we are also free to introduce a map $i_{e,e'} : T_p(M) \to T_{p'}(M')$ which is constructed simply identifying the elements of the two ordered basis.

The strategy is now to exploit the fact that the exponential map is a diffeomorphism (hence invertible) in a geodesic neighbourhood to introduce a map $\imath_{e,e'} : \mathcal{O}_p \to \mathcal{O}'_{p'}$ such that

$$\imath_{e,e'} \doteq \exp_{p'} \circ\, i_{e,e'} \circ \exp_p^{-1} . \tag{1}$$

It is important to stress a few further aspects of this last definition:

- it is a priori not guaranteed that $i_{e,e'} \circ \exp_p^{-1}(\mathcal{O}_p) \subset \widetilde{\mathcal{O}}_{p'}$, but we can always consider a sufficiently smaller subset of $\mathcal{O}_p$, retaining all its properties, where the above inclusion holds true. Therefore for the sake of notational simplicity, we shall not introduce a further symbol, while we shall assume to have automatically performed such operation whenever needed.

- the map $\imath_{e,e'}$, which maps a sufficiently small $\mathcal{O}$ to $\mathcal{O}'$, is not unique, in the sense that it depends on the chosen orthonormal frames e and e'. We have always the freedom to act with an element of the structure group of the fibre (be it $SO(3,1)$ or $SO_0(3,1)$ depending from the considered scenario) which maps an orthonormal basis into a second one, and this either on $T_p(M)$ or $T_{p'}(M')$. Such arbitrariness cannot be mod out and, for this reason, we have explicitly indicated the two frames in the mapping $\imath_{e,e'}$. This freedom will actually play a relevant role in the discussion of section 4.

As a related point, notice that, if the spacetime M is isometrically embedded into M', a scenario close to the hypotheses in [BFV03], each isometry $\phi : M \to M'$ induces an isomorphism between the orthonormal frame bundle $F[SO(3,1), \pi', M]$ and $F[SO(3,1), \pi', M']$ since the metric structure is preserved. In this case every local character of the manifold M is preserved under ϕ (see for example §3 of [ON83]) and, hence, one can consider a sufficiently small subset of the normal neighbourhood of any $p \in M$ as well as of $\phi(p)$ so that our construction yields the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_p & \xrightarrow{\exp_p^{-1}} & T_pM \\ \phi \big\downarrow & & \big\downarrow i_{e,(\phi_* \circ e)} \\ \mathcal{O}_{\phi(p)} & \xleftarrow[\exp_{\phi(p)}]{} & T_{\phi(p)}M' \end{array}$$

Notice that the presence of $\phi_* \circ e$ in place of a generic e' can be justified as follows: if we call $(,)_p$ the internal product between vectors in $T_p(M)$, then for any $v, w \in T_p(M)$, it holds $(v, w)_p = (\phi_*(v), \phi_*(w))_{\phi(p)}$, which, upon the introduction of a local frame $e : \mathbb{R}^4 \to T_p(M)$, yields $(v,w)_p = (e(v_i), e(w_i))_p = (\phi_* \circ e(v_i), \phi_* \circ e(w_i))_{\phi(p)}$ where $v_i, w_i \in \mathbb{R}^4$. Moreover, if a generic e' is used in the place of $\phi_* \circ e$ there is no guaranty that the previous diagram commute. A counter example can be actually constructed considering two isometrically related non rotationally invariant spacetime and taking for e', $\phi_* \circ e$ rotated by some generic angle.

2.2 Double cones and their past boundary

The analysis of the previous subsection represents a first step towards the set-up of a full-fledged procedure which allows to locally compare quantum field theories on different spacetimes. We shall now single out a preferred codimension one submanifold on which to apply a bulk-to-boundary reconstruction procedure.

To this avail we have to be sure in the first place that we can consistently assign to the background M we consider a well-defined quantum field theory. Since we shall only be interested in local quantities, the usual hypothesis of global hyperbolicity of the spacetime can be mildly relaxed and, henceforth, we shall assume that M is *a strongly causal* [Wa84], that is for every point $p \in M$ and for all open set $\mathcal{O}_p$, there exists a subset $\mathcal{O}'_p \subset \mathcal{O}_p$ such that no causal curve intersects $\mathcal{O}'_p$ more than once, or, in other words, $\mathcal{O}'_p$ itself is globally hyperbolic.

From a physical point of view, this simply forces us to require that, ultimately, the theory must coincide with the usual quantization procedure on each of these subset, while from a geometrical perspective, the discussion of the preceding section still holds true since we are entitled to select $\mathcal{O}_p$ as a normal neighbourhood of p in such a way that the exponential map is a local diffeomorphism also on $\mathcal{O}'_p$. Furthermore a rather useful class of sets is constructed out of the so-called *double cones*

$$\mathscr{D}(p', q) = I^+(p') \cap I^-(q) \subset M,$$

where $I^\pm$ stands respectively for the chronological future and past while $q \in \mathcal{O}'_{p'}$. Notice that both p' and q can be arbitrary but, for our construction, we shall always suppose that at least one of them coincides with p and henceforth $p' \equiv p$. It is also interesting to pinpoint that $\mathscr{D}(p,q)$ is an open and still globally hyperbolic subset of $\mathcal{O}'_p$. In the forthcoming discussion it will also be relevant the boundary of this region and we wish to point out that the closure $\overline{\mathscr{D}(p,q)}$ is a compact set - see for example §8 in [Wa84]- which coincides with $(J^+(p) \cap J^-(q)) \cup \{p\} \cup \{q\}$ since, we employ the definition of causal future and past of a point such that, $p \notin J^+(p)$ and $q \notin J^-(q)$. Furthermore it is also important to recall both that the set of (the closure of the) double cones can be used as a base of the topology of $\mathcal{O}'_p$ and that, under the previous assumptions, we can also freely consider the image of $\overline{\mathscr{D}(p,q)}$ under the inverse exponential map $\exp_p^{-1}$ which we shall denote as $U(p,q)$. A potential reader should bear in mind that the $U(p,q)$ is not necessarily the closure of a double cone in $T_pM \sim \mathbb{R}^4$ with respect to the flat metric since, as commented in the previous section, only (portions of) cones in T_pM, having p as their own tip, are mapped in (portions of) those in $\mathcal{O}_p$ and vice versa .

Nonetheless this construction allows us to identify the main geometrical structure that we need, since the very existence of $\mathscr{D}(p,q)$ and the properties of this set as well as of $J^+(p)$ under the exponential map suggest us to consider $\mathscr{C}_p^+ \doteq \partial J^+(p) \cap \overline{\mathscr{D}(p,q)}$ as the natural boundary on which to encode data from a bulk field theory. The bulk is here meant as $\mathscr{D}(p,q)$ which is a genuine globally hyperbolic submanifold of M on which it is indeed possible to define a full-fledged quantum field theory.

From a geometrical point of view, a few interesting intrinsic properties of $\mathscr{C}_p^+$ can be readily inferred, namely, to start with, $\mathscr{C}_p^+$ is generated by future-directed null geodesics stemming out of p. Notice particularly that the latter are not complete since the set we are interested in is constrained to lie in $\overline{\mathscr{D}(p,q)} \subset \mathcal{O}'_p$ and, therefore, its image under $\exp_p^{-1}$ in T_pM identifies a portion of a future directed null cone C^+ constructed with respect to the flat metric η, where this portion is topologically equivalent to $I \times \mathbb{S}^2$, with $I \subseteq \mathbb{R}$. Yet all these properties are universal, thus they do not depend on the choice of a specific frame e at p, while it is not the case for the form of the image of $\mathscr{C}_p^+$ in C^+ under $\exp_p^{-1}$ or the pull-back of the metric in normal coordinates under $\exp_p^*$. This clearly depend upon the considered coordinate system (individuated by e), hence it is certainly worth a more detailed discussion on the possible choices of e and hence of coordinates on $\mathscr{C}_p^+$.

If one starts from the realization that the double cones, we are interested in, all lie in a normal neighbourhood, a first natural guess would be to select the standard normal coordinates constructed out of the frame e. In this setting the metric can be expanded as

$$g_{\mu\nu}(q) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\alpha\nu\beta}(p)\sigma^\alpha(q,p)\sigma^\beta(q,p) + O(\sigma^{3/2}),$$

where $\sigma(q,p)$ is the so-called Synge's function, namely half of the square of the geodesic distance between p and q. Moreover, we indicated by upper σ^α and σ^β the covariant derivative of σ performed on p.

Unfortunately both the coordinate system and the expansion are not well suited to be used in the analysis of the geometry of the null cone $\mathscr{C}_p^+$, since one would like to have a local chart where it is even manifest that $\mathscr{C}_p^+$ is a null hypersurface. Furthermore, for our later purposes, we also need to discuss some properties of the metric near the whole $\mathscr{C}_p^+$ and not only in a neighbourhood of p. Therefore, to this end, it is useful to employ the so called "retarded coordinates" as introduced in [Po04, PrPo06]. We also refer to the review [Po03], which has the added advantage to clearly discuss the explicit relation between these new coordinates and the normal ones (or, also, the Fermi-Walker ones).

Let us briefly recall their construction and let us start considering a timelike geodesic $\tilde{\gamma}$ through p with unit tangent vector u; in this setting one can define a coordinate r as the field

$$r(q) = -\sigma_\alpha(q,p')u^\alpha(p'),$$

where $q \in \mathcal{O}$ and $p' \in \tilde{\gamma}$ are connected by a light-like geodesic steaming out of p' and pointing towards the future. The covariant derivative in $\sigma_\alpha(q,p')$ is taken with respect to the normal coordinates arising out of the choice of a frame e at p. The net advantage of r is that, on $\mathscr{C}_p^+$, it can be read as an affine parameter of the null geodesics emanating from p. The promotion to coordinate is achieved setting $r = 0$ on p' and its well posedness can be inferred by its very construction which depends only on u^α. Hence, once an orthonormal frame e is chosen in such a way that $e^0(p') = u$, the scalar field r on $C^+_{p'}$ is unambiguously fixed.

We can now define the full retarded coordinates as (u, r, x^A), where u labels the family of forwarded null cones whose tip lies on $\tilde{\gamma}$, and

$$\mathscr{C}^+_p = \left\{ p' \in \mathscr{D}(p,q) \mid u(p') = 0 \right\},$$

while x^A are local coordinates on $\mathbb{S}^2$. Notice that one could alternatively switch to the more common local chart (θ, φ) of $\mathbb{S}^2$ at p' and we shall do it whenever needed throughout the discussion.

Moreover, in the introduced coordinate system, the most generic form of the metric reads [CBCMG09]

$$ds^2 = -\alpha du^2 + 2\upsilon_A du dx^A - 2e^{2\beta} du dr + g'_{AB} dx^A dx^B, \tag{2}$$

where $\alpha, \upsilon_A, \beta$ and g_{AB} are all smooth functions depending on the coordinates. Moreover the x-coordinates on the sphere are such that its volume element with respect to (2) takes the form

$$\sqrt{|g'_{AB}|}\, dx^A \wedge dx^B = \sqrt{|g_{AB}|}\, |\sin\theta| d\theta \wedge d\varphi \;, \tag{3}$$

where the symbol $|\cdot|$ under the square root is kept to recall that we are actually referring to the determinant of the involved matrices. Notice also that, depending on the chosen coordinates x^A on $\mathbb{S}^2$, the switch to (θ, φ) yields an harmless addition contribution to the metric coefficients; this justifies the two symbols g'_{AB} and g_{AB}, although, henceforth, we shall mostly stick to this last one.

It is also remarkable that, whenever $R_{\mu\nu}\dot{\gamma}^\mu\dot{\gamma}^\nu = 0$, γ being any generator of $\mathscr{C}^+_p$, then one can prove that (2) simplifies as (still refer to [CBCMG09])

$$ds^2 = -\alpha du^2 - 2du dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{4}$$

where the standard coordinates (θ, φ) on the 2-sphere are here introduced in place of x^A. Besides being much simpler this form is more closely connected to the standard Bondi one which is canonically used in the implementation of bulk-to-boundaries techniques as devised in [DMP06, DMP07] for a large class of asymptotically flat and of cosmological spacetimes. Unfortunately, contrary to what happened in these last cited papers, the scenario we consider is much more complicated and, furthermore, the cone seems not to display any particular symmetry group to exploit, such as for example the BMS in [DMP06]. Yet the situation is not as desperate as one might think, since, ultimately, for our purposes it will be only relevant that the metric on p becomes the Minkowski one, being our coordinates constructed out of an orthonormal frame at p. Particularly this means that, at p, $\sqrt{|g_{AB}|}$ will become proportional to r, which, in this special scenario, can be here seen both as the affine null parameter above introduced, or, equivalently, the standard radial coordinate in Minkowski spacetime constructed out of the orthonormal frame in $T_pM \sim \mathbb{R}^4$.

Before we conclude this section, we would like to briefly compare (4) with the one stemming from Minkowski spacetime, namely, here the flat metric can be written as

$$ds^2 = -dU^2 + 2dUdr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{5}$$

where $U = t + r$ is the light coordinate constructed out of the time and spherical ones. The cone whose tip is centred in 0 is individuated by $U = 0$ and, also in this case, r is an affine parameter along the null geodesics emanating from 0. It is also important to stress that the pull-back of (2), under $\exp_p^*$, tends to (5) towards the point $\exp_p(0) = p$.

Finally, to conclude the section, we wish to stress the behaviour of $\mathscr{D}(p,q)$ under (1). As mentioned before $\exp^{-1}$ does not map a double cone in M into one on T_pM, but, nonetheless, we can still adapt the choice of q in such a way that $\imath_{e,e'}(\mathscr{D}(p,q))$ is properly contained into a sufficiently large double cone $\mathscr{D}(p,q') \subset \mathcal{O}'$.

3 On the bulk and on the boundary algebras of observables

In the previous section, we focused our attention on the introduction and on the analysis of the main geometric tools we shall need and, particularly, we recall once more that we shall focus on the double cones $\mathscr{D}(p,q)$ which are globally hyperbolic spacetimes on their own. Since, in the forthcoming discussion, neither p nor q will play a distinguished role, we shall omit them, hence referring to $\mathscr{D}$ in place of $\mathscr{D}(p,q)$. More importantly, we are now entitled to introduce a well defined classical field theory and, for the sake of simplicity, we shall henceforth only focus on a free real scalar field.

We now recollect some standard properties of such a physical system along the lines, for example, of [Wa95]. To start with we shall refer to $\varphi : \mathscr{D} \to \mathbb{R}$ which fulfils the following equation of motion:

$$P\varphi \doteq \left(\Box_g + \xi R + m^2\right)\varphi = 0, \quad m^2 > 0 \text{ and } \xi \in \mathbb{R} \tag{6}$$

where $\Box_g = -\nabla^\mu\nabla_\mu$ is the d'Alembert wave operator constructed out of the metric g while R is the scalar curvature. Since this is a second order hyperbolic partial differential equation, each solution can be constructed as the image of the following map

$$\Delta : C_0^\infty(\mathscr{D}) \to C^\infty(\mathscr{D}), \tag{7}$$

where Δ is called the *causal propagator* defined as the difference of the advanced and the retarded fundamental solution. Furthermore each $\varphi_f \doteq \Delta(f)$ satisfies the following support property

$$supp(\varphi_f) \subseteq J^+(supp(f)) \cap J^-(supp(f)),$$

and, if we call $\mathfrak{S}(\mathscr{D})$ the set of solutions of (7) with smooth compactly supported initial data on any Cauchy surface Σ of $\mathscr{D}$, then this turns out to be a symplectic space if endowed with the following strongly non degenerate symplectic form:

$$\sigma(\varphi_f, \varphi_h) = \int\limits_\Sigma d\mu(\Sigma)\left(\varphi_f\nabla_n\varphi_h - \varphi_h\nabla_n\varphi_f\right) = i\int_{\mathscr{D}} d\mu(\mathscr{D})\,(f\Delta h). \quad \forall f, h \in C_0^\infty(\mathscr{D}) \tag{8}$$

Here Σ is any Cauchy surface, the integral, being independent from such a choice, whereas $d\mu(\Sigma)$, $d\mu(\mathscr{D})$ and n are respectively the metric induced measures and the normal vector to Σ.

As a last ingredient, we can exploit all these last properties as well as the statement that, per construction, $\mathscr{D}$ is contained in a larger globally hyperbolic open set - $\mathcal{O}'$ in the notation of the previous section -, in order to conclude that φ_f can be unambiguously extended to a solution of the very same equation in the whole $\mathcal{O}'$. This can be proved identifying the initial data supported on a common portion of a Cauchy surface of both $\mathscr{D}$ and $\mathcal{O}'$, following almost slavishly proposition 2.3 in [DMP06]. As a consequence of this last property we are entitled to consider the restriction of φ_f on $\mathscr{C}_p^+$ which yields

$$\varphi_f|_{\mathscr{C}_p^+} \in C^\infty(\mathscr{C}_p^+). \tag{9}$$

3.1 Quantum algebras on $\mathscr{D}$

After the set-up of a classical field theory, we are free to consider a suitable quantization scheme and this is can be described as a two-fold process: as a first step we shall select a suitable algebra of fields which fulfils the needed commutation relations and, later, we choose a quantum state as a functional on the said algebra in order to be able to compute the expectation values of the relevant observables.

Let us thus proceed in logical sequential step starting from $\mathscr{D}$, our bulk spacetime where we can introduce $\mathscr{F}_b(\mathscr{D})$ as the subset of sequences with a finite number of elements lying in

$$\bigoplus_{n\geq 0} \otimes_s^n C_0^\infty(\mathscr{D}),$$

where $n=0$ yields per definition $\mathbb{C}$ while $\otimes_s$ stands for the symmetric tensor product. According to the said definition it is customary to denote a generic $F \in \mathscr{F}_b(\mathscr{D})$ as a finite sequence $\{F_n\}_n$ where each $F_n \in \otimes_s^n C_0^\infty(\mathscr{D})$. We can now promote $\mathscr{F}_b(\mathscr{D})$ to a *-algebra equipping it with

- a tensor product $\cdot_S$ such that

$$(F \cdot_S G)_n = \sum_{p+q=n} \mathcal{S}(F_p \otimes G_q),$$

  where $\mathcal{S}$ is the operator which realizes total symmetrization.

- a $*-$operation out of complex conjugation, *i.e.*, $\{F_n\}_n^* = \{\overline{F}_n\}_n$ for all $F \in \mathscr{F}_b(\mathscr{D})$.

Although more traditional, the above realization of $\mathscr{F}_b(\mathscr{D})$ could be replaced with a novel point of view, fully accounted for in [BrFr09, BDF09], namely we consider $\mathscr{F}_b(\mathscr{D})$ as a suitable subset of the functionals over $C^\infty(\mathscr{D})$, the smooth field configurations. To wit $F \in \mathscr{F}_b(\mathscr{D})$ yields a functional $F: C^\infty(\mathscr{D}) \to \mathbb{R}$ out of $\langle , \rangle$ the standard pairing between $C^\infty(\mathscr{D})$ and $C_0^\infty(\mathscr{D})$

$$F(\varphi) = \sum_n \frac{1}{n!}\langle F_n, \varphi^n\rangle \ . \tag{10}$$

In order to grasp the connection between the two perspectives, it is useful to introduce a Gâteaux derivative:

$$F^{(n)}(\varphi)h^{\otimes n} = \left.\frac{d^n}{d\lambda^n}F(\varphi+\lambda h)\right|_{\lambda=0}, \quad \forall h\in C^\infty(\mathscr{D})$$

so that $F_n \equiv F^{(n)}(0)$. A potential reader should be warned that we shall use alternatively both pictures in the forthcoming analysis.

The key subsequent point in the quantization scheme consists of the modification of the algebraic product $\cdot_S$ into a new one, $\star$, which is constructed out of the unambiguously defined causal propagator Δ introduced in (7):

$$(F\star G)(\varphi) = \sum_n \frac{i^n}{2^n n!}\langle F^{(n)}(\varphi), \Delta^{\otimes n}G^{(n)}(\varphi)\rangle\ . \qquad \forall F,G\in\mathscr{F}_b(\mathscr{D}) \tag{11}$$

Per direct inspection one can realize that $F\star G$ still lies in $\mathscr{F}_b(\mathscr{D})$ and, more importantly, that $(\mathscr{F}_b(\mathscr{D}),\star)$ pertains the structure of $*$-algebra under the operation of complex conjugation.

It is important to notice that, up to now, we have not used the existence of the equation of motion (6) and, therefore, we can refer to $\mathscr{F}_b(\mathscr{D})$ as an *"off shell"* algebra. Conversely, if one wants to encompass also the dynamics of the classical system, one needs only to quotient $\mathscr{F}_b(\mathscr{D})$ with respect to the ideal $\mathscr{I}$ which is nothing but the set of elements in $\mathscr{F}_b(\mathscr{D})$ generated by those of the form $P_jF_n(x_1,\ldots,x_n)$, where P_j is the operator in (6) applied to j-th variable in $F_n\in\otimes^n_s C^\infty_0(\mathscr{D})$. The outcome is the *"on shell"*-algebra $\mathscr{F}_o(\mathscr{D}) \doteq \frac{\mathscr{F}_b(\mathscr{D})}{\mathscr{I}}$ which inherits the $*$-operation from $\mathscr{F}_b(\mathscr{D})$ and it is nothing but the more commonly used Borchers-Uhlmann algebra.

At this stage, it is important to remark that neither $\mathscr{F}_b(\mathscr{D})$ nor its on shell version $\mathscr{F}_o(\mathscr{D})$ contain all the needed elements to fully analyse the underlying quantum field theory. As a matter of fact, objects, such as for example the components of the stress energy tensor, involve the products of fields evaluated at the very same spacetime point, an operation which is a priori not well defined due to the distributional nature of the fields themselves. To circumvent this obstruction, a standard procedure calls for the regularization of these ill-defined objects by means of a suitable scheme which goes under the name of Hadamard prescription. We shall not dwell here into the technical details giving only some highlights in the appendix and, therefore, we suggest an interested reader to refer to [HoWa01, HoWa02] for a full account.

On the opposite, in the functional language that we have used before, the mentioned problem translates in the impossibility to include in $\mathscr{F}_b(\mathscr{D})$ objects of the form

$$F(\varphi) = \int_{\mathscr{D}} d\mu(g)f(x)\varphi^2(x),$$

where $d\mu(g)$ is the metric-induced volume form, while f is a test function in $C^\infty_0(\mathscr{D})$ and $\varphi\in C^\infty(\mathscr{D})$. Actually the star product (11) applied to a couple of such fields involve the ill-defined pointwise product of Δ with itself.

To solve this problem, we shall follow the same line of reasoning as in [BrFr09], namely we introduce a new class of functionals, $\mathscr{F}_e(\mathscr{D})$, which must have a finite number of non vanishing derivatives and the n-th one must be a symmetric element of the following space: compactly supported distributions $\mathcal{E}'(\mathscr{D}^n)$, whose wave front set should, moreover, satisfy the following restriction

$$WF(F_n) \cap \left\{ \left(\mathscr{D} \times \overline{V}^+ \right)^n \cup \left(\mathscr{D} \times \overline{V}^- \right)^n \right\} = \emptyset, \tag{12}$$

where $\overline{V}^{\pm}$ correspond respectively to the forward and to the backward causal cone in the tangent space. The closure symbol indicates that we are also including the tip of the cone to the set of future or past directed causal vectors.
We can make $\mathscr{F}_e(\mathscr{D})$ a $*$-algebra if we extend naturally the $*$-operation of $\mathscr{F}_b(\mathscr{D})$ and if we endow it with a new product, $\star_H$, whose well-posedness was first proved in [BFK95, BrFr00, HoWa01, HoWa02] and whose explicit form is realized as follows

$$(F \star_H G)(\varphi) = \sum_n \frac{1}{n!} \langle F^{(n)}(\varphi), H^{\otimes n} G^{(n)}(\varphi) \rangle \ , \tag{13}$$

where $H \in \mathcal{D}'(\mathscr{D}^2)$ is the so-called Hadamard bi-distribution. We shall briefly introduce and discuss it in the appendix, but, to our purposes, it is important to recall that, on the one hand, it satisfies the microlocal spectrum condition, hence yielding a natural substitute for the notion of positivity of energy out of its wave front set, while on the other hand it suffers from an ambiguity. To wit, at a level of integral kernel, only the antisymmetric part of the Hadamard bi-distribution is fixed to "$i/2$" times the causal propagator Δ as well as the singular structure which is unambiguously determined by the choice of the background. Otherwise there always exists the freedom to add a smooth symmetric function to it and, in our scenario, this means that, if H, H' are two Hadamard distributions, then the integral kernel of $H - H'$ is a symmetric element of $C^\infty(\mathscr{D}^2)$. Yet, as far as the algebra is concerned, this freedom boils down to an algebraic isomorphism $\alpha : (\mathscr{F}_e(\mathscr{D}), \star_H) \to (\mathscr{F}_e(\mathscr{D}), \star_{H'})$, namely [HoWa01, BDF09]

$$\begin{aligned} \imath_{H',H} &= \alpha_{H'} \circ \alpha_H^{-1} \ , \\ \alpha_H(F) &\doteq \sum_n \frac{1}{n!} \langle H^{\otimes n}, F^{(2n)} \rangle. \end{aligned} \tag{14}$$

As for the algebra generated by compactly supported smooth functions, also the extended one $\mathscr{F}_e$ has its on shell counterpart that we indicate as $\mathscr{F}_{eo}$ and it is found out of the quotient with the ideal generated by the equation of motion applied to the elements of $\mathscr{F}_e$.

One of the net advantages of the employed formalism is the possibility to easily transcribe the overall construction in terms of categories, hence yielding a crystal clear mathematical picture of the relevant structures and of their relations.

This was first advocated and employed in the seminal papers [BFV03], where the principle of general local covariance was first formulated in this language and we shall also stick to it. Particularly we shall now recast the above discussion in this different perspective, while the actual relation with [BFV03] will be only later outlined in section 4. Hence, we shall use the following categories:

DoCo: the objects are the oriented and time oriented double cones $\mathscr{D} \equiv \mathscr{D}(p,q)$ contained in some normal neighbourhood of any spacetime, as introduced in the previous section. Since they are globally hyperbolic, the choice of time and space orientation is always possible. The morphisms are instead the maps $\imath_{e,e'}|_{(\mathscr{D},\mathscr{D}')}$ introduced in (1) when the restriction is possible and meaningful.

DoCo_{iso}: it is the subcategory of DoCo obtained keeping the same objects but restricting the possible morphisms of DoCo to those which are isometric embeddings.

Alg_i: the objects are unital $*$-algebras $\mathscr{F}_i$ with $i = b, o, e, eo$, constructed above, while the morphisms are injective $*$-homomorphisms among them.

That said, since the key ingredient to construct both $\mathscr{F}_b(\mathscr{D})$ and $\mathscr{F}_o(\mathscr{D})$ are just the causal propagator Δ from (7) and the operator (6) realizing the equations of motion, their uniqueness in any $\mathscr{D}$ suggests that the association to each double cone of a suitable algebra

$$\mathscr{F} : \mathscr{D} \to \mathscr{F}_i(\mathscr{D}) \,. \qquad i = b, o \tag{15}$$

is a functor between DoCo_{iso} and Alg_i. It is important to notice that (14) grants us that the ambiguity in the choice of the Hadamard bi-distribution does not spoil the well-posedness of (15) when $i = e$. All these assertion can be proved noticing that, due to the discussion presented after (1), the category DoCo_{iso} is just a sub category of the category of local manifolds introduced in [BFV03] where similar results were discussed.

It would be desirable to extend the functor (15) to the category DoCo that has a larger group of morphisms. Unfortunately this is not straightforward and, actually, not even possible. If we consider two generic globally hyperbolic regions $\mathscr{D}$ and $\mathscr{D}'$ in DoCo, related by $\imath_{e,e'}$ as in (1), we can draw the following diagram

$$\begin{array}{ccc} \mathscr{D} & \xrightarrow{\mathscr{F}} & \mathscr{F}_i(\mathscr{D}) \\ {\scriptstyle \imath_{e,e'}}\big\downarrow & & \\ \mathscr{D}' & \xrightarrow[\mathscr{F}]{} & \mathscr{F}_i(\mathscr{D}') \end{array} \tag{16}$$

In order to have a well defined functor between DoCo and some Alg_{iso} it is necessary to close (16) with an additional arrow from $\mathscr{F}_i(\mathscr{D})$ to $\mathscr{F}_i(\mathscr{D}')$ which makes it commutative. Unfortunately this is not possible, the reason being that $\imath_{e,e'}$ is not an isometry in general and, thus, it will neither map solutions of (6) in $\mathscr{D}$ into those of the same equation (but out of a different metric) in $\mathscr{D}'$, nor it will preserve the causal propagator. Hence it will spoil the $\star$ operation and it will not preserve the singular structure of the Hadamard bi-distribution which depends only upon the underlying geometry.

As a side remark, a positive answer to the present question will be possible only considering the off shell classical $*-$algebras $(\mathscr{F}_b, \cdot_S)$, but, as soon as quantum algebras are employed, the situation looks grim. Yet it is possible to circumvent this obstruction making a profitable use of the geometric properties of the portion of future directed light cone in any $\mathscr{D}$ in order both to set up a bulk-to-boundary correspondence and to later compare the outcome on the boundary of different spacetimes.

3.2 Quantum field theory on the boundary

In order to fulfil the goals, we set at the end of the previous section, it is mandatory as a first step to understand how to construct a full-fledged quantum field theory on the light cone and this will be the main aim of this section. The procedure we shall employ stems from the gathered experience in similar scenarios where a field theory on a null surface was constructed such as, for example, in [DMP06, DMP07, DMP08, MoPi04] (see also [He08, Sc09, ChKi09] for further analyses in similar contexts).

Therefore, following the same philosophy as in these cited papers, we shall first show that it is possible to assign to the boundary a natural field algebra and that we can also perform a "natural" choice for the relevant quantum state. To this avail, we shall consider in this section the cone as an abstract manifold on its own, not seeing it as a particular portion of the boundary of a specific global hyperbolic double cone $\mathscr{D}$, since the connection with the bulk theory will be only presented later. Nevertheless, we have to keep in mind that the algebra we are going to construct has to be large enough in order to contain the image of some suitable projection of all the elements of the algebra in $\mathscr{D}$. This will be the most tricky point in the whole construction because it is not sufficient to consider an algebra generated by compactly supported data on the cone, we shall extend this set to more generic elements.

Let us start to introduce the three distinct sets in $\mathbb{R}\times\mathbb{S}^2\subset\mathbb{R}^4$ relevant for the following construction, namely, employing the standard coordinates, we can start from

$$\mathscr{C}_p^+ = \left\{(V,\theta,\varphi)\in\mathbb{R}\times\mathbb{S}^2 \mid V\in(0,V_0(\theta,\varphi))\subset\mathbb{R},\ (\theta,\varphi)\in\mathbb{S}^2\right\}, \tag{17}$$

where $V_0(\theta,\varphi)$ is a positive, bounded smooth function on the sphere. The remaining two regions will be denoted as $\mathscr{C}_p$ and $\mathscr{C}$ where, respectively, the coordinate V is allowed to run on $(0,\infty)$ and on the full real line, so that $\mathscr{C}_p^+\subset\mathscr{C}_p\subset\mathscr{C}$. We stress that, with a small abuse of notation, we employ the symbol $\mathscr{C}_p^+$ as in the previous sections although we are not referring to an actual cone since, ultimately, (17) will indeed coincide to $J_p^+\cap\overline{\mathscr{D}}$, if we employ the same conventions and nomenclatures as in the preceding analysis.

As a next natural step, we need to identify a suitable space of functions on the boundary and, to this avail, viewing $\mathscr{C}_p$ immersed in $\mathbb{R}^4$, we define

$$\mathscr{S}(\mathscr{C}_p) \doteq \left\{\psi\in C^\infty(\mathscr{C}_p),\ \psi = h\, f|_{\mathscr{C}_p},\ f\in C_0^\infty(\mathbb{R}^4) \text{ and } h\in C^\infty(\mathscr{C}_p)\right\}, \tag{18}$$

where h must vanish as $V\to 0$ uniformly in the coordinate on $\mathbb{S}^2$, while each derivative along V tends to a constant uniformly in the angles. As far as this section is concerned we can safely choose h to be always equal to V. Furthermore $\mathscr{S}(\mathscr{C}_p)$ turns out to be a symplectic space if endowed with the following strongly non-degenerate symplectic form:

$$\sigma_{\mathscr{C}}(\psi,\psi') \doteq \int_{\mathscr{C}_p}\left[\psi\frac{d\psi'}{dV} - \frac{d\psi}{dV}\psi'\right] dV\wedge d\mathbb{S}^2\ ,\quad \forall\psi,\psi'\in\mathscr{S}(\mathscr{C}_p) \tag{19}$$

where $d\mathbb{S}^2$ is the standard measure on the unit 2-sphere.

The reason for such an apparently strange choice for $\mathscr{S}(\mathscr{C}_p)$ is related with the need to later relate the theory on these sets with those in the bulk of a double cone. For this reason the most natural choice of compactly supported smooth functions on $\mathscr{C}_p$ would not fit into the overall picture since a general solution of the Klein-Gordon equation (6) with smooth compactly supported initial data on some $\mathscr{D}$ would propagate on the light cone $\mathscr{C}_p$ to a function which is also supported on the tip. This point would coincide to $V = 0$ in the above picture and, thus, it does not lie in $\mathscr{C}_p$. On the opposite, we shall show in the next section that (18) is indeed the natural counterpart on the boundary which arises from the set of solutions of (6).

In order to introduce the relevant algebra of observables, we follow the same philosophy as in section 3.1, thus introducing $\mathscr{F}_B(\mathscr{C}_p)$, whose generic element F' is a sequence $\{F'_n\}_n$ with a finite number of elements in

$$\bigoplus_{n\geq 0} \otimes_s^n \mathscr{S}(\mathscr{C}_p), \tag{20}$$

where the subscript $\otimes_s$ still stands for the symmetrized tensor product and the first element in the previous sum is still $\mathbb{C}$. Notice that the $'$-superscript is introduced in this section in order to avoid a potential confusion with the similar symbols used for the counterpart in the bulk. In order to promote (20) to a full $*$-algebra, we must endow it with

- a $*$-operation which is nothing but the complex conjugation, *i.e.*, $\{F'_n\}^*_n = \{\overline{F}'_n\}_n$ for all $F' \in \mathscr{F}_B(\mathscr{C}_p)$.
- an operation between algebra elements such that, for any $F', G' \in \mathscr{F}_B(\mathscr{C}_p)$
$$(F' \cdot_S G')_n = \sum_{p+q=n} \mathcal{S}(F'_p \otimes G'_q).$$

Although well-defined, this algebra is not suited to be put in relation with data in the bulk and, thus, we must once more deform the above product. To this avail, we employ the functional point of view as in (10), namely $F' \in \mathscr{F}_B(\mathscr{C}_p)$ yields a functional $F' : C^\infty(\mathscr{C}_p) \to \mathbb{R}$ out of $\langle , \rangle$ the pairing between $C^\infty(\mathscr{C}_p)$ and $\mathscr{S}(\mathscr{C}_p)$

$$F'(\Phi) = \sum_n \frac{1}{n!}\langle F'_n, \Phi^n\rangle \ . \quad \forall \Phi \in C^\infty(\mathscr{C}_p).$$

Out of a direct inspection of the definition of $\mathscr{S}(\mathscr{C}_p)$ in (18), notice that $\langle , \rangle$ is nothing but the standard internal product on $(0,\infty) \times \mathbb{S}^2$ between compactly supported functions and smooth one, taken with respect to the measure $dV \wedge d\mathbb{S}^2$.

Although the theory on $\mathscr{C}_p$ has no equation of motion built in and, hence, no causal propagator such as (7), we can nonetheless introduce a new $\star_B$-product on $\mathscr{F}_B(\mathscr{C}_p)$, namely

$$(F' \star_B G')(\Phi) = \sum_n \frac{i^n}{2^n n!}(F'^{(n)}(\Phi), \Delta^n_{\sigma_{\mathscr{C}}} G'^{(n)}(\Phi)) \ , \quad \forall \Phi \in C^\infty(\mathscr{C}_p) \tag{21}$$

where $\Delta_{\sigma_{\mathscr{C}}}$ is nothing but the integral kernel of (19), namely

$$\Delta_{\sigma_{\mathscr{C}}}((V,\theta,\varphi),(V',\theta',\varphi')) = -\frac{\partial^2}{\partial V \partial V'}\, sign(V-V')\; \delta(\theta,\theta'), \tag{22}$$

where $\delta(\theta,\theta')$ is a short cut for $\delta(\theta-\theta')\delta(\varphi-\varphi')$ and the derivatives have to be taken in the weak sense. Notice that above $\Delta_{\sigma_{\mathscr{C}}}$ is defined as a distribution on $C_0^\infty(\mathscr{C}_p^2)$ which, per direct inspection, can be extended also on $\mathscr{S}^2(\mathscr{C}_p^2)$, finally, $\star_B$ is well posed because only a finite number of elements appear in the sum on the right hand side of (22) and, thus, convergence problems are not an issue here. We can hence conclude this subsection with a proposition whose prove follows from the preceding discussion.

Proposition 3.1. *The couple $(\mathscr{F}_B(\mathscr{C}),\star_B)$ equipped with the $*-$operation introduced above, is a well defined $*-$algebra.*

In the next we shall discuss the form of a certain class of quantum states on this algebra. Eventually we shall use them in order to extend the boundary algebra of observables in analogy to what it is done for the extended local one in $\mathscr{D}$.

3.3 Natural boundary states

The next natural step in our construction would be the introduction of an extended algebra, but we should recall that, in the scenario we are now considering, there is no natural definition of an Hadamard state or bi-distribution and, thus, this lack of a class of a priori physically relevant states hinders a slavish analysis stemming from the function H as in (13). Therefore we need a natural bi-distribution on $\mathscr{C}_p^+$ and, for this purpose, our choice is the following weak-limit

$$\omega((V,\theta,\varphi),(V',\theta',\varphi')) \doteq -\frac{1}{\pi}\lim_{\epsilon\to 0^+}\frac{1}{(V-V'-i\epsilon)^2}\delta(\theta,\theta')\ , \tag{23}$$

which has the further advantage of being at the same time a well-defined element of $\mathcal{D}'(\mathscr{C}^2)$, where $\mathscr{C}\sim\mathbb{R}\times\mathbb{S}^2$ and where $\mathcal{D}'$ stands for the space of distributions over the test-functions in $C_0^\infty(\mathscr{C})$.

Such an expression already appeared in different, albeit related scenarios where a scalar quantum field theory was studied [DMP06, DMP08, DMP09, MoPi04, KaWa91, Sc09, Se82]. It is important to remark that in the first two of theses cited papers, (23) was actually used as the building block to construct a quasi-free pure state for a scalar field theory built on a three dimensional null cone which represented the conformal boundary of a suitable class of spacetimes. In all these cases, the particular geometric structure as well as the presence of a particular symmetry group yields that (23) fulfils suitable uniqueness properties and, furthermore, it gives origin to a full-fledged Hadamard state in the bulk. For these reasons we shall employ the above expression as the natural candidate bi-distribution on the boundary, proving in the next sections that, when we realize $\mathscr{C}_p^+$ as part of the boundary of a double cone, we can also construct a physically meaningful state on $\mathscr{D}$ out of (23).

Therefore the above bi-distribution can be read as a functional $\omega : \mathscr{S}(\mathscr{C}_p) \times \mathscr{S}(\mathscr{C}_p) \to \mathbb{R}$, but it is actually much more convenient to recall that $\mathscr{C}_p \subset \mathscr{C}$. Within this perspective, since $\mathscr{C}$ is topologically the full $\mathbb{R} \times \mathbb{S}^2$ and since each element in $\mathscr{S}(\mathscr{C}_p)$ also lies in $L^2(\mathscr{C}, dV \wedge d\mathbb{S}^2)$, the following expression for the 2-point function is meaningful

$$\omega(\psi, \psi') = -\frac{1}{\pi} \lim_{\epsilon \to 0} \int_{\mathbb{R}^2 \times \mathbb{S}^2} dV dV' d\mathbb{S}^2(\theta, \varphi) \frac{\psi(V, \theta, \varphi)\psi'(V', \theta, \varphi)}{(V - V' - i\epsilon)^2}, \tag{24}$$

where we have already integrated out the delta-function over the angular coordinates. The distribution (24) satisfies a suitable continuity condition, as for example shown in [Mo08],

$$|\omega(\psi, \psi')| \leq C \left(\|\psi\|_{L^2} + \|\partial_V \psi\|_{L^2}\right) \left(\|\psi'\|_{L^2} + \|\partial_V \psi'\|_{L^2}\right) < \infty, \tag{25}$$

which permits to extend ω on the space of square integrable functions whose derivative along the V-coordinate is also L^2. Furthermore, these last remarks also entail that it is possible to perform a Fourier-Plancherel transform along V (see appendix C in [Mo08]) and this operation yields a much more manageable form for (24)

$$\omega(\psi, \psi') = \int_{\mathbb{R} \times \mathbb{S}^2} 2k \; \Theta(k) \; \overline{\widehat{\psi}(k, \theta, \varphi)} \widehat{\psi}'(k, \theta, \varphi) \; dk d\mathbb{S}^2(\theta, \varphi) \;, \tag{26}$$

where $\Theta(k)$ is the step function such that it is equal to 1 if $k \geq 0$ and it vanishes otherwise. It should be stressed that the presence of $\Theta(k)$ mimics the physical intuition of taking only positive frequencies, also because, on the cone, the only causal directions are the lines at constant angular variables. This idea has under special circumstances a clear connection with the geometrical bulk data as well as with the Hadamard property of a bulk state constructed out of (23) - see for example [Mo06, Mo08, DMP08].

As a last step in this section, we wish to underline that the above analysis entails two relevant remarks. The first one concerns the wave front set of the bi-distribution ω on $\mathscr{C}^2 \sim (\mathbb{R} \times \mathbb{S}^2)^2$. This was already studied in Lemma 4.4. of [Mo08] yielding

$$WF(\omega) \subseteq A \cup B$$

where

$$\begin{aligned} A = \big\{ &(V, \theta, \varphi, \zeta_V, \zeta_\theta, \zeta_\varphi), (V', \theta', \varphi', \zeta'_V, \zeta'_\theta, \zeta'_\varphi) \in (T^*\mathscr{C})^2 \setminus \{0\}, \mid V = V', \theta = \theta', \varphi = \varphi' \\ &0 < \zeta_V = -\zeta'_V, \; \zeta_\theta = -\zeta'_\theta, \; \zeta_\varphi = -\zeta'_\varphi \big\} \end{aligned} \tag{27}$$

and

$$\begin{aligned} B = \big\{ &(V, \theta, \varphi, \zeta_V, \zeta_\theta, \zeta_\varphi), (V', \theta', \varphi', \zeta'_V, \zeta'_\theta, \zeta'_\varphi) \in (T^*\mathscr{C})^2 \setminus \{0\}, \mid \theta = \theta', \varphi = \varphi' \\ &\zeta_V = \zeta'_V = 0, \; \zeta_\theta = -\zeta'_\theta, \; \zeta_\varphi = -\zeta'_\varphi \big\} \; . \end{aligned} \tag{28}$$

Although, at this stage, this result represents only a side remark, it will play a pivotal role in the discussion of section 3.4 and 3.5. Particularly, if we recall that $\mathscr{C}_p \subset \mathscr{C}$, it turns out that the wave front of ω on $C_0^\infty(\mathscr{C}_p^2)$ can only be smaller or equal to $A \cup B$, and actually it correspond to $A \cup B$ restricted on $(T^*\mathscr{C}_p)^2$.

As a second remark, we can see that it is possible to construct a new algebra, say $\mathscr{F}_B(\mathscr{C})$ on the full $\mathscr{C}$ starting from (18) and considering the set $L^2(\mathscr{C}, dV \wedge d\mathbb{S}^2)$ in place of $\mathscr{S}(\mathscr{C}_p)$, while keeping the same $*$-operation and composition rule. On the one hand it straightforwardly holds that $\mathscr{F}_B(\mathscr{C}_p)$ is sub $*$-algebra of $\mathscr{F}_B(\mathscr{C})$ while, on the other hand, we can see that the two-point function ω as in (23) can be used as a building block of a quasi-free state for $\mathscr{F}_B(\mathscr{C})$. Hence the same conclusion can be drawn for $\mathscr{F}_B(\mathscr{C}_p)$ since the antisymmetric part of ω is, per construction, equal to $\frac{i}{2}\Delta_\sigma$. The only possible issue is positivity, but this is solved by direct inspection of (26) whose right hand side is manifestly greater than 0 once $\psi = \psi'$. It is important to point out, for the sake of completeness, that an almost identical analysis appears in [DMP06, DMP08], though performed at a level of Weyl algebras. To summarize,

Proposition 3.2. *The Gaussian (quasi-free) state constructed out of the distribution ω enjoys the following properties*

1. *It is a well defined algebraic state on $\mathscr{F}_B(\mathscr{C}_p)$ and on $\mathscr{F}_B(\mathscr{C})$.*
2. *It is a vacuum with respect to ∂_V, in the sense given in [SaVe00].*
3. *It is invariant under the change of the local frame, hence it is invariant under the action of $SO_0(1,3)$.*

Proof. The first point can be analysed checking, linearity, positivity and normalizability of the state on $\mathscr{F}_B(\mathscr{C}_p)$. Since the state is quasi free, it is enough to check these properties for the functional ω on $\mathscr{S}(\mathscr{C}_p) \times \mathscr{S}(\mathscr{C}_p)$ and, in this last case, they follow from the previous discussion.

The proof of the second point descends from the observation both that ω, seen as a state for $\mathscr{F}_B(\mathscr{C})$, is invariant under translations, and from the fact that the Fourier Plancherel transform of the integral kernel of ω along the V direction contains only positive frequency, as appear clear from (26).

The third point can be proved recalling a result proved in [DMP06], namely ω on $\mathscr{F}_B(\mathscr{C})$ can be shown to be invariant under the action of an infinite dimensional group, the so-called Bondi-Metzner-Sachs group (BMS). In a few words, if one switches from the coordinates (V, θ, φ) to $(V, z, \bar{z})$ obtained out of a stereographic projection, the BMS maps

$$\begin{cases} z \to z' = \Lambda(z) \doteq \frac{az+b}{cz+d}, \quad ad - bc = 1 \quad a, b, c, d \in \mathbb{C}, \\ V \to V' = K_\Lambda(z, \bar{z})(V + \alpha(z, \bar{z})), \end{cases}$$

where, $\bar{z}$ transform as the complex conjugate transformation of z, $\alpha(z, \bar{z}) \in C^\infty(\mathbb{S}^2)$ and where

$$K_\Lambda(z, \bar{z}) \doteq \frac{1 + |z|^2}{|az + d|^2 - |bz + c|^2}.$$

Hence, per direct inspection of the above formulas, one can realize that the BMS group is the regular semidirect product $SL(2, \mathbb{C}) \ltimes C^\infty(\mathbb{S}^2)$. Most notably one can observe that there exists a proper subgroup which is homomorphic to $SO(3, 1)$ and thus the state turns out to be invariant under the sought group. □

3.4 Extended algebra on the boundary

In the previous subsection, we introduced the boundary algebra together with a suitable notion of $\star$-product, but this is still not sufficient to intertwine the boundary data with the bulk one because we lack a counterpart for the extended algebra of observables on $\mathscr{C}_p$. Yet, thanks to the achieved results, we have all the needed ingredients to construct it.

Therefore, as a starting point, we define the building block of the extend algebra as follows

Definition 3.1. *We call $\mathscr{A}^n$ the set of elements $F'_n \in \mathcal{D}'(\mathscr{C}_p^n)$ that fulfils the following properties:*

1. **compactness:** *the F'_n are compact towards the future, that is the support of F'_n is contained in a compact subset of $\mathscr{C}^n \sim (\mathbb{R}\times\mathbb{S}^2)^n$,*

2. **causal non monotonic singular directions:** *the wave front set of F'_n contains only causal non monotonic directions which means that*

$$WF(F'_n) \subseteq W_n \doteq \Big\{(x,\zeta)\in (T^*\mathscr{C}_p)^n \setminus \{0\}, (x,\zeta)\notin \overline{V}_n^+ \cup \overline{V}_n^-, (x,\zeta)\notin S_n\Big\} \tag{29}$$

   where $(x,\zeta)\equiv(x_1,\dots,x_n,\zeta_1,\dots,\zeta_n)\in\overline{V}_n^+$ if, employing the standard coordinates on $\mathscr{C}_p$, for all $i=1,...,n$, $(\zeta_i)_V>0$ or ζ_i vanish, the subscript V here referring to the component along the V-direction on $\mathscr{C}_p$. Analogously we say $(x,\zeta)\in\overline{V}_n^-$ if every $(\zeta_i)_V<0$ or ζ_i vanish. Furthermore $(x,\zeta)\in S_n$ if there exists an index i such that, at the same time, $\zeta_i\neq 0$ and $(\zeta_i)_V=0$.

3. **factorization:** *The distribution F'_n can be factorized into the tensor product between a smooth function and an element of $\mathscr{A}^{n-1}$ when localized in a neighbourhood of $V=0$. To wit, there exists a compact set $\mathcal{O}\subset\mathscr{C}_p$ such that, if we pick $\Theta\in C_0^\infty(\mathscr{C}_p)$, so that it is equal to 1 on $\mathcal{O}$, and if we call $\Theta'\doteq 1-\Theta$,*

$$\Theta'_{x_i}F'_n = f(x_i)u_{x_1,\dots\hat{x}_i,\dots x_n} \tag{30}$$

   where $u_{x_1,\dots\hat{x}_i,\dots x_n}\in\mathscr{A}^{n-1}$ in the variables $x_1,\dots x_n$, though leaving out x_i. Furthermore, both f and $\partial_V f$ must lie in $C^\infty(\mathscr{C}_p)\cap L^2(\mathscr{C}_p, dV\wedge d\mathbb{S}^2)\cap L^\infty(\mathscr{C}_p)$ while the limit of f as V tends to 0 must vanish pointwisely in the coordinates over $\mathbb{S}^2$.

Let us notice that $\mathscr{S}(\mathscr{C}_p)$ is strictly contained in $\mathscr{A}$ and that the candidate to play the role of extended algebra on $\mathscr{C}_p$ is thus

$$\mathscr{F}_{B,E}(\mathscr{C}_p)) = \bigoplus_{n\geq 0}\mathscr{A}_s^n \ ,$$

where $\mathscr{A}_s^n$ is the subset of totally symmetric elements in $\mathscr{A}^n$ defined in 3.1. We can now endow this set with the structure of $*$-algebra introducing the $*$-operation $\{F'_n\}^* = \{\overline{F'_n}\}$ for all $F'\in\mathscr{F}_{B,E}$. The composition law instead arises out of a modification of $\star_B$ by means of the state constructed starting from (23). It is a priori clear that such a procedure depends intrinsically on the particular ω, one considers. Nonetheless, despite this setback, our choice will be later justified thanks both to its connection with the bulk data and to the well-posedness of the new structure.

If we stick to the functional representation, we can thus introduce

$$\star_\omega : \mathscr{F}_{B,E}(\mathscr{C}_p) \times \mathscr{F}_{B,E}(\mathscr{C}_p) \to \mathscr{F}_{B,E}(\mathscr{C}_p)$$
$$(F' \star_\omega G')(\Phi) = \sum_n \frac{1}{n!}(F'^{(n)}(\Phi), \omega^n G'^{(n)}(\Phi)) \; , \tag{31}$$

for all $F', G' \in \mathscr{F}_{B,E}(\mathscr{C}_p)$ and for all $\Phi \in C^\infty(\mathscr{C}_p)$. The following proposition aims to dispel the possible doubts on the well posedness of the preceding operation.

Proposition 3.3. *The operation* (31) *is a well-defined product in* $\mathscr{F}_{B,E}$.

Proof. As a starting point, we notice that (31) is per construction bilinear and, per definition of $\mathscr{F}_{B,E}(\mathscr{C}_p)$, there are only a finite number of non vanishing elements $F'^{(n)}$ and $G'^{(n)}$. Equivalently, (31) is made up only of finite linear combinations of terms such as

$$\mathcal{S} \int\limits_{\mathscr{C}_p^{2k}} F_j(x_1, \ldots, x_j)\, \omega(x_1, y_1) \ldots \omega(x_k, y_k)\, G_l(y_1, \ldots, y_l) \prod_{i=1}^k d\mu(x_i) d\mu(y_i) \tag{32}$$

with $k \leq j$ and $k \leq l$, while $\mathcal{S}$ realizes symmetrization in the non integrated variables (x_{k+1}, $\ldots, x_j$, y_{k+1}, $\ldots, y_l$) and $d\mu$ is the measure $dV \wedge d\mathbb{S}^2(\theta, \varphi)$ on $\mathscr{C}_p$, here written in the usual coordinates. Therefore the proof amounts to show that (32) yields an element of $\mathscr{A}^{j+l-2k}$.

First of all, we shall try to follow the traditional proof [HoWa01], and to this avail, to see if (32) can be seen as the distribution $\omega^{\otimes k} \otimes I^{\otimes(j+l-2k)} \in \mathcal{D}'(\mathscr{C}^{j+l})$ tested on $F_j \otimes G_l \in \mathscr{A}^{j+l}$, where I stands for the identity operator on $\mathscr{A}$. Therefore well-posedness of this operation can be checked looking at the structure of the wave front set of the single objects verifying that their composition never contains the zero section. The key ingredients in this game can be readily inferred using theorem 8.2.9 in [Hö89]:

$$WF(F'_j \otimes G'_l) \subset (W_j \cup \{0\}) \times (W_l \cup \{0\}) \setminus \{0\}, \tag{33}$$

and

$$WF(\omega^{\otimes k} \otimes I^{j+l-2k}) \subset (A \cup B \cup \{0\})^k \times \{0\} \setminus \{0\} \; , \tag{34}$$

where, as usual, we have not specified the dimension of the $\{0\}$ zero section in the cotangent space. Furthermore A, B and W_j are respectively defined in (27), (28) and (29). It is now possible to apply theorem 8.2.10 in [Hö89] since the above wave front sets never sum up to the zero section. This is tantamount to notice that for every n and m, since $A^n \subset \overline{V}_n^+ \times \overline{V}_n^-$ and $\overline{V}_n^\pm \cap W_n = \emptyset$, $B^n \times \{\mathbb{R}^3\}^m \cap W_{2n+m} = \emptyset$ and that $A^n \cap (W_n \times W_n) = \emptyset$. The outcome is that

(32), seen as the pointwise product of $F'_j \otimes G'_l \in \mathcal{D}'(\mathscr{C}_p^{j+l})$ with $\omega^{\otimes k} \otimes I^{j+l-2k} \in \mathcal{D}'(\mathscr{C}_p^{j+l})$, is still a well-defined element in $\mathcal{D}'(\mathscr{C}_p^{j+l})$, whose wave front set must satisfy the following inclusion

$$WF(F'_j \otimes G'_l \cdot \omega^{\otimes k} \otimes I^{j+l-2k}) \subset WF(F_j \otimes G_l) \cup \{0\} + WF(\omega^{\otimes k} \otimes I^{j+l-2k}) \cup \{0\}, \qquad (35)$$

where, as usual, the sum of two wave front set is defined as the sum on the fibres of the cotangent spaces. Unfortunately this does not suffice to conclude the proof of the theorem since, being F'_j and G'_l not compactly supported respectively on $\mathscr{C}_p^j$ and $\mathscr{C}_p^l$, their product does not lie in $\mathcal{E}'(\mathscr{C}_p^{j+l})$. Hence we cannot directly test $(F'_j \otimes G'_l) \cdot (\omega^{\otimes k} \otimes I^{j+l-2k})$ on the identity on $\mathscr{C}_p^{2k}$ in order to infer that (32) is an element in $\mathscr{A}^{j+l-2k}$.

Therefore we need once more to resort to an analysis of the wave front sets to show directly the wanted property and, to this avail, it is sufficient to consider just the case $k = 1$, since all others descend out of a recursive application of the very same procedure and eventually to the application of an operator realizing the total symmetrization. In other words we shall be interested in

$$\int_{\mathscr{C}_p^2} (F'_j \otimes G'_l) \cdot (\omega \otimes I^{j+l-2}) d\mu(x_1) d\mu(y_1),$$

where we recall that $F'_j \in \mathscr{A}^j$ and $G'_l \in \mathscr{A}^l$. We can exploit property 3 of definition 3.1. Particularly we notice that, if the factorization therein holds for a compact set $\mathcal{O}$, it must also hold for every larger compact set $\mathcal{O}_1$ containing $\mathcal{O} \in \mathscr{C}_p$. We can thus find a common set $\mathcal{O}_1$ for which the factorization property stands true at the same time for $F'_j = (\Theta + \Theta')F'_j$ and $G'_l = (\Theta + \Theta')G'_l$ with respect to a common compactly supported function Θ equal to 1 on $\mathcal{O}_1$. We have effectively divided the above integral in the sum of four different ones, which we shall now analyse separately.

Part I) the first term, we consider, is

$$\int_{\mathscr{C}_p^2} (\Theta(x_1) F'_j \otimes \Theta(y_1) G'_l) \cdot (\omega \otimes I^{j+l-2}) d\mu(x_1) d\mu(y_1), \qquad (36)$$

In this case the integral can be read as the smearing of a distribution in $\mathcal{D}'(\mathscr{C}_p^{j+l})$ with a test function in $C_0^\infty(\mathscr{C}^2)_p$. Hence Theorem 8.2.12 of [Hö89] guaranties us that, using the notation introduced therein, the result of (36) is a distribution whose wave front set, is contained in $W_{j+l-2} \cup (W_j \times W_l) \circ (A \times \{0\}) \subset W_{j+l-2}$ as given in (29). Notice that, in the proof of the last inclusion, we have used (33) and (34). That said, property 1 in definition 3.1 is automatically satisfied since, per hypothesis, $F'_j \in \mathscr{A}^j$ and $G'_l \in \mathscr{A}^l$, while property 3 holds true for the resulting distribution, by noticing that (30) holds a priori in all variables and, thus, it is left untouched for those which have not been integrated out in (36).

Part II) the second term we consider is

$$\int_{\mathscr{C}_p^2} (\Theta'(x_1) F'_j \otimes \Theta'(y_1) G'_l) \cdot (\omega \otimes I^{j+l-2}) d\mu(x_1) d\mu(y_1).$$

The analysis is rather simple if we make a profitable use of (30) in the integrated variables, namely we set $\Theta'(x_1)F'_j = f(x_1)t_{j-1}$ and $\Theta'(x_{j+1})G'_l = f'(x_{j+1})t'_{l-1}$ where t_{j-1} and t'_{l-1} are in $\mathscr{A}^{j-1}$ and $\mathscr{A}^{l-1}$ respectively while f, f' and their derivative along the V-coordinate are square integrable on $\mathscr{C}$ in particular. The wanted result straightforwardly arises since $t_{j-1} \otimes t'_{l-1} \in \mathscr{A}^{j+l-2}$ while the remaining operation $\omega(f, f')$ is well defined due to the continuity property (25) satisfied by ω.

Part III & IV) the remaining two terms are substantially identical and we shall treat only one of them, the other following suit. Hence let us consider

$$\int_{\mathscr{C}_p^2} (\Theta F'_j \otimes \Theta' G'_l) \cdot (\omega \otimes I^{j+l-2}) d\mu(x_1) d\mu(y_1). \tag{37}$$

In order to cope with this integral we introduce a new larger factorization $\eta + \eta' = 1$ with $\eta \in C_0^\infty(\mathscr{C}_p)$ and such that $\eta = 1$ on a large compact set containing properly the closure of $supp(\Theta)$ so that both $supp(\eta'\Theta') \cap supp(\Theta) = \emptyset$ and $\eta'\Theta' = \eta'$. If now substitute G'_l with $(\eta + \eta')G'_l$, we obtain another splitting: on the one hand, since $\Theta'\eta \in C_0^\infty(\mathscr{C}_p)$, the analysis of $\Theta F'_j \otimes \Theta' \eta G'_l$ boils down to that of case I, while, on the other hand, $\frac{\Theta(V)[\Theta'\eta'](V')}{(V-V')^2}$ turns out to be smooth on $\mathscr{C}_p$, since per construction $(V - V')^2 > 0$ for V on the support of Θ and V' on the one of η'. Hence, if we indicate the factorization (30) introduced by η as $\eta'(x_{j+1})G'_l = f(x_{j+1})t_{l-1}$ where $t_{l-1} \in \mathscr{A}^{l-1}$, we obtain that $u \doteq \Theta\omega(f)$ is a compactly supported smooth function on $\mathscr{C}$, thus yielding in place of (37)

$$\lambda \doteq F'_j(u) \otimes t_{l-1} \ ,$$

where the smearing of $F'_j \in \mathscr{A}^j$ is a well-posed operation being u compactly supported. Furthermore, due to theorem 8.2.12 in [Hö89], we also know that the wave front set of the result λ is contained in W_{j+l-2} given in (29). In order to conclude that λ is contained in $\mathscr{A}^{j+l-2}$, we have to check property 3 in definition 3.1, which holds true just applying (30) either for F'_j before smearing it with u or directly to the t_{l-1}. □

The end point of this subsection is that $(\mathscr{F}_{B,E}, \star_\omega)$ is a full-fledged $*$-algebra and, thus, we are ready to discuss the intertwining relations between bulk and boundary data.

3.5 Interplay between the algebras and the states on $\mathscr{D}$ and on $\mathscr{C}_p$

We are now in place to discuss a connection between the field theories we have up to now described, hence setting up a bulk-to-boundary correspondence and identifying an Hadamard state in the bulk. We shall devote the whole subsection to this issue, but, as a starting point, we need to recapitulate the geometric structure in order to clearly relate the structures in subsection 2.2 and 3.2.

Let us thus briefly recall that we consider the globally hyperbolic subset $\mathcal{O}'$ contained in a geodesic neighbourhood of any but fixed point p in a strongly causal spacetime M. We extract in $\mathcal{O}'$ a double cone $\mathscr{D} \equiv \mathscr{D}(p, q)$, which plays the role of the bulk spacetime, while the set

$\partial J^+(p) \cap \overline{\mathscr{D}}$, the latter symbol standing for the closure of $\mathscr{D}$, is our chosen boundary. Up to the choice of an orthonormal frame in p, the latter can be seen as the locus $u = 0$ in the natural coordinate system (u, r, x^A) introduced in subsection 2.2 and it is furthermore endowed with the metric (2). In terms of the structure of subsection 3.2, we can identify the boundary as $\mathscr{C}_p^+$ with a small caveat with respects to the used coordinates. While it is always possible to switch from x^A $(A = 1, 2)$ to the standard (θ, φ), the role of V as a coordinate is played by r, the affine parameter on the null geodesics of the cone. As a last point, the role of the function h in (18) is played in general by $\sqrt[4]{|g_{AB}|}$, where g_{AB} are the metric components appearing in (2) evaluated at $u = 0$ and $|\cdot|$ is kept in order to recall that we are actually considering the determinant. It is interesting to notice, that, whenever the conditions for the reduction of (2) to (4) are fulfilled, then h can be set to $V \equiv r$, (see also the relation between the volume elements of the sphere in different coordinates (3)). Furthermore, in the used retarded coordinates, the exponential map becomes an identity, hence, if not strictly necessary, we shall not indicate it anymore.

We shall now proceed in two steps. The first one will consist of proving that it is possible to introduce a well-defined map from the extended algebra in $\mathscr{D}$ to the one on $\mathscr{C}_p^+ \subset \mathscr{C}_p$, while, in the second, we shall prove that the found map is also well-behaved with respect to the algebra structures.

Theorem 3.1. *Let $\mathscr{D}$ be a double cone and let the portion $\mathscr{C}_p^+$ of the boundary be seen as contained in a cone $\mathscr{C}_p$. Let us introduce the linear map $\Pi : \mathscr{F}_e(\mathscr{D}) \to \mathscr{F}_{B,E}(\mathscr{C}_p)$ such that*

$$\Pi_n(F_n) \doteq \sqrt[4]{|g_{AB}|}_1 \dots \sqrt[4]{|g_{AB}|}_n \, \Delta^{\otimes n}(F_n)\big|_{\mathscr{C}_p^n} \;, \tag{38}$$

where Δ is the causal propagator (7), $|_{\mathscr{C}_p}$ stands for the restriction on $\mathscr{C}_p$ and the subscripts $1, ..., n$ entail dependence of the root on the coordinates on the i-th cone. Then, the following properties hold true:

1. *The integral kernel of Π_n is an element in $\mathcal{D}'\left((\mathscr{C}_p \times \mathscr{D})^n\right)$.*
2. *The image of $\mathscr{F}_e(\mathscr{D})$ under Π lies in $\mathscr{F}_{E,B}(\mathscr{C}_p)$.*

Proof. We shall prove the above two properties in two separate steps:

1) *Construction of $(\sqrt[4]{|g_{AB}|}\Delta_{\mathscr{C}})^{\otimes n}$ and its wave front set*

As a starting point, let us consider $\Delta^{\otimes n}$, Δ being as in (7). Thanks to our geometric hypothesis, this is a distribution in $\mathcal{D}'(\mathcal{O}'^{2n})$, where $\mathcal{O}'$ is the global hyperbolic region which contains $\mathscr{D}$, and whose WF is contained in $\{WF(\Delta) \cup \{0\}\}^n$ where

$$WF(\Delta) = \left\{(x_1, \zeta_1; x_2, \zeta_2) \in T^*\mathscr{D}^2 \setminus \{0\}; (x_1, \zeta_1) \sim (x_2, -\zeta_2).\right\} \tag{39}$$

The equivalence relation $(x_1, \zeta_1) \sim (x_2, \zeta_2)$ means that there exists a null geodesic γ with respect to the metric g in $\mathscr{D}$ which contains both x and y and such that $g^{\mu\nu}(\zeta_1)_\nu$ and $g^{\mu\nu}(\zeta_2)_\nu$ are the tangent vector to γ respectively in x and y. This form was worked out for example in [Rad96a, Rad96b].

Furthermore, we can now slavishly apply the same reasoning as in proposition 4.3 in [Mo08] or in [Ho00] to use theorem 8.2.4 in [Hö89] to restrict on $\mathscr{C}_p^+$ one entry of the causal propagator, while leaving the other localized in $\mathscr{D}$. If we indicate as χ the restriction map and as $\Delta_{\mathscr{C}}$ the image of Δ under χ, still theorem 8.2.4 of Hörmander yields that $WF(\Delta_{\mathscr{C}}) \subset \chi^* WF(\Delta)$. Particularly this entails that, if $(x, \zeta_x; y, \zeta_y) \in WF(\Delta_{\mathscr{C}})$ with $x \in \mathscr{C}$ and $y \in \mathscr{D}$ both ζ_x and ζ_y cannot vanish at the same time and furthermore $(\zeta_x)_r$ is never equal to zero. At this point, we need only to multiply every $\Delta_{\mathscr{C}}$ with $\sqrt[4]{|g_{AB}|}$. If we notice that, in a geodesic neighbourhood, the square root of the determinant of the metric is the inverse of the Van-Vleck Morette determinant (see for example section 3.1 in [Po03]), which is smooth and locally of definite sign, we can claim that $\sqrt[4]{|g_{AB}|}$ is also smooth on $\mathscr{C}_p$. Hence we can safely multiply it to $\Delta_{\mathscr{C}}$ without changing the wave front set and thus $\Pi_n \doteq (h\Delta_{\mathscr{C}})^n \in \mathcal{D}'(\{\mathscr{C}_p \times \mathscr{D}\}^n)$, where $h = \sqrt[4]{|g_{AB}|}$.

2) *on the image of* Π

Let us notice that, since every $F \in \mathscr{F}_e(\mathscr{D})$ is composed by a finite number of F_n, it is sufficient to prove that the generic F_n is mapped to an element of $\mathscr{A}_s$ by Π or, better, by Π_n. Moreover, the pointwise product of Π_n and $I^n \otimes F_n$, I being the identity in $\mathcal{D}'(\mathscr{C}_p)$, is well defined because their wave front set does not sum up to the zero section, as one can infer from (12) and (39). Hence the Hörmander criterion for the multiplication of distributions, *i.e.*, theorem 8.2.10 in [Hö89], is fulfilled and, moreover, the resulting distribution $(\Pi_n) \cdot (I^n \otimes F_n)$ can be tested on any compactly supported smooth characteristic function χ of the support of F_n ($\chi = 1$ on the support of F_n), hence yielding the sought $\Pi_n(F_n)$. Theorem 8.2.12 still in [Hö89] guarantees that $\Pi_n(F_n) \in \mathcal{D}'(\mathscr{C}^n)$ and that the wave front set is contained in W_n, as in (29). This is tantamount to verify the second condition in definition 3.1.

As far as the first one is concerned, this can be shown to hold true since, per construction $supp(F_n) \subset K \subset \mathscr{D}^n$, K being a compact set and this holds also for $J^-(\mathscr{D}) \cap \mathscr{C}_p^+$ thanks to the support property of Δ. Thus, per definition of Π_n, the wanted statement is verified.

The third and last requirement can also be shown to hold thanks to the fact that the singular support of the causal propagator (7) is contained in the set of the null geodesics and those emanating from the support of any F_n (recall that $supp(F_n) \subset K$) intersect $\mathscr{C}_p$ on a compact set that is disjoint from p in particular. Hence the causal propagator is a smooth function whenever one entry is smoothly localized[1] on the support of F_n and the other on a neighbourhood of p. Furthermore such smooth function, even after multiplication both per the function Θ' as in definition 3.1 and per $\sqrt[4]{|g_{AB}|}$, is square integrable and bounded, together with its V-derivative, in a suitable open set of $\mathscr{C}_p \sim \mathbb{R}^+ \times \mathbb{S}^2$ such that $V \in (0, V_0)$.

Finally let us notice that $\Pi_n(F_n)$ is totally symmetric whenever F_n has such property, and this complete the proof that $\Pi_n(F_n) \in \mathscr{A}_s^n$. □

Remark: Notice that the ideal $\mathscr{I}$ generated by the equations of motion (6) is mapped by Π to $0 \in \mathscr{F}_{B,E}(\mathscr{C})$, because Δ is a weak solution of (6). Hence the image of both $\mathscr{F}_b(\mathscr{D})$ and $\mathscr{F}_o(\mathscr{D})$ under Π lie in $\mathscr{F}_{B,E}(\mathscr{C})$; actually they coincide.

[1] The localization is realized by pointwise multiplication by smooth functions with suitable supports.

If we bear in mind this remark, we should also stress another important property enjoyed by Π

Proposition 3.4. *The map Π is injective when acting on the on shell extended algebra $\mathscr{F}_{eo}(\mathscr{D})$*

Proof. Due to the continuity of Π in the relevant topology and since $\Pi_n = \Pi_1^{\otimes n}$, it is sufficient to prove injectivity for compactly supported smooth functions. To wit, we have to show that $\Pi_1 : C_0^\infty(\mathscr{D}) \to \mathscr{A}_s$ is injective if we consider the quotient of $C_0^\infty(\mathscr{D})$ with respect to $\mathscr{I} \doteq \{Pf,\ f \in C_0^\infty(\mathscr{D})\}$ where P is the operator in (6). A closer look to the form of Π_1 tells us that, up to the multiplicative factor, the image of $\frac{C_0^\infty(\mathscr{D})}{\mathscr{I}}$ under Π_1 are, according to (9), elements of $C^\infty(\mathscr{C}_p)$. Suppose, per absurdum, that there exist two representative elements of f, g of two different equivalence classes in $\frac{C_0^\infty(\mathscr{D})}{\mathscr{I}}$ such that $\Pi_1(f-g) = 0$, then we could use $\psi_f = \Pi_1(f) \doteq \sqrt[4]{|g_{AB}|}\Delta(f)|_{\mathscr{C}}$ and $\psi_g = \Pi_1(g) \doteq \sqrt[4]{|g_{AB}|}\Delta(g)|_{\mathscr{C}}$ as the initial data of a Goursat problem, which yields a unique smooth solution of (6) in $J^+(p)$. Since these new solutions of (6) must coincide with those constructed out of f and g in the bulk due to uniqueness, the only possibility to have $\psi_f = \psi_g$ is that $f = g$. □

Before continuing with the analysis of the map Π which acts on the extended algebra $\mathscr{F}_e(\mathscr{D})$, we shall show that the pull-back both of the symplectic form $\sigma_{\mathscr{C}}$ and of the boundary state ω have nice interplay with the symplectic form in the bulk and with the Hadamard states in general. Let us start with the analysis of the pull-back of the symplectic form.

Proposition 3.5. *The projection $\Pi : \mathscr{F}_b(\mathscr{D}) \to \mathscr{F}_{B,E}(\mathscr{C}_p)$ is a simplectomorphism, that is, for every $f, h \in C_0^\infty(\mathscr{D})$,*

$$\sigma(\varphi_f, \varphi_h) = \sigma_{\mathscr{C}}\left(\Pi_1 f, \Pi_1 h\right), \tag{40}$$

where σ is here taken as in (8).

Proof. As a starting point, let us indicate $\varphi_f = \Delta f$ and $\varphi_h = \Delta h$ where Δ is as in (7). Let us also consider both a Cauchy surface Σ of $\mathscr{D}$ and the portion of $\mathcal{O}_1 \doteq \mathscr{D} \cap I^-(\Sigma)$ whose boundary is formed by the null surface $\mathscr{C}_p^+$ and by Σ. Let us now consider the current

$$J_\mu \doteq \varphi_f \partial_\mu \varphi_h - \varphi_h \partial_\mu \varphi_f;$$

if we indicate with n^μ the unit future directed vector normal to Σ, than $\int_\Sigma d\mu(\Sigma) n^\mu J_\mu = \sigma(\varphi_f, \varphi_h)$. We can hence apply the divergence theorem to J_μ in $\mathcal{O}_1$ seen as a subregion of a larger globally hyperbolic spacetime, namely $\mathcal{O}'$ that contains $\mathscr{D}$. The end point is that, since $\nabla^\mu J_\mu = 0$ in $\mathcal{O}_1$ in particular, the following identity holds

$$\sigma(\varphi_f, \varphi_h) = \int_{\mathscr{C}_p^+} d\mu(\mathscr{C}_p^+) n^\mu J_\mu\ .$$

Furthermore, the right hand side of the preceding equation can be rewritten in the retarded coordinates on $\mathscr{C}_p^+$ and, if one uses the relation between the volume elements on the sphere (3) in spherical and local coordinates, it becomes

$$\int_{\mathbb{R}^+\times\mathbb{S}^2} \sqrt{|g_{AB}|}\left[\varphi_f \frac{\partial}{\partial r}\varphi_h - \varphi_h \frac{\partial}{\partial r}\varphi_f\right] dr \wedge d\mathbb{S}^2 \ , \tag{41}$$

where both φ_f and φ_h are meant evaluated on $\mathscr{C}_p^+$, a legitimate operation as explained at the beginning of the section. Finally, due to the antisymmetry of the preceding expression, we can consider $\sqrt[4]{|g_{AB}|}\varphi_f = \Pi_1 f$ as well as $\sqrt[4]{|g_{AB}|}\varphi_h = \Pi_1 h$ and it holds per direct inspection that (41) equals $\sigma_{\mathscr{C}}(\Pi_1 f, \Pi_1 g)$ as given in (19) setting $r = V$. □

The next proposition deals with the singular structure of the state ω when pulled back in the bulk.

Proposition 3.6. *Under the assumptions of theorem 3.1, it holds that*

$$H_\omega \doteq \Pi^*\omega, \tag{42}$$

is an Hadamard bi-distribution constructed as the pull-back of ω *as in* (23) *under* Π *as in* (38).

Proof. The proof of this proposition can be achieved just restricting our attention to the compactly supported smooth functions on $\mathscr{D}$. In this case the antisymmetric part of H_ω equals the symplectic form (8) which is preserved by action of Π as per proposition 3.5 and H_ω satisfies weakly the equation of motion. Hence, due the the work of Radzikowksi [Rad96a], it is only necessary to check that the wave front set of H_ω satisfies the microlocal spectral condition. This can be verified following almost slavishly the procedure envisaged in [Mo08, Ho00]. We shall not enter in a pointless repetition of all the details and, instead, we just recall the main part of the proof, which consists of showing that

$$WF(H_\omega) \subset \{(x_1,\zeta_1;x_2,\zeta_2) \in T^*\mathscr{D}^2 \setminus \{0\}, (x_1,\zeta_1) \sim (x_2,\zeta_2), \zeta_1 \triangleright 0\}, \tag{43}$$

where $\sim$ is the same equivalence relation as in (39), while $\zeta_1 \triangleright 0$ means that ζ_1 is a future directed vector. The key point stands in localizing Π_2 on a compact set $K^2 \subset \mathscr{D}^2$ multiplying it by a partition of unit on $\mathscr{C}_p$, *i.e.*, $\Theta_K + \Theta'_K = 1$, such that $\Theta_K \in C_0^\infty(\mathscr{C}_p)$ is equal to 1 on the compact set $J^-(K) \cap \mathscr{C}_p$. With such a construction in mind, H_ω becomes the sum of four terms $\omega(\Theta_K + \Theta'_K)\Pi_1 \otimes (\Theta_K + \Theta'_K)\Pi_1$ and, if K is chosen sufficiently large, the only one which has a non vanishing wave front set is $\omega\Theta_K\Pi_1 \otimes \Theta_K\Pi_1$. The wanted inclusion arises out of an application of theorem 8.2.13 of [Hö89]. □

We can now proceed to prove a second theorem which focuses on the effect of the map Π on the algebraic structures and on the boundary state. Particularly we stress that we shall individuate an Hadamard state in the bulk.

Theorem 3.2. *Under the assumptions of theorem 3.1, it also holds,*

1. *there always exists a product* $\star_{H_\omega}$ *in* $\mathscr{F}_e(\mathscr{D})$ *such that* Π *is a* $*$*-homomorphism between* $(\mathscr{F}_e(\mathscr{D}), \star_{H_\omega})$ *and* $(\mathscr{F}_{B,E}(\mathscr{C}), \star_\omega)$.

2. Π *is an injective* $*-$*homomorphism when acting "on shell" on* $(\mathscr{F}_{eo}(\mathscr{D}), \star_{H_\omega})$.

Proof. We shall here prove only the first statement because the second one descends per direct inspection from it and from the proposition 3.4. We can first notice that Π automatically preserves the $*-$operation because $\overline{\Pi}_n = \Pi_n$ and hence

$$\Pi_n(F_n)^* = \Pi_n(F_n^*) \ .$$

Thus we only need to verify the statement on the $\star$-products. Particularly we look for $\star_{H_\omega}$: $\mathscr{F}_e(\mathscr{D}) \times \mathscr{F}_e(\mathscr{D}) \to \mathscr{F}_e(\mathscr{D})$ such that, for every $F, G \in \mathscr{F}_e(\mathscr{D})$

$$\Pi(F \star_{H_\omega} G) = (\Pi F) \star_\omega (\Pi G) \tag{44}$$

and, moreover, such that $(\mathscr{F}_e, \star_{H_\omega})$ is isomorphic to $(\mathscr{F}_e, \star_H)$.

The natural candidate arises from the analysis performed in proposition 3.6 and particularly from the distribution H_ω introduced in (42) which we aim to plug in (13) in place of H. This is a well-defined procedure since H_ω is of Hadamard form as proved in proposition 3.6 and hence $(\mathscr{F}_e, \star_H)$ turns out to be isomorphic to $(\mathscr{F}_e, \star_{H_\omega})$, the isomorphism being realized as in (14). We are thus left with (44), to be verified for every F and G in $\mathscr{F}_e(\mathscr{D})$. If we exploit the bilinearity of all the involved $\star$-products, this reduces to show that

$$(\Pi_l F_l \otimes \Pi_m G_m)(\omega^{\otimes k}) = \Pi_{l+m-2k}\left((F_l \otimes G_m)({H_\omega}^{\otimes k})\right)$$

for $l + m - 2k \geq 0$. The last inequality directly stems from bilinearity, (42) and from the fact that $\Pi_k = \Pi_{k_1} \otimes \Pi_{k_2}$ for all $k_1, k_2 > 0$ such that $k_1 + k_2 = k$. □

Remark: Notice that it is possible to turn the injective homomorphism into a bijective one if we restrict our attention to the local von Neumann algebras defined as the double commutant of the C^*-algebra generated by the local Weyl operators constructed out of the symplectic form (8) and (19) respectively in $\mathscr{D}$ and on the boundary $\mathscr{C}_p^+$. This last claim is based on the invertibility of Π_2 on the weak solutions of the Klein Gordon equation [BrMo09], while we recall that the Goursat problem with compact initial data on $\mathscr{C}_p$ yields, in general, a solution of (6) whose restriction on any Cauchy surface of $\mathscr{D}$ is not compact.

Alas, the mentioned von Neumann algebra does not contain some relevant physical observables such as the component of the regularized stress energy tensor or the regularized squared fields, objects we would like to use in order to extract information about the local geometric data such as the scalar curvature.

4 Interplay with general covariance and comparison between spacetimes

We are now in place to collect all our results in a single body which will allow us to show both a nice interplay with the principle of local covariance, as devised in [BFV03], and a possibility to compare quantum field theories on different backgrounds both at a level of algebras and of states.

To this avail the construction in subsection 3.5 will play a pivotal role and the natural language we shall adopt is that of categories, which was already introduced in subsection 2.1. Particularly we noticed that the construction of an extended algebra of observables on a double cone $\mathscr{D}$ can be realized as a suitable functor between the categories DoCo_{iso} and Alg, although it was not possible to extend such a functor to DoCo.

Despite this obstruction, the additional structure which arises from both the boundary and the field theory defined thereon, we allow us to circumvent the above problem in a way that will also put us in a position to compare field theories in different spacetimes. To this avail, we need to introduce a further category beyond those of section 3.1, namely

BAlg: it is the category whose objects are the extended boundary algebras presented in subsection 3.4, constructed on all possible $\mathscr{C}_p^+$, while the morphisms are the $*-$homomorphisms.

The key point consists of making a profitable use of the $*-$homomorphisms Π introduced in theorem 3.1, in order to notice that $\Pi \circ \mathscr{F}$ defines really a functor between the two categories DoCo and BAlg, such that the following diagram commute.

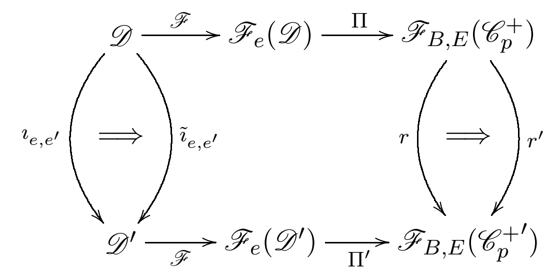
$$\begin{array}{ccccc} \mathscr{D} & \xrightarrow{\mathscr{F}} & \mathscr{F}_e(\mathscr{D}) & \xrightarrow{\Pi} & \mathscr{F}_{B,E}(\mathscr{C}_p^+) \\ \imath_{e,e'}\big\downarrow & & & & \big\downarrow r \\ \mathscr{D}' & \xrightarrow[\mathscr{F}]{} & \mathscr{F}_e(\mathscr{D}') & \xrightarrow[\Pi']{} & \mathscr{F}_{B,E}(\mathscr{C}_p^{+\prime}) \end{array} \tag{45}$$

Here the arrow r traces is origin back to the analysis started in subsection 3.2 where, it was shown, that the boundary theory can be constructed and later analysed independently from the specific bulk we consider. The information of the latter is subsequently projected on the cone via the map Π. Hence r here is nothing but a restriction map which accounts for the fact that, in the geometric construction we set up, it holds that $\imath_{e,e'}(\mathscr{C}_p^+) \subseteq \mathscr{C}_p^{+\prime}$.

As far as the connection with the principle of general local covariance is concerned, we should start recalling that, in the most general case, it is not possible to find a direct relation between $\mathscr{F}(\mathscr{D})$ and $\mathscr{F}(\mathscr{D}')$, unless the embedding $\Pi : \mathscr{F}(\mathscr{D}') \to \mathscr{F}(\mathscr{C}')$ can be inverted on the image of Π composed with r. This is indeed what happens whenever, for example, $\imath_{e,e'}$ is an isometry (or, at worst, even a conformal isometry - see [Pi09]) which preserves the base point p, hence we are working in DoCo_{iso}.

Under this assumption, the discussion about causality, which is usually an integrated part of the reasoning, such as in [BFV03, BrFr09], does not need to be directly performed, since its essence is already encoded in analysis of the properties of the map Π. A similar statement holds also for the time slice axiom; particularly, since the theory on the boundary is, to a certain extent, non dynamical, there is no such axiom in our boundary framework and, that in the bulk, is automatically assured by Π and by its properties.

It is important to realize that the above diagram does not catch the whole picture, since, it does not give information about different choices of frames e and e' both in $\mathscr{D}$ and $\mathscr{D}'$. Actually one could exploit the bundle structure in order to act on e and e' in the definition (1) of $\imath_{e,e'}$ by means of two independent $SO_0(3,1)$-transformations, say Λ and Λ', yielding $\widetilde{e} \doteq \Lambda e$ and $\widetilde{e}' \doteq \Lambda' e$. The end point would be another arrow in DoCo, say $\imath_{\widetilde{e},\widetilde{e}'}$ and, at the same time, a different restriction map, say r', in BAlg. These would be connected to the original ones by a morphism which traces back its origin to (Λ, Λ') thus giving rise to two 2-category structure which could be summarized in the following commutative diagram



(46)

It is important to notice that the choice of a different frame is tantamount in a physical language to choose a different observer at the point $p \in M$ and, therefore we expect that the bulk state identified in the previous section remains untouched under a change of frame, thus guaranteeing us that the choice we performed is compatible also with the above diagram. In the next subsection we shall indeed show that this is the case, hence strengthening the validity of our results.

4.1 Comparison of expectation values in different spacetimes

The importance of the previous subsection lies in (45) which is the heart of our analysis on the local algebraic structure of a quantum scalar field theory on different spacetimes. Nonetheless we are still falling one step short from our final goal since we need still to clarify in which sense one can compare two field theories on two different backgrounds. The forthcoming discussion will be totally devoted to this point and we shall first clarify the procedure abstractly and, then, we shall give a concrete example.

Therefore, if we start from (45), we can straightforwardly see that, whenever we assign a state ω on $\mathscr{F}_{E,B}(\mathscr{C}'_p)$, we can pull it back either on $\mathscr{F}_e(\mathscr{D}')$ via Π' or on $\mathscr{F}_e(\mathscr{D})$ via $r \circ \Pi$. The meaningfulness of this operation is guaranteed by the commutativity of the diagram itself and it is noteworthy that the information of the bulk geometry is indeed restored by Π and Π'.

This is a rather general feature which holds true regardless of the global structure of the spacetimes in which $\mathscr{D}$ and $\mathscr{D}'$ are embedded. Yet, on a practical ground, if one wants to make a crystal clear use of the above observation, a first natural step would be to choose that one of the two double cones, we use, is embedded in the four dimensional Minkowski spacetime, where our capability of performing explicit computations of physical quantities is enhanced mostly due to the large symmetry of the background.

Let us try to be a little bit more specific on this point: hence let us consider, a double cone $\mathscr{D}$ as a subset of $(\mathbb{R}^4, \eta)$ while $\mathscr{D}'$ is left lying in a generic strongly causal spacetime M'. The only further assumption we ask for is that all the above objects are chosen in such a way that there exist two frames e and e' respectively in $T_p\mathbb{R}^4$ and $T_{p'}M'$ such that $\imath_{e,e'} : \mathscr{D} \to \mathscr{D}'$ fulfils all the conditions set in section 2. At this point we can slavishly apply the construction discussed for the local fields and algebras which are related on the boundary via the map r.

As a next step, following also the general philosophy of [BFV03], we can consider observables constructed out of the same local fields either on $\mathscr{D}$ or on $\mathscr{D}'$, but, in our scenario, we have the further advantage that we can evaluate them on the pull-back of a suitable boundary state yielding an Hadamard counterpart in the bulk. In general the difference between the computed expectation values depends on the geometric data of both $\mathscr{D}$ and $\mathscr{D}'$, but, being the former an open set of Minkowski spacetime, it turns out that only the geometry of $\mathscr{D}'$ can play a role and we can probe it just out of this procedure. In other words we are comparing quantum field theories on different backgrounds.

Nonetheless this procedure is still too involved since one has to cope with the singular structure of the chosen state. Even if we restrict to those fulfilling the Hadamard condition, one can easily avoid the need of taking care of the regularization procedure if one consider two of such Hadamard states constructed out of the pull-backs of different boundary states and employs their difference. In this case the integral kernel of the two-point function of such a difference is known to be smooth, but the price to pay is the need to introduce a second state. Yet the situation is not so grim since, after all, we have at our disposal a natural candidate as a distinguished reference, namely ω arising form (42) discussed in proposition 3.6.

Before we discuss an explicit example of the procedure we have in mind, we need to further stress the properties both of ω and of its pull-back in the bulk, say $\Pi^*\omega$. These can be inferred out of both proposition 3.2 and theorem 3.2; we shall now discuss those which are more remarkable for our scopes:

1. *Local Lorentz invariance:* According to the third point of proposition 3.2, ω turns out to be invariant under a large set of geometric transformations on the full cone $\mathscr{C}$ which contains the boundary. Particularly this yields that ω is invariant under the natural action of $SO(3,1)$ or, in other words, it is independent from the choice of a particular orthonormal frame. Since map Π is substantially constructed out of the causal propagator (7) in a geodesic neighbourhood, its action on ω via pull-back does not spoil the above property, *i.e.*, $\Pi^*\omega$ is invariant under change of an orthonormal frame. Let us stress that this entails the well-posedness of our scheme even with respect to the structure of 2-category (46).

2. *Microlocal structure:* The wave front set of ω is contained in the union of (27) and (28) and,

most notably, it does not contain past directed direction. Particularly this allows to prove proposition 3.6 according to which $\Pi^*\omega$ satisfies in the bulk double cone the Hadamard property.

3. *Behaviour as a "vacuum":* the boundary state ω turns out to be invariant under rigid translations of the V-coordinate, or, in other words, it is a vacuum with respect to the transformation generated by the vector ∂_V. This statement can be proved exactly as in [Mo06, DMP08] for the counterpart on the conformal boundary of an asymptotically flat or of a cosmological spacetime, namely from the explicit form of the two-point function (26). This also means that the energy computed on the cone with respect to ∂_V is minimized. Unfortunately such a nice property has not a strong counterpart in the bulk but, if the bulk can be realized as an open set in $(\mathbb{R}^4, \eta)$, then $\Pi^*\omega$ can be seen to coincide with the Minkowski vacuum [BrMo09].

4.2 An application: extracting the curvature

In this subsection we shall present an explicit application which follows the guidelines above given. For simplicity let us consider on the one hand a double cone $\mathscr{D}$ realizable as an open subset of Minkowski $(\mathbb{R}^4, \eta)$ where the metric η is here meant in the standard diagonal form with respect to the Cartesian coordinates (t, x, y, z) induced by the standard orthonormal basis/frame e of $\mathbb{R}^4$. As $\mathscr{D}'$, we shall instead consider a double cone which can be embedded in a homogeneous and isotropic solution of Einstein's equation with flat spatial section. This is a Friedmann Robertson Walker spacetime (M', g) where $g = a^2(t)\eta$ and $a(t) \in C^\infty(I, \mathbb{R}^+)$, where $I \subseteq \mathbb{R}$ and $a(0) = 1$. Here t refers to the so-called conformal time and thus we are still considering the coordinates (t, x, y, z) induced by the standard basis/frame of $\mathbb{R}^4$, which will be indicated as e' to distinguish it from the previous one.

Since the underlying spacetimes are conformally related, their causal structure coincides and also the double cones in particular. Let us consider two points $p = (0, 0, 0, 0)$ and $q = (t', 0, 0, 0)$ and the corresponding double cones $\mathscr{D}(p, q) \subset \mathbb{R}^4$, and $\mathscr{D}'(p, q) \subset M'$. In this framework the map $\imath_{e,e'} : \mathscr{D}(x_0, x_1) \to \mathscr{D}'(x_0, x_1)$ turns out to be trivial. As next datum, we shall take a minimally coupled real scalar field theory, namely

$$\phi : \mathscr{D} \to \mathbb{R}, \quad \Box\phi = 0 \ ,$$

where $\Box$ is the d'Alembert wave operator constructed out of the metric in $\mathscr{D}$. We wish to stress once more that we are considering the very same equation also in $\mathscr{D}'$. If we follow the guideline in section 3.1 of the previous section, we can construct the quantum algebras $\mathscr{F}_e(\mathscr{D})$ and $\mathscr{F}_e(\mathscr{D}')$ and their counterpart on the boundaries $\mathscr{C}_p$ and $\mathscr{C}'_p$. As outlined in the previous subsection, we shall now consider two algebraic states on $\mathscr{F}_{B,E}$, one, ω, is the reference one, while the other can be arbitrary provided that the pull-back to the bulk via Π still fulfils the Hadamard condition. This request is not so restrictive since, for example, any state which differs from ω by a smooth function on the boundary, which vanishes in a neighbourhood of the tip, is an admissible choice.

Particularly let us thus consider another Gaussian state $\omega' : \mathscr{F}_{B,E}(\mathscr{C}_p) \to \mathbb{C}$, whose two-point function has the following form

$$\omega'(\psi_1, \psi_2) = \omega(\psi_1, \psi_2) + \frac{1}{4\pi} \int_{\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{S}^2} \psi_1(r, \theta, \varphi) \psi_2(r', \theta, \varphi) \, dr dr' d\mathbb{S}^2(\theta, \varphi) \; . \tag{47}$$

As a first step we should check that the pull-back of the state ω' is of Hadamard form, since the integral in the right hand side entails that the integral kernel of $\omega' - \omega$ is not smooth containing a δ-like singularity in the angular coordinates. Despite this fact ω' can be pulled back to every spacetime still yielding an Hadamard bi-distribution; the reason boils down to the symmetric structure of $\omega' - \omega$ is symmetric which, thanks to the results of §8 of [Hö89] of which we also adopt the nomenclature, it holds

$$WF(\Pi_2)' \circ WF(\omega' - \omega) = \emptyset \; .$$

Hence the same proof as in [Mo08, Ho00] which guarantees that ω is of Hadamard form can be here repeated, ultimately giving the same result.

We are now ready to look at the expectation values of suitable observables from the point of view of both $\mathscr{D}$ and $\mathscr{D}'$ and, the most natural one is the expectation value $\phi^2(f)$, where $f \in C_0^\infty(\mathscr{D})$ (and hence in this case also in $C_0^\infty(\mathscr{D}')$), with respect to the bulk states constructed out of ω' and ω. Notice that $\phi^2(f)$ is a short cut for saying that we are actually considering $u_f = f(x)\delta(x - y) \in \mathscr{A}_s^2(\mathscr{D}) \subset \mathscr{F}_e(\mathscr{D})$. One of the advantage of the construction we propose is that, in this case, we are allowed to keep a more general stance, namely we can substitute $f(x)$ with a Dirac function peeked at the point $x_t = (t, 0, 0, 0)$ with $t \in (0, t')$. This is tantamount to consider $: \phi^2 : (x_t)$ where $u = \delta(x - x_t)\delta(x - y) \in \mathscr{A}_s^2(\mathscr{D})$.

We can now use (38) to evaluate $\Pi_2 u$ in both $(\mathbb{R}^4, \eta)$ and (M', g) by means of the explicit form of the causal propagator. In Minkowski it looks like (see [Fr75] or [Po03])

$$\Delta(x, x') \doteq -\frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|} + \frac{\delta(t - t' + |\mathbf{x} - \mathbf{x}'|)}{4\pi|\mathbf{x} - \mathbf{x}'|} \tag{48}$$

where t is the time coordinate while $\mathbf{x}$ is the three dimensional spatial vector in Euclidean coordinates. the counterpart of Δ in $\mathscr{D}'$ can be directly evaluated exploiting the conformal transformation between (M', g) and $(\mathbb{R}^4, \eta)$, namely the d'Alembert wave equation in the first spacetime (M', g) corresponds in the flat one to

$$\Box \phi + \frac{a''}{a} \phi = 0 \; , \tag{49}$$

where $'$ stands for time derivation and $\Box = -\nabla\nabla$. Furthermore the causal propagator $\widetilde{\Delta}$ of this partial differential equation is related to the sought one in M' as

$$\Delta_{M'}(x, y) = \frac{1}{a(t_x) a(t_y)} \widetilde{\Delta}(x, y) \; . \tag{50}$$

If we follow the procedure discussed in chapter 4 of [Po03], we find that

$$\widetilde{\Delta}(x,x') = \Delta(x,x') + \frac{V(x,x')}{4\pi}\left(\Theta(t-t'-|\mathbf{x}-\mathbf{x}'|) - \Theta(t-t'+|\mathbf{x}-\mathbf{x}'|)\right)$$

where $V(x,x')$ is a smooth function whose explicit form descends from the Hadamard recursive relations for (49). Particularly it also holds that $V(x,x) = \frac{a''(x)}{a(x)}$.

We are now ready to compare the expectation values in the desired state. The Minkowski side of this operation yields per direct computation

$$(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u) = \frac{1}{4}$$

while in the cosmological setting

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[(4\pi)\int_0^\infty \frac{\widetilde{\Delta}(x_t, r_*)}{a(t)a(r_*)} r_* a(r_*) a(r_*)^2 dr_*\right]^2$$

where we have rewritten the integral in the r-variable in terms of r_*, the affine parameter of the null cone in Minkowski spacetime. The defining relation between the two variables is

$$dr = a^2(r_*)dr_*.$$

The above integral can be rewritten by means of (50) as

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[4\pi \int_0^\infty \Delta(x_t, r_*)\frac{a(r_*)^2}{a(t)} r_* dr_* + \int_0^{t/2} V(x_t, r_*)\frac{a(r_*)^2}{a(t)} r_* dr_*\right]^2,$$

in which the first integral yields via (48)

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\int_0^\infty \delta(t - 2r_*)\frac{a(r_*)^2}{a(t)} dr^* + \int_0^{t/2} V(x_t, r_*)\frac{a(r_*)^2}{a(t)} r_* dr_*\right]^2.$$

Let us now expand $a(t)$ in a power series around the point x_0:

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\frac{1}{2}\frac{a(t/2)^2}{a(t)} + a''(x_0)\frac{t^2}{4} + O(t^3)\right]^2$$

where, in the derivation, we have exploited that, at first order in t, it holds

$$V(x_t, r_*) = \frac{a''(x_0)}{a(x_0)} + O(t),$$

also thanks to the rotational symmetry of M'. If we now expand in a Taylor series both $a(t)$ and $a(t/2)$, we obtain

$$(\omega' - \omega)(\Pi_2^{M'} u) = \left[\frac{a}{2}\left[1 + \left(-\frac{1}{2}\frac{a''}{a} + \frac{3}{2}\left(\frac{a'}{a}\right)^2\right)t^2\right] + a''\frac{t^2}{4} + O(t^3)\right]^2 = \left[\frac{a}{2} + \frac{3a}{4}\left(\frac{a'}{a}\right)^2 t^2 + O(t^3)\right]^2,$$

where all the functions a together with their derivatives are thought as evaluated at x_0. We can summarize the discussion finally calculating the difference between $(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u)$ and $(\omega' - \omega)(\Pi_2^{M'} u)$:

$$(\omega' - \omega)(\Pi_2^{\mathbb{R}^4} u) - (\omega' - \omega)(\Pi_2^{M'} u) = \frac{3}{4}(a'(x_0))^2 t^2 + O(t^3),$$

where we have also used that $a(x_0) = 1$. The interpretation of this result is that, exactly as we foresaw, the above comparison procedure yields a result which, at first order, allows us to extract via a measurement a precise information on the a priori unknown geometric data, such as, in this case, the derivative of the scale factor at the point x_0 in a Friedmann-Robertson-Walker universe.

5 Summary and Outlook

We can summarize the content of this paper claiming that we achieved a twofold goal: on the one hand we proposed a novel way to look at the properties of a local quantum field theory in a suitable curved background, while, on the other hand, the very same construction yields a nice mechanism which allows to compare the expectation values of field observables in different spacetimes.

More in detail, we started from a careful analysis of the underlying geometry and we realized that our scopes could be reached just thanks to rather mild assumption, namely our general setting consisted of an arbitrary strongly causal manifold M in which we identified any but fixed double cone $\mathscr{D} \equiv \mathscr{D}(p, q) = I^+(p) \cap I^-(q)$ strictly contained in a normal neighbourhood of p. Since $\mathscr{D}$ is globally hyperbolic, we can consider therein a real scalar field theory along the lines of (6) and, we therefore could follow the general quantization scheme which particularly calls for the association to the chosen system of a Borchers-Uhlmann algebra of observables. The latter can be further extended both enlarging the set of its element and the defining product in order to encompass also for a priori more singular objects, such as the Wick polynomials, which are gathered together to from the so-called extended algebra. The very deep reason for choosing $\mathscr{D} \subset M$ lies in its boundary and more properly on the portion of $J^+(p)$ which it contains. This is a differentiable codimension 1 submanifold on which it is possible to construct a genuine free scalar field theory, following exactly the same procedure successfully employed for the causal boundary of an asymptotically flat or of a cosmological spacetime in [DMP06, DMP07]. The main novel result in this framework arises from the construction also for the boundary theory of an extended algebra - $\mathscr{F}_{B,E}(\mathscr{C}_p)$ in the main body - whose well-posedness is justified both by its mathematical properties and by its relation with the bulk counterpart; to wit, the latter is embedded in $\mathscr{F}_{B,E}(\mathscr{C}_p)$ by means of Π, an injective $*$-homomorphism.

The net advantage of this picture is the possibility to make a nice use of a long tradition, stemming from [KaWa91], which allows to exploit the nice geometrical properties of the boundary to identify for the algebra thereon a natural state which can be pulled-back to the bulk via Π yielding a counterpart which satisfy the microlocal spectrum condition hence it is of Hadamard form. This guarantees that we can identify a local state in $\mathscr{D}$ which is physically well-behaved and, furthermore, it turns out to be insensible to the various performed choice, most notably that of the frame at the point p (in physical terms this means it is the same for all inertial observers at p). In other words the bulk state, as well as the boundary one, is invariant under a natural action of $SO_0(3,1)$, thus allowing us to identify it as a sort of local vacuum on a curved spacetime independent from the frame. In order to support this viewpoint we notice that if the bulk is any double cone in the Minkowski spacetime, the state correspond to the Minkowksi vacuum [BrMo09].

The second goal we reached was outlined in the previous section, namely the above depicted construction can allow us to go one step further. More precisely we can consider not just one but actually two regions as above in two a priori different spacetimes M and M'. The construction of the field theories proceeds as usual, but, now we can, moreover, make a novel use of the invertibility of the exponential map in geodesic neighbourhoods, in order to engineer the double cones $\mathscr{D} \subset M$ and $\mathscr{D}' \subset M'$ so that we can map via a local diffeomorphism the boundary we use in $\mathscr{D}$ to that in $\mathscr{D}'$. This procedure can be brought at a level of boundary extended algebra which thus can be related by means of a suitable restriction homomorphism. The net advantage of this remark stems from the previously unknown possibility to make a careful use of the distinguished state identified on each $\mathscr{C}_p$ in order to compare the bulk expectation values of field observables constructed for the theories in $\mathscr{D}$ and $\mathscr{D}'$. The important point to stress is that this new perspective is completely compatible with the standard principle of general local covariance when applicable as devised in [BFV03], and, actually, it complements it enhancing its significance.

Furthermore, since nothing prevents from choosing one of the spacetimes as the Minkowski one, one can concretely appreciate how the proposed machinery allows to compare the expectation values of the field observables making manifest therein the role and the magnitude of the geometric quantities. We stressed this point by means of a simple example in the previous subsection involving a massless minimally couple field in the flat and in a cosmological spacetime, but, we feel safe to claim that there are several open possibilities to apply our procedure to many other cases of physical interest. These are certainly not the only roads left open and actually even the identified bulk Hadamard state could be studied more in detail and, as a matter of fact, we feel it would be rather interesting to understand whether it is connected in any way with the state of minimum energy which are appearing in Friedmann-Robertson-Walker spacetimes [Ol07]. We leave this as well as the myriad of other questions for the future works.

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A Hadamard states

The aim of this appendix is to shortly recollect some properties of Hadamard states which are used throughout the text. Since most of the material has been already proven in several different alternative ways in the literature, we shall just limit ourselves to give the main statements and the necessary references. Let us stress that, from a physical perspective, these are the natural candidates as physical ground states for a quantum field theory on a curved background since their ultraviolet behaviour mimics that of the Minkowski vacuum at short distances and, furthermore, they guarantee that the quantum fluctuations of the expectation values of observables, such as the smeared components of the stress-energy tensor, are finite.

In the subsequent discussion we also always assume that we are dealing with a quasi-free state on a suitable field algebra constructed on a globally hyperbolic spacetime (M,g) where a field fulfilling an equation of motion such as (6) lives. We stick to this assumption because it is consistent with the main body of the paper, but a potential reader should keep in mind that such hypothesis could be relaxed (see for example [Sa09]). As a starting point we shall state a global criterion to characterize Hadamard states [Rad96a, Rad96b]:

Definition A.1. *A state ω satisfies the* **Hadamard condition** *and is thus called a* **Hadamard state** *if and only if*

$$WF(\omega) = \left\{(x,y,k_x,-k_y) \in T^*M^2 \setminus 0, \mid (x,k_x) \sim (y,k_y), \quad k_x \triangleright 0\right\},$$

where, in this expression, ω actually stands for the integral kernel of the two-point function associated to ω. Here, $(x,k_x) \sim (y,k_y)$ implies that there exists a null geodesic γ connecting x to y such that k_x is coparallel and cotangent to γ at x and k_y is the parallel transport of k_x from x to y along γ. Finally, $k_x \triangleright 0$ means that the covector k_x is future-directed.

The above condition on the wave front set is, despite all appearances, rather useful and it is often employed on a practical ground to check whether a given state is really Hadamard or not. Nonetheless it is also possible to provide another definition via the so-called *Hadamard form*, which has been rigorously introduced in [KaWa91].

Definition A.2. *A state ω is said to be of the (local)* **Hadamard form** *if and only if in any convex normal neighbourhood the integral kernel of the associated two-point function can be* *written as*

$$\omega(x,y) = H(x,y) + W(x,y),$$

where

$$H(x,y) = \lim_{\epsilon \to 0^+} \frac{U(x,y)}{\sigma_\epsilon(x,y)} + V(x,y) \ln \frac{\sigma_\epsilon(x,y)}{\lambda^2}, \tag{51}$$

and the limit has to be understood in the weak sense. Here, U, V, as well as W are smooth functions and V while λ is a reference length; furthermore $\sigma_\epsilon(x,y) \doteq \sigma(x,y) \pm 2i\epsilon\,(T(x) - T(y)) + \epsilon^2$ with $\epsilon > 0$. In the above formula, T is a time function, such that ∇T is timelike and future pointing on the full spacetime (M,g). Furthermore if we apply (6) both either in the x or in the y-variable, the result must be a smooth function

The existence of a time function T is guaranteed on any globally hyperbolic manifold [BS03, BS04] as these can be decomposed as $\Sigma \times \mathbb{R}$, where Σ is a smooth Cauchy surface and $\mathbb{R}$ is the very range of the time function T.

Furthermore, a completely satisfactory definition of the Hadamard form requires some more work to rule out spacelike singularities, to circumvent convergence problems of the series V, which is only asymptotic, and, finally, to assure that the definition does not depend neither on a special choice of the temporal function T nor on the employed convex normal neighbourhood.

Particularly, in strict terms, we have only defined the local Hadamard form here. A stronger and more satisfactory definition, the so-called global Hadamard form, has been introduced in [KaWa91], and it reinforces the local form as it extends it from the convex normal neighbourhoods to certain "causally-shaped" neighbourhoods of a Cauchy surface, thereby ruling out spacelike singularities. However, in [Rad96b], it has been shown that the local Hadamard form already implies the global Hadamard form.

Another important fact is that the singular structure (51) is completely determined by the geometry of the background and the equation of motions. This of course does not hold for W which encodes the full state dependance.

References

[BS03] A. N. Bernal and M. Sanchez, *"On smooth Cauchy hypersurfaces and Geroch's splitting theorem,"* Commun. Math. Phys. **243** (2003) 461 [arXiv:gr-qc/0306108].

[BS04] A. N. Bernal and M. Sanchez, *"Smoothness of time functions and the metric splitting of globally hyperbolic spacetimes,"* Commun. Math. Phys. **257** (2005) 43 [arXiv:gr-qc/0401112].

[BGP07] C. Bär, N. Ginoux and F. Pfäffle, *"Wave Equations on Lorentzian Manifolds and Quantization"* (2007) European Mathematical Society.

[BDF09] R. Brunetti, M. Duetsch and K. Fredenhagen, *"Perturbative Algebraic Quantum Field Theory and the Renormalization Groups,"* arXiv:0901.2038 [math-ph].

[BrFr00] R. Brunetti and K. Fredenhagen, *"Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds,"* Commun. Math. Phys. **208** (2000) 623 [arXiv:math-ph/9903028].

[BrFr09] R. Brunetti and K. Fredenhagen, *"Quantum Field Theory on Curved Backgrounds,"* Part of Lecture Notes in Physics 786, Published by Springer.

[BFK95] R. Brunetti, K. Fredenhagen and M. Köhler, *"The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes,"* Commun. Math. Phys. **180** (1996) 633 [arXiv:gr-qc/9510056].

[BFV03] R. Brunetti, K. Fredenhagen and R. Verch, *"The generally covariant locality principle: A new paradigm for local quantum physics,"* Commun. Math. Phys. **237**, 31 (2003) [arXiv:math-ph/0112041].

[BrMo09] R. Brunetti and V. Moretti, private communication and work in progres.

[ChKi09] W. Chmielowiec and J. Kijowski, *"Hamiltonian description of radiation phenomena: Trautman-Bondi energy and corner conditions,"* Rep. Math. Phys. **64** (2009) 223-240

[CBCMG09] Y. Choquet-Bruhat, P. T. Chrusciel and J. M. Martin-Garcia, *"The light-cone theorem,"* Class. Quant. Grav. **26** (2009) 135011 [arXiv:0905.2133 [gr-qc]].

[DMP06] C. Dappiaggi, V. Moretti and N. Pinamonti, *"Rigorous steps towards holography in asymptotically flat spacetimes,"* Rev. Math. Phys. **18** (2006) 349 [arXiv:gr-qc/0506069].

[DMP07] C. Dappiaggi, V. Moretti and N. Pinamonti, *"Cosmological horizons and reconstruction of quantum field theories,"* Commun. Math. Phys. **285** (2009) 1129 [arXiv:0712.1770 [gr-qc]].

[DMP08] C. Dappiaggi, V. Moretti and N. Pinamonti, *"Distinguished quantum states in a class of cosmological spacetimes and their Hadamard property,"* J. Math. Phys. **50** (2009) 062304 [arXiv:0812.4033 [gr-qc]].

[DMP09] C. Dappiaggi, V. Moretti and N. Pinamonti, *"Rigorous construction and Hadamard property of the Unruh state in Schwarzschild spacetime,"* arXiv:0907.1034 [gr-qc].

[Fr75] F. G. Friedlander, *"The wave equation on a curved space-time."* Cambridge Univeristy Press (1975).

[Ge68] R. P. Geroch, *"Spinor Structure of Space-Times in General Relativity. I"* J. Math. Phys. **9** (1968) 1739.

[Ge70] R. P. Geroch, *"Spinor Structure Of Space-Times In General Relativity. II,"* J. Math. Phys. **11**, (1970) 343.

[He08] A. Herdegen, *"Infrared problem and spatially local observables in electrodynamics,"* Annales Henri Poincare **9** (2008) 373 [arXiv:0710.0500 [hep-th]].

[Ho00] S. Hollands: *"Aspects of Quantum Field Theory in Curved Spacetime"*, Ph.D. Thesis - University of York (2000), advisor B.S. Kay, unpublished.

[HoWa01] S. Hollands and R. M. Wald, *"Local Wick polynomials and time ordered products of quantum fields in curved spacetime,"* Commun. Math. Phys. **223**, 289 (2001).

[HoWa02] S. Hollands and R. M. Wald, *"Existence of local covariant time ordered products of quantum fields in curved spacetime,"* Commun. Math. Phys. **231**, (2002) 309 .

[Hö89] L. Hörmander, *"The Analysis of Linear Partial Differential Operators I"*, second edition, Springer-Verlag, Berlin, Germany (1989).

[Hus96] D. Husemoller, *"Fibre Bundles"* Springer-Verlag (1996) third edition,

[KaWa91] B. S. Kay and R. M. Wald, *"Theorems on the Uniqueness and Thermal Properties of Stationary, Nonsingular, Quasifree States on Space-Times with a Bifurcate Killing Horizon,"* Phys. Rept. **207** (1991) 49.

[KoNo96] S. Kobayashi and K. Nomitsu, *"Foundations of Differential Geometry. Vol I."* (1996) Wiley-Interscience

[Mo06] V. Moretti, *"Uniqueness theorem for BMS-invariant states of scalar QFT on the null boundary of asymptotically flat spacetimes and bulk-boundary observable algebra correspondence,"* Commun. Math. Phys. **268** (2006) 727 [arXiv:gr-qc/0512049].

[Mo08] V. Moretti, *"Quantum ground states holographically induced by asymptotic flatness: Invariance under spacetime symmetries, energy positivity and Hadamard property,"* Commun. Math. Phys. **279** (2008) 31 [arXiv:gr-qc/0610143].

[MoPi04] V. Moretti and N. Pinamonti, *"Holography, SL(2,R) symmetry, Virasoro algebra and all that in Rindler spacetime,"* J. Math. Phys. **45** (2004) 230 [arXiv:hep-th/0304111]

[Ol07] H. Olbermann, *"States of Low Energy on Robertson-Walker Spacetimes,"* Class. Quant. Grav. **24** (2007) 5011 [arXiv:0704.2986 [gr-qc]].

[ON83] B. O'Neill, *"Semi-Riemannian Geometry"* (1983) Academic Press.

[Pi09] N. Pinamonti, *"Conformal generally covariant quantum field theory: The scalar field and its Wick products,"* Commun. Math. Phys. **288** (2009) 1117 [arXiv:0806.0803 [math-ph]].

[Po03] E. Poisson, *"The motion of point particles in curved spacetime,"* Living Rev. Rel. **7** (2004) 6 [arXiv:gr-qc/0306052].

[Po04] E. Poisson, *"Retarded coordinates based at a world line, and the motion of a small black hole in an external universe,"* Phys. Rev. D **69** (2004) 084007 [arXiv:gr-qc/0311026].

[PrPo06] B. Preston and E. Poisson, *"Light-cone coordinates based at a geodesic world line,"* Phys. Rev. D **74** (2006) 064009 [arXiv:gr-qc/0606093].

[Rad96a] M. J. Radzikowski, *"Micro-Local Approach To The Hadamard Condition In Quantum Field Theory On Curved Space-Time,"* Commun. Math. Phys. **179** (1996) 529.

[Rad96b] M. J. Radzikowski, *"A Local to global singularity theorem for quantum field theory on curved space-time,"* Commun. Math. Phys. **180** (1996) 1.

[SaVe00] H. Sahlmann and R. Verch, *"Passivity and microlocal spectrum condition,"* Commun. Math. Phys. **214** (2000), 705.

[Sa09] K. Sanders, *"Equivalence of the (generalised) Hadamard and microlocal spectrum condition for (generalised) free fields in curved spacetime,"* arXiv:0903.1021 [math-ph].

[Sc09] B. Schroer, *"BMS symmetry, holography on null-surfaces and area proportionality of 'light-slice' entropy,"* arXiv:0905.4435 [hep-th].

[Se82] G. L. Sewell, *"Quantum fields on manifolds: PCT and gravitationally induced thermal states,* Ann. Phys. **141**, (1982) 201.

[Wa84] R. M. Wald, *"General Relativity"* (1984) Chicago University Press.

[Wa95] R. M. Wald, *"Quantum field theory in curved space-time and black hole thermodynamics,"* Chicago University Press (1994) 205.