Remarks on two notions of spectral minimal partitions

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Abstract

In continuation of previous work, we analyze the properties of spectral minimal partitions. We focus on the comparison between two definitions of minimal partitions and give some simple rather generic criterion implying that they cannot coincide. We illustrate this criterion in the case of simple convex examples like the rectangle and the equilateral triangle. This partially answers a question posed by Bucur-Buttazzo-Henrot in 1998.

1 Introduction

This paper is motivated by questions related to some conjecture of Bishop [1] discussed more recently in [14] and also by an article of Bucur-Buttazzo-Henrot [6]. We analyze the relations between different definitions of minimal partitions. The analysis presented in [14] was on the sphere but we will focus in this note on domains in $\mathbb{R}^2$, where paradoxically we know less !

Before presenting our main questions, let us recall a few definitions that the reader can for example find in [13]. For $1 \leq k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^2$, we call a k-**partition** of Ω a family $\mathcal{D} = \{D_i\}_{i=1}^k$ of pairwise disjoint open domains such that

$$\cup_{i=1}^k D_i \subset \Omega\,. \tag{1.1}$$

It is called **strong** if

$$\mathrm{Int}\,(\overline{\cup_{i=1}^k D_i}) \setminus \partial\Omega = \Omega\,. \tag{1.2}$$

We denote by $\mathfrak{O}_k$ the set of such partitions.
For each open set $\omega \subset \Omega$, we denote by $H(\omega)$ the Dirichlet realization of the Laplacian in ω and by $\lambda_j(\omega)$ the increasing sequence of eigenvalue counted with multiplicity. For $\mathcal{D} \in \mathfrak{O}_k$, we introduce

$$\Lambda_k(\mathcal{D}) = \max_i \lambda(D_i)\,, \tag{1.3}$$

where $\lambda(D_i) = \lambda_1(D_i)$, and

$$\mathfrak{L}_k(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda_k(\mathcal{D})\,. \tag{1.4}$$

We call a spectral minimal k-partition, a k-partition $\mathcal{D} \in \mathfrak{O}_k$ such that

$$\mathfrak{L}_k(\Omega) = \Lambda_k(\mathcal{D})\,.$$

More generally we can consider (see [8, 9, 10, 13]) for $p \in [1, +\infty[$

$$\Lambda_{k,p}(\mathcal{D}) = (\frac{1}{k}\sum_i \lambda(D_i)^p)^{\frac{1}{p}}\,, \tag{1.5}$$

and

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k} \Lambda_{k,p}(\mathcal{D})\,. \tag{1.6}$$

We write $\mathfrak{L}_{k,\infty}(\Omega) = \mathfrak{L}_k(\Omega)$ and recall the monotonicity property

$$\mathfrak{L}_{k,p}(\Omega) \leq \mathfrak{L}_{k,q}(\Omega) \text{ if } p \leq q\,. \tag{1.7}$$

The notion of p-minimal k-partition can be extended accordingly by minimizing $\Lambda_{k,p}(\mathcal{D})$.
Our main question in this paper is to discuss whether the equality $\mathfrak{L}_{k,1}(\Omega) = \mathfrak{L}_{k,\infty}(\Omega)$ holds or not. This question was discussed when $k = 2$ in [16, 6] in the equivalent form (see below (1.10)) :

When do we have $\mathfrak{L}_{2,1}(\Omega) = \lambda_2(\Omega)$?

But we will already bring a new insight in this case.

We first recall that the existence and the regularity of minimal partitions has been proved in a series of papers [8, 9, 10, 13, 7] (and other references therein). We do not recall here all the notions but present specific notation and results which are directly useful in our analysis. For any real-valued eigenfunction u of $H(\Omega)$, we introduce the nodal set of u as

$$N(u) = \overline{\{x \in \Omega \mid u(x) = 0\}} \tag{1.8}$$

and call the components of $\Omega \setminus N(u)$ the nodal domains of u. The number of nodal domains of such a function will be called $\mu(u)$.

If $\mathcal{D}$ is a regular partition (in the sense of [13]), we say that $D_i, D_j \in \mathcal{D}$ are **neighbors** if

$$D_{ij} : \mathrm{Int}\,(\overline{D_i \cup D_j}) \setminus \partial\Omega$$

is connected. Attached to a regular partition $\mathcal{D}$ we can associate its boundary $N(\mathcal{D})$ which is defined in $\overline{\Omega}$ by

$$N(\mathcal{D}) = \overline{\bigcup_i (\partial D_i \cap \Omega)}\,. \tag{1.9}$$

Courant's nodal Theorem says that the number of nodal domains $\mu(u)$ of an eigenfunction u associated with $\lambda_k(\Omega)$ satisfies $\mu(u) \leq k$. As in [13], we say that u is **Courant-sharp** if $\mu(u) = k$. For any integer $k \geq 1$, we denote by $L_k(\Omega)$ the smallest eigenvalue for which there is an eigenfunction in the associated eigenspace with k nodal domains. We recall that for $k = 2$,

$$\lambda_2(\Omega) = \mathfrak{L}_2(\Omega) = L_2(\Omega)\,. \tag{1.10}$$

The main result of [13] which will be helpful for us is the following :

Theorem 1.1
Suppose that $\Omega \subset \mathbb{R}^2$ is regular. Then we have, for any k

$$\lambda_k(\Omega) \leq \mathfrak{L}_k(\Omega) \leq L_k(\Omega)\,. \tag{1.11}$$

If $\mathfrak{L}_k(\Omega) = L_k(\Omega)$ or $\lambda_k(\Omega) = \mathfrak{L}_k(\Omega)$, then

$$\lambda_k(\Omega) = \mathfrak{L}_k(\Omega) = L_k(\Omega)\,,$$

and any minimal k-partition is nodal and admits a representative which is the family of nodal domains of some eigenfunction u associated with $\lambda_k(\Omega)$.

Nothing similar is known for $\mathfrak{L}_{k,p}$. Nethertheless, we also prove in [14] the "fermionic" inequality :

$$\frac{1}{k}\left(\sum_{j=1}^{k} \lambda_j(\Omega)\right) \leq \mathfrak{L}_{k,1}(\Omega)\,, \tag{1.12}$$

which in some sense is the optimal substitute of (1.11). But this inequality is strict when Ω is connected. This inequality (and its extension to $p \in [1, +\infty[$) can actually be proved more easily by direct comparison between the spectrum of Ω and the spectrum of $\Omega \setminus N(\mathcal{D}^{k,p})$ where $\mathcal{D}^{k,p}$ is a p-minimal k-partition.
This leads to

$$\left(\frac{1}{k}\sum_{j=1}^{k} \lambda_j(\Omega)^p\right)^{\frac{1}{p}} \leq \Lambda_{k,p}(\mathcal{D}^{k,p}) = \mathfrak{L}_{k,p}(\Omega)\,. \tag{1.13}$$

Note also that already, when $k = 2$ and $p < \infty$, $\mathfrak{L}_{2,p}(\Omega)$ is unknown, except for the case of dimension 1 and the case of the sphere which is a result mentioned in Bishop (see [1], [14] and references therein) :

$$\mathfrak{L}_{2,p}(\mathbb{S}^2) = \lambda_2(\mathbb{S}^2) = 2\,. \tag{1.14}$$

For $k = 3$, it has been shown in [14] that :

$$\mathfrak{L}_{3,\infty}(\mathbb{S}^2) = \frac{15}{4}\,. \tag{1.15}$$

Bishop conjectures that $\mathfrak{L}_{3,1}(\mathbb{S}^2) = \frac{15}{4}$.

2 Main results on $\mathfrak{L}_{k,p}(\Omega)$

2.1 Non-convex examples

A natural question is to determine under which condition the infimum of $\Lambda_{2,1}(\mathcal{D})$ for $\mathcal{D} \in \mathfrak{O}_2$ is realized for a pair (D_1, D_2) such that $\lambda(D_1) = \lambda(D_2)$. Let us illustrate the question by a simple example which can be found in [6] or [14]. If we consider two disks C_1 and C_2 such that $\lambda(C_1) < \lambda(C_2) < \lambda_2(C_1)$, it is not too difficult to see that if we take Ω as the union of these two disks and of a thin channel joining the two disks, then $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$ will be very close to $\lambda(C_2)$ and the infimum of $\Lambda_{2,1}(\mathcal{D})$ will be less than $\frac{1}{2}(\lambda(C_1)+\lambda(C_2))$. Hence we will have strict inequality if the channel is small enough.

2.2 A necessary condition for $\mathfrak{L}_{k,1}(\Omega) = \mathfrak{L}_{k,\infty}(\Omega)$

Our aim is to show the following proposition :

Proposition 2.1
Let Ω be a domain in $\mathbb{R}^2$ and $k \geq 2$. Let $\mathcal{D}$ a k-minimal partition for $\mathfrak{L}_k(\Omega)$ and suppose that there is a pair of neighbors (D_i, D_j) in $\mathcal{D}$ such that a second eigenfunction ϕ_{ij} of $H(D_{ij})$ having D_i and D_j as nodal domains satisfies

$$\int_{D_i} |\phi_{ij}(x,y)|^2 dxdy \neq \int_{D_j} |\phi_{ij}(x,y)|^2 dxdy\,. \tag{2.1}$$

Then

$$\mathfrak{L}_{k,1}(\Omega) < \mathfrak{L}_k(\Omega)\,. \tag{2.2}$$

Proof

We consider a deformation of domains (like for example in [15]) associated with a compactly supported vector field V by

$$\Phi(t) = I + tV\,, \tag{2.3}$$

where $\operatorname{supp} V$ is contained in a neighborhood of some point x_{ij} of $\partial D_i \cap \partial D_j \cap \Omega$ and

$$V\cdot\nu \geq 0 \text{ on } \partial D_i \cap \partial D_j\,, \tag{2.4}$$

with strict inequality at x_{ij}. Here ν is the interior normal oriented from D_i to D_j.

We now compute $\lambda(D_j(t))$ where

$$D_j(t) = \Phi(t)(D_j)\,. \tag{2.5}$$

Replacing D_i and D_j by $D_i(t)$ and $D_j(t)$ and keeping the other open sets of the minimal partition unchanged, we obtain a new k-partition $\mathcal{D}(t)$, such that

$$\Lambda_{k,\infty}(\mathcal{D}(t)) \geq \mathfrak{L}_{k,\infty}(\Omega)\,. \tag{2.6}$$

Let us now estimate $\Lambda_{k,1}(\mathcal{D}(t))$ for t small.

Coming back to the definition of $\Lambda_{k,1}$ and having in mind that $\Lambda_{k,1}(\mathcal{D}) = \Lambda_{k,+\infty}(\mathcal{D})$, we have

$$\begin{aligned}\Lambda_{k,1}(\mathcal{D}(t)) &= \Lambda_{k,1}(\mathcal{D}) + \tfrac{1}{k}\left(\lambda(D_i(t)) - \lambda(D_i)\right) + \tfrac{1}{k}\left(\lambda(D_j(t)) - \lambda(D_j)\right)\\ &= \mathfrak{L}_{k,\infty}(\Omega) + \tfrac{1}{k}\left(\lambda(D_i(t)) - \lambda(D_i)\right) + \tfrac{1}{k}\left(\lambda(D_j(t)) - \lambda(D_j)\right)\,.\end{aligned} \tag{2.7}$$

Hence it remains to compute the two last terms. Here the Hadamard Formula (see Theorem 5.1 in [15]) gives :

$$\lambda(D_i(t)) = \lambda(D_i) + t\int_{\partial D_i\cap\partial D_j} (V\cdot\nu)\,(\partial_\nu\psi_i)^2 d\sigma + \mathcal{O}(t^2)\,, \tag{2.8}$$

where ψ_i is the positive normalized ground state[1] of the Dirichlet Laplacian in D_i.

This can be rewritten in the form

$$\lambda(D_i(t)) = \lambda(D_i) + t a_i^{-2}\int_{\partial D_i\cap\partial D_j} (V\cdot\nu)\,(\partial_\nu\phi_{ij})^2 d\sigma + \mathcal{O}(t^2)\,, \tag{2.9}$$

with a_i defined by

$$\phi_{ij} = a_i\psi_i \text{ in } D_i\,. \tag{2.10}$$

[1]Sometimes, we also denote by ψ_i the extension of ψ_i by 0 outside of D_i.

Note here that

$$v_{ij} := \int_{\partial D_i \cap \partial D_j} (V \cdot \nu)\,(\partial_\nu \phi_{ij})^2 d\sigma > 0\,. \tag{2.11}$$

Similarly, we obtain

$$\lambda(D_j(t)) = \lambda(D_j) - t a_j^{-2} \int_{\partial D_i \cap \partial D_j} (V \cdot \nu)\,(\partial_\nu \phi_{ij})^2 d\sigma + \mathcal{O}(t^2)\,. \tag{2.12}$$

Summing up, we obtain

$$\mathfrak{L}_{k,1}(\Omega) \leq \mathfrak{L}_{k,\infty}(\Omega) + \frac{t}{k}(a_i^{-2} - a_j^{-2}) v_{ij} + \mathcal{O}(t^2)\,. \tag{2.13}$$

Then we are done by choosing t such that $t(a_i^2 - a_j^2) < 0$ and $|t|$ small enough.

Remark 2.2
As observed in Section 4 of [6], this analysis is related to a paper of Kawohl [16], which is a weak reformulation of our statement that $\mathfrak{L}_2(\Omega) = \lambda_2(\Omega)$. Note that in [6] a small error (line 7, p. 9) in the computation of the derivative of the functional (which is corrected above) makes their discussion in the lines 8-11 in p. 9 irrelevant.

2.3 More on $\mathfrak{L}_{k,p}(\Omega)$

In the case $p \in [1, +\infty[$, we obtain, using again the Hadamard Formula (see (2.8)) and the property that the derivative of $t \mapsto \Lambda_{1,p}(\mathcal{D}(t))$ should vanish at $t = 0$ for any vector field V, we obtain that

$$\lambda(D_i)^{p-1} \partial_\nu \psi_i = -\lambda(D_j)^{p-1} \partial_\nu \psi_j\,,$$

on $\partial D_i \cap \partial D_j$. Hence we obtain the following :

Proposition 2.3
If $\mathcal{D}$ is a p-minimal k-partition of Ω, then for any pair of neighbors (D_i, D_j) of $\mathcal{D}$, $\lambda(D_i)^{p-1}\psi_i - \lambda(D_j)^{p-1}\psi_j$ belongs to $W^{2,2}(D_{ij})$,

Hence, a p-minimal k-partition ($p > 1$) for which there exists a pair of neighbors satisfying $\lambda(D_i) \neq \lambda(D_j)$ cannot be a q-minimal k-partition for $q < p$. This implies :

Proposition 2.4
For any $k \geq 2$, there exists $p^*(k) \in [1, +\infty]$, such that $p \mapsto \mathfrak{L}_{k,p}(\Omega)$ is strictly increasing on $[1, p^*(k)[$ and constant for $p \in [p^*(k), +\infty]$.

We first observe that $p \mapsto \mathfrak{L}_{k,p}(\Omega)$ is a continuous function of $p \in [1, +\infty]$. This follows from

$$\mathfrak{L}_{k,q} \leq \mathfrak{L}_{k,p} \leq k^{\frac{1}{q}-\frac{1}{p}} \mathfrak{L}_{k,q}\,,\ \forall q \leq p\,. \tag{2.14}$$

Suppose now that we have a pair (p, p') with $p < p'$ and $\mathfrak{L}_{k,p} = \mathfrak{L}_{k,p'}$. Then necessarily $\mathfrak{L}_{k,p'} = \mathfrak{L}_{k,\infty}$ because all the $\lambda(D_j)$ of the p'-minimal k-partition must be equal. Hence $\mathfrak{L}_{k,p} = \mathfrak{L}_{k,\infty}$, and by monotonicity we have stationarity on $[p, +\infty]$.

3 Application for some family of rectangles

The aim in this section is to give explicit examples of open set Ω's where $\mathfrak{L}_{k,1}(\Omega) < \mathfrak{L}_{k,\infty}(\Omega)$ (with $k = 2, 3$). We are going to study a family of rectangles $]-\frac{a}{2}, \frac{a}{2}[\times]-\frac{b}{2}, \frac{b}{2}[$ $(b \geq a)$ for which the quotient b/a is close to the minimal value for which the third eigenvalue is Courant sharp.

3.1 Main results

We will consider the case of the rectangle

$$\mathcal{R}_\epsilon :=]-\frac{\pi}{2}\epsilon, +\frac{\pi}{2}\epsilon[\times]-\frac{\pi}{2}, \frac{\pi}{2}[$$

and consider the limiting value

$$\epsilon = \sqrt{\frac{3}{8}}$$

which corresponds (see [13] and [3]) to the case when the minimal 3-partitions are still nodal. Hence we have

$$\mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon) = \lambda_3(\mathcal{R}_\epsilon) = \lambda_4(\mathcal{R}_\epsilon)\,. \tag{3.1}$$

Our aim is to prove

Proposition 3.1
For $\epsilon = \sqrt{\frac{3}{8}}$, we have

$$\mathfrak{L}_{3,1}(\mathcal{R}_\epsilon) < \mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon)\ . \tag{3.2}$$

3.2 The perturbative approach

As we have proved in [13] and analyzed further in [3], we have for this specific value of ϵ a continuous family of nodal 3-partitions for $\mathcal{R}_\epsilon$ which are minimal 3-partitions. Each one is associated with the eigenfunction :

$$\Psi_\beta(x,y) = \cos\frac{x}{\epsilon}\cos 3y + \beta\sin\frac{2x}{\epsilon}\cos y\,. \tag{3.3}$$

For the proof it is easier to make the change of variable $\hat{x} = \frac{x}{\epsilon}$, $\hat{y} = y$. In the new coordinates, $\mathcal{R}_\epsilon$ becomes the square $]-\frac{\pi}{2}, +\frac{\pi}{2}[^2$ and the eigenfunction reads

$$\widehat{\Psi}_\beta(\hat{x},\hat{y}) = \cos\hat{x}\cos 3\hat{y} + \beta\sin 2\hat{x}\cos\hat{y}\,. \tag{3.4}$$

From now on we omit the hats.

Let us determine the nodal domains. We first use the factorization :

$$\Psi_\beta(x,y) = \cos x\cos y\left((1-4\sin^2 y) + 2\beta\sin x\right)\,. \tag{3.5}$$

Then, for $\beta \in [0,\frac{1}{2}]$ and in the new coordinates, the square is divided in three parts D_1, D_2, D_3 (from down to up) delimited by the two curves

$$\sin y = \pm\frac{1}{2}\sqrt{(1+2\beta\sin x)}\,. \tag{3.6}$$

For $\beta = 0$, we have of course the two curves $y = \pm\frac{\pi}{6}$. For $\beta = \frac{1}{2}$, the two curves touch at $(-\frac{\pi}{2}, 0)$.

For the proof of our proposition, we will work for β small and in order to apply the criterion, compare the L^2-norms of Ψ_β in each nodal domain. Actually, it is enough to show the

Lemma 3.2
There exists $\beta_0 > 0$ such that, for $\beta \in]0,\beta_0]$,

$$q(\beta) := \int_Q (\Psi_\beta)_+(x,y)^2 dxdy / \int_Q (\Psi_\beta)(x,y)^2 dxdy \neq \frac{1}{3}\,. \tag{3.7}$$

Here $(\Psi_\beta)_+$ is the positive part of Ψ_β.

We first verify that $q(0) = \frac{1}{3}$ and that $q'(0) = 0$. Hence the crucial point is to compute $q''(0)$.

Let us first compute the second derivative of $\int_Q (\Psi_\beta)_+(x,y)^2 dxdy$ with respect to β. We obtain

$$\left(\int_Q (\Psi_\beta)_+(x,y)^2 dxdy\right)''(0) = 2\int_{]-\frac{\pi}{2},\frac{\pi}{2}[\times]-\frac{\pi}{6},\frac{\pi}{6}[} (\sin 2x)^2 (\cos y)^2 dy\,.$$

We now observe that

$$\int_{]-\frac{\pi}{6},+\frac{\pi}{6}[} (\cos y)^2 dy = \frac{\pi}{6} + \cos\frac{\pi}{3} \text{ and } \int_{]-\frac{\pi}{2},+\frac{\pi}{2}[} (\sin 2x)^2 dx = \frac{\pi}{2}\,.$$

We then obtain that

$$\left(\int_Q (\Psi_\beta)_+(x,y)^2 dxdy / \int_Q (\Psi_\beta)(x,y)^2 dxdy\right)''(0) = 4\frac{\cos\frac{\pi}{3}}{\pi}\,,$$

hence

$$\int_Q (\Psi_\beta)_+(x,y)^2 dxdy / \int_Q (\Psi_\beta)(x,y)^2 dxdy = \frac{1}{3} + 2\beta^2 \frac{\cos\frac{\pi}{3}}{\pi} + \mathcal{O}(\beta^3)\,. \quad (3.8)$$

This proves the lemma.

Remark 3.3
From the previous result we obtain also $\mathfrak{L}_{3,1}(\mathcal{R}_\epsilon) < \mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon)$ *for* ϵ *larger (but close to)* $\sqrt{\frac{3}{8}}$*. We know indeed that in this case*

$$\mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon) \geq \lambda_3(\mathcal{R}_\epsilon) = \pi^2\left(1 + \frac{4}{\epsilon^2}\right)\,,$$

and that

$$\mathfrak{L}_{3,1}(\mathcal{R}_\epsilon) \leq \mathfrak{L}_{3,1}(\mathcal{R}_{\sqrt{\frac{3}{8}}}) < \mathfrak{L}_{3,\infty}(\mathcal{R}_{\sqrt{\frac{3}{8}}}) = \lambda_3(\mathcal{R}_{\sqrt{\frac{3}{8}}}) = \frac{35\pi^2}{3}\,.$$

Hence we have the strict inequality $\mathfrak{L}_{3,1}(\mathcal{R}_\epsilon) < \mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon)$ *for* $\epsilon \in [\sqrt{\frac{3}{8}}, \epsilon_1[$ *with* ϵ_1 *given by*

$$\pi^2\left(1 + \frac{4}{\epsilon_1^2}\right) = \mathfrak{L}_{3,1}(\mathcal{R}_{\sqrt{\frac{3}{8}}})\,.$$

We can also treat the case when ϵ *is smaller than* $\sqrt{\frac{3}{8}}$*. Using the minimal partition of* $\mathcal{R}_{\sqrt{\frac{3}{8}}}$ *for* $\mathfrak{L}_{3,1}$ *we can shift it (by dilation) into a 3-partition of* $\mathcal{R}_\epsilon$ *and obtain by the domain continuity of the eigenvalues*

$$\mathfrak{L}_{3,1}(\mathcal{R}_\epsilon) \leq \mathfrak{L}_{3,1}(\mathcal{R}_{\sqrt{\frac{3}{8}}}) + \mathcal{O}(|\epsilon - \sqrt{\frac{3}{8}}|)\,.$$

Now

$$\mathfrak{L}_{3,\infty}(\mathcal{R}_\epsilon) = \pi^2 \left(9 + \frac{1}{\epsilon^2} \right) ,$$

for $\epsilon \leq \sqrt{\frac{3}{8}}$.

Hence the strict inequality is preserved for ϵ *sufficiently close to* $\sqrt{\frac{3}{8}}$.

This shows an example where the minimal 3*-partition for* $\mathfrak{L}_{3,\infty}$ *satisfies* (2.1)*, hence* $\mathfrak{L}_{3,1} < \mathfrak{L}_{3,\infty}$.

Remark 3.4

By taking the union of two neighboring nodal domains (with $\beta \in [0, \beta_0]$*), we also obtain an example of a domain for which* $\mathfrak{L}_{2,1} \neq \mathfrak{L}_{2,\infty}$*. This example is not convex.*

This motivates us for completing the analysis by analyzing more carefully $\beta = \frac{1}{2}$. See the next subsection.

3.3 The case with $\beta = \frac{1}{2}$

This case is not a particular case of the previous study which was perturba-

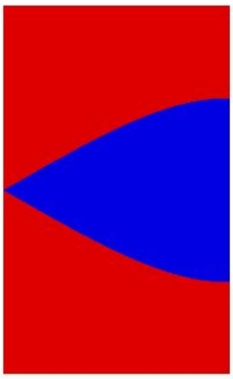

Figure 1: Nodal sets of $\varphi_{1,\frac{1}{2}}(x,y) = \cos\frac{x}{\epsilon}\,\cos 3y\,+\frac{1}{2}\sin\frac{2x}{\epsilon}\,\cos y$.

tive in β, but will permit us to produce an explicit convex set Ω, for which

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega) = \lambda_2(\Omega)\,.$$

This gives a negative answer to a question whether for convex Ω, $\mathfrak{L}_{2,1}(\Omega) = \lambda_2(\Omega)$ posed in [6].
We keep the same hat -coordinates and we have to compute $q(\frac{1}{2})$ (with q defined in (3.7)). This leads to the computation (after integration with respect to y) of six terms

$$
\begin{aligned}
A &:= \tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 \arcsin\sqrt{\tfrac{1+\sin x}{4}}\, dx\,;\\
B &:= \tfrac{1}{12}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 \sin\left[6\arcsin\sqrt{\tfrac{1+\sin x}{4}}\right] dx\,;\\
C &:= \tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 (\sin x)^2 \arcsin\sqrt{\tfrac{1+\sin x}{4}}\, dx\,;\\
D &:= \tfrac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos x)^2 (\sin x)^2 \sin\left[2\arcsin\sqrt{\tfrac{1+\sin x}{4}}\right] dx\,;\\
E &:= \tfrac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x (\cos x)^2 \sin\left[2\arcsin\sqrt{\tfrac{1+\sin x}{4}}\right] dx\,;\\
F &:= \tfrac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x (\cos x)^2 \sin\left[4\arcsin\sqrt{\tfrac{1+\sin x}{4}}\right] dx\,.
\end{aligned}
$$

$q(\frac{1}{2})$ is obtained by dividing the sum of these 6 terms by $\frac{5\pi^2}{32}$.
Numerical computations kindly done by J. Tidblom give :

$$
\begin{aligned}
&A \sim 0.398578\,, && B \sim 0.00221767\,, && C \sim 0.0958989\,,\\
&D \sim 0.0731831\,, && E \sim 0.0665301\,, && F \sim -0.0399181\,.
\end{aligned}
$$

This leads to

$$
q(\frac{1}{2}) \sim 0.38\,. \tag{3.9}
$$

So $q(\frac{1}{2}) \neq \frac{1}{3}$ and the corresponding nodal partition satisfies (2.1).
Now the claim is that $\Omega = D_{12}$ (with $D_{12} = \mathrm{Int}\,(\overline{D_1} \cup \overline{D_2})$ and D_1, D_2 the two lowest nodal domains) is convex and satisfies

$$
\mathfrak{L}_{2,1}(D_{12}) < \mathfrak{L}_{2,\infty}(D_{12}) = \lambda_2(D_{12})\,. \tag{3.10}
$$

To prove this, it is enough to verify that the function $y = y(x)$ which is the non negative solution of

$$
4\sin y^2 = 1 + \sin x\,,
$$

such that $y(\frac{-\pi}{2}) = 0$ and $y(\frac{\pi}{2}) = \frac{\pi}{4}$ is concave. This is done by an elementary computation.

4 The equilateral triangle

For the equilateral triangle the eigenfunctions and eigenvalues can be worked out analytically, see [19] for a detailed and geometrical description[2]. Lamé in the 19th century was probably the first who calculated the eigenfunctions and the eigenvalues for equilateral triangles. To be explicit, we consider an equilateral triangle $\mathcal{T}$, where the corners have the coordinates $(-1/3, 0)$, $(1/3, 0)$, and $(0, 1/\sqrt{3})$. One can explicitely give the first antisymmetric eigenfunction (with respect to the y-axis) :

$$u^{AS}(x,y) = \sin(\pi x)\,\sin(3\pi\sqrt{3}y) + \sin(4\pi x)\,\sin(2\pi\sqrt{3}y) + \sin(5\pi x)\,\sin(\pi\sqrt{3}y)\,, \tag{4.1}$$

which corresponds to the eigenvalue $28\pi^2$.
According to [19] (Fig. 8.5, p. 279), the same formula gives also the eigenfunction of the Dirichlet Laplacian on the rectangle $[0,1] \times [0, \frac{1}{\sqrt{3}}]$, each of the six nodal domains being equal after rotation-translation to a half $\mathcal{T}$. For this problem, the eigenvalue $28\pi^2$ has multiplicity 3 and an associated basis of eigenfunctions is given by

$$\begin{aligned} \phi_{13}(x,y) &= \sin(\pi x)\,\sin(3\pi\sqrt{3}y) \\ \phi_{4,2}(x,y) &= \sin(4\pi x)\,\sin(2\pi\sqrt{3}y) \\ \phi_{5,1}(x,y) &= \sin(5\pi x)\,\sin(\pi\sqrt{3}y)\,. \end{aligned}$$

Moreover

$$v(x,y) = u^{AS}(x + \frac{2}{3}, y)\,, \tag{4.2}$$

is also an eigenfunction of the equilateral triangle corresponding also to a rotation of the previous one by $-\frac{\pi}{3}$.
Hence we can construct a symmetric eigenfunction orthogonal to u^{AS} in the two-dimensional eigenspace of the Dirichlet Laplacian in the equilateral triangle with eigenvalue $28\pi^2$ by considering :

$$u^S(x,y) = \frac{1}{\sqrt{3}}\,(v(x,y) + v(-x,y))\ . \tag{4.3}$$

We find

$$u^S(x,y) = \cos(\pi x)\,\sin(3\pi\sqrt{3}y) + \cos(4\pi x)\,\sin(2\pi\sqrt{3}y) - \cos(5\pi x)\,\sin(\pi\sqrt{3}y)\,. \tag{4.4}$$

[2]One standard reference [20] contains unfortunately misprints (see formula (2.4) where one should read $(2m - n)$ instead of $(2n - m)$).

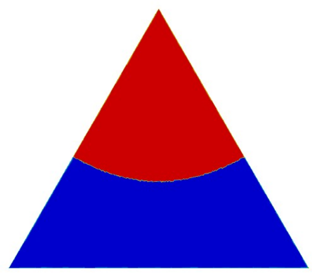

Figure 2: Nodal set of the symmetric second eigenfunction.

Working in the same spirit as for our rectangle, we compute the quantity

$$I = \int_{x>0,(x,y)\in\mathcal{T}} u^{AS}(x,y)u^{S}(x,y)\, dxdy\,. \tag{4.5}$$

If $I \neq 0$, we will obtain by considering the nodal 2-partition of $u^{AS} + \beta u^{S}$ for β small, a 2-partition which satisfies (2.1). With help of M. Levitin and Maple, we obtain :

$$I = \frac{7}{160}\frac{\sqrt{3}}{\pi}\,. \tag{4.6}$$

Hence, we have :

Proposition 4.1
For the equilateral triangle $\mathcal{T}$, we have

$$\mathfrak{L}_{2,1}(\mathcal{T}) < \mathfrak{L}_{2,\infty}(\mathcal{T})\,.$$

Remark 4.2
More directly, we can compare the square of the L^2-norm in the lower and upper nodal components of u^S.
Numerical integration[3] *gives that*

$$\int_{u^S>0} |u^S(x,y)|^2\, dxdy \sim 0.0284\,, \int_{u^S<0} |u^S(x,y)|^2\, dxdy \sim 0.0437\,. \tag{4.7}$$

Hence we can lower the sum of the eigenvalues for a 2-partition of the equilateral triangle by starting of the nodal 2-partition of u^S and by deforming the nodal line of u^S.

[3]again with the help of M. Levitin

In view of the previous considerations it would be interesting to investigate the following conjecture :

Conjecture 4.3
Take a simply connected bounded domain $\Omega \subset \mathbb{R}^2$ that is invariant with respect to inversion, i.e. $x \in \Omega$ implies $-x \in \Omega$. Suppose the eigenspace associated with $\lambda_2(\Omega)$ consists of functions which are antisymmetric with respect to inversion, then $\lambda_2(\Omega) = \mathfrak{L}_{2,1}(\Omega)$.

5 Conclusion

Of course it would be interesting to have numerical computations available in order to formulate other conjectures. But many of the recent computations are for minimal partitions for $\mathfrak{L}_{k,\infty}$ (see [2, 4]) and cannot be extended to 1-minimal spectral partitions. A numerical determination of the 1-minimal 2-partition for the equilateral triangle would be quite interesting.

The pictures given in [11] indicate that, for the square $\mathcal{R}_1$, the equality $\mathfrak{L}_{k,1}(\mathcal{R}_1) = \mathfrak{L}_{k,\infty}(\mathcal{R}_1)$ holds for $k = 1, 2, 3, 4, 5$. In [11], the authors also compute for a rectangle such that $\frac{b}{a} = 1.3$ the minimal partition for $\mathfrak{L}_{5,1}$ (see their Figure 2). This minimal partition has a boundary with three critical points and is symmetric with respect to the perpendicular bisector of the largest side of the rectangle. Moreover, see their Figure 3, they found

$$\lambda(D_1) = \lambda(D_2) < \lambda(D_3) < \lambda(D_4) = \lambda(D_5)\,.$$

This partition is consequently not minimal for $\mathfrak{L}_{5,\infty}$.

One could hope that the numerical methods used by [5], although developed at the moment for the torus, could be efficient on this problem.

Another approach has been developed by N. Landais [17] in his PHD thesis using a genetic algorithm. At the moment, the method seems to have a lack of accuracy and slowly convergent but can be complemented by a method of boundary variation [18]. As an application of their methods, the authors treat various examples :

- The case of the triangle $\mathcal{T}_{ABC}$ with $A = (0, 0)$, $B = (9, 0)$, $C = (5.33, 5.96)$.

In this case, they find approximately

$$\mathfrak{L}_{2,1}(\mathcal{T}_{ABC}) = 1.85\,,$$

with for the corresponding 1-minimal 2-partition

$$\lambda(D_1) = 1.62\,,\ \lambda(D_2) = 2.08\,.$$

Because $\lambda(D_1) \neq \lambda(D_2)$, we have

$$\mathfrak{L}_{2,1}(\mathcal{T}_{ABC}) < \mathfrak{L}_{2,\infty}(\mathcal{T}_{ABC})\,.$$

Numerical computations performed by V. Bonnaillie-Noël give

$$\mathfrak{L}_{2,\infty}(\mathcal{T}_{ABC}) = \lambda_2(\mathcal{T}_{ABC}) = 2.0557\,.$$

- The case of a rectangle $\mathcal{R}_{\frac{1}{2}}$.
  Here the 1-minimal 2-partition seems to be two squares. Hence this suggests that $\mathfrak{L}_{2,1}(\mathcal{R}_{\frac{1}{2}}) = \mathfrak{L}_{2,\infty}(\mathcal{R}_{\frac{1}{2}})$.
- The case of the square and of the disk.
  Here $\mathfrak{L}_3$ is considered. Although not sufficiently precise, the result seems to confirm the candidates for the square and the disc discussed respectively in [3] and [12].

In conclusion we hope to have demonstrated that the equality between $\mathfrak{L}_{k,1}$ and $\mathfrak{L}_{k,\infty}$ can only occur in very particular situations. But obviously a deeper understanding of the qualitative (and quantitative) properties of minimal partitions, in particular for $p = 1$, is still missing.

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References

[1] C.J. Bishop. Some questions concerning harmonic measure. Dahlberg, B. (ed.) et al., Partial Differential equations with minimal smoothness and applications. IMA Vol. Math. Appl. 42 (1992), p. 89-97.

[2] V. Bonnaillie-Noël and B. Helffer. Numerical analysis of nodal sets for eigenvalues of Aharonov-Bohm hamiltonians on the square and application to minimal partitions. In preparation.

[3] V. Bonnaillie-Noël, B. Helffer and T. Hoffmann-Ostenhof. Aharonov-Bohm Hamiltonians, isospectrality and minimal partitions. Journal of Physics A : Math. Theor. 42 (18), 185203 (2009).

[4] V. Bonnaillie-Noël, B. Helffer and G. Vial. Numerical simulations for nodal domains and spectral minimal partitions. To appear in COCV (2009).

[5] B. Bourdin, D. Bucur, and E. Oudet. Optimal partitions for eigenvalues. Preprint January 2009. To appear in Siam Journal on Scientific Computing.

[6] D. Bucur, G. Buttazzo, and A. Henrot. Existence results for some optimal partition problems. Adv. Math. Sci. Appl. 8 (1998), p. 571-579.

[7] L.A. Caffarelli and Fang Hua Lin. An optimal partition problem for eigenvalues. Journal of Scientific Computing 31 (1/2) (2006), p. 5-18.

[8] M. Conti, S. Terracini, and G. Verzini. An optimal partition problem related to nonlinear eigenvalues. J. of Funct. Anal. 198 (2003), p. 160-196.

[9] M. Conti, S. Terracini, and G. Verzini. A variational problem for the spatial segregation of reaction-diffusion systems. Indiana Univ. Math. J. 54 (3) (2005), p. 779-815.

[10] M. Conti, S. Terracini, and G. Verzini. On a class of optimal partition problems related to the Fucik spectrum and to the monotonicity formula. Calc. Var. 22 (2005), p. 45-72.

[11] O. Cybulski, V. Babin, R. Hołyst. Minimization of the Renyi entropy production in the space-partitioning process. Phys. Rev. E (3) **71**(4) (2005) 046130-10.

[12] B. Helffer, T. Hoffmann-Ostenhof. On spectral minimal partitions : new properties and applications to the disk. To appear in Proceedings of the conference in Montreal. CRM Lectures (2008).

[13] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. Nodal domains and spectral minimal partitions. Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009), p. 101-138.

[14] B. Helffer, T. Hoffmann-Ostenhof, S. Terracini. Nodal domains and spectral minimal partitions : the case of the sphere. To appear in a volume in honor of V. Maz'ya.

[15] A. Henrot. *Extremum Problems for Eigenvalues of Elliptic Operators.* Birkhäuser, Basel, Boston, London, 2006.

[16] B. Kawohl. Remarks on some old and current eigenvalue problems. In Partial Differential Equations of elliptic type, A. Alvino, E. Fabes, G. Talenti eds, Symposia Mathematica, Vol. XXXV, Cambridge University Press (1994).

[17] N. Landais. Problèmes de régularité en optimisation de forme. PHD Thesis (2007). IRMAR.

[18] N. Landais, G. Vial, and M. Pierre. Utilisation d'algorithmes génétiques et de méthodes de variation de frontière pour des problèmes de partages optimaux. Prépublication de l'IRMAR (2007).

[19] B.J. McCartin. Eigenstructure of the equilateral triangle, Part I: The Dirichlet problem. Siam Review 45(2), p. 267-287 (2003).

[20] M.A. Pinsky. The eigenvalues of an equilateral triangle. Siam J. Math. Anal. 11 (5), p. 819-827 (1980).

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