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Connecting Points in Irrducible Stein Spaces by Irreducible Analytic Curves

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Connecting points in irreducible Stein spaces by irreducible analytic curves

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1 Introduction

In this paper we are concerned with the geometry of Stein spaces and some consequences in characterization of holomorphic q-hulls in top degrees. We show:

Theorem 1 *Any two points p, q of an irreducible Stein space X can be connected by an irreducible analytic curve C. In fact if $n = dim\ X$ there are $n-1$ holomorphic functions $h_1, \ldots, h_{n-1} \in \mathcal{O}(X)$ such that $C = \{h_1 = \cdots = h_{n-1} = 0\}$.*

It will be clear from the proof that this theorem holds for an arbitrary discrete subset of points of X. Moreover we may allow $h_1, \ldots, h_{n-1}$ to be global holomorphic sections in a given holomorphic line bundle L on X.

Note that if $H^2(X, \mathbf{Z}) \neq 0$ there may be irreducible analytic curves that cannot be given by $n-1$ equations. For instance, the famous example of Oka, cited in the book of Range [6, p.239 − 240] works also here. X is the domain of holomorphy in $\mathbf{C}^2$ given by $X = \{(z_1, z_2)\,;\, 3/4 < z_j < 5/4,\ j = 1, 2\}$ and $C = \{z \in X\,;\, z_2 - z_1 + 1 = 0\}$. Then there does not exists a holomorphic function $f : X \longrightarrow \mathbf{C}$ such that $f^{-1}(0) = C$.

On the other hand there are examples of non-compact connected two dimensional complex manifolds that do not have complex curves at all (Take a two dimensional complex torus without complex curves and then delete one point).

We mention that a similar version to Theorem 1 (in a weaker form) was earlier proved by Baran in [1], namely; Through an arbitrary discrete sequence of points of a connected Stein space passes a connected complex curve.

Theorem 1 is a straightforward consequence of the following

Theorem 2 *Let X be an irreducible Stein space and p, q two arbitrary points of X. Then there is $f \in \mathcal{O}(X)$ such that $Z_f = \{x \in X \mid f(x) = 0\}$ is an irreducible hypersurface that passess through the points p, q.*

As an application of the above theorem we prove in sect. 4 a theorem which asserts that the holomorphic n-hull of compact subsets in a purely n-dimensional Stein spaces X is simply obtained by filling the holes of K, which is the natural n-dimension extension of Behnke's generalization of Runge's theorem to an open Riemann surface. (See Theorem 3, sect. 4)

The main ideas of this paper stem from the article of Demailly [2] where he produces irreducible hypersurfaces in $\mathbf{C}^n$ with given singular locus.

2 Preliminaries

All complex spaces are reduced with countable topology.

Let X be a complex space. Then $\mathcal{O}(X)$ endowed with the topology of compact convergence is a Fréchet space.

Now suppose X be purely n-dimensional. We define $\mathcal{U} :=$ the subset of $\mathcal{O}(X)$ made of all holomorphic functions f not identically zero such that

$$\dim Z_f \cap \mathrm{Sing}\,(X) \leq n-2. \tag{$\clubsuit$}$$

Then $\mathcal{U}$ is a dense open subset of $\mathcal{O}(X)$.

($\clubsuit$) allow us to define the multiplicity of f on the irreducible components of Z_f. Indeed let Z_1 be an irreducible component of Z_f. Then $H := Z_1 \setminus (\mathrm{Sing}\,(Z_1) \cup \mathrm{Sing}\,(X))$ is a connected smooth hypersurface of the complex manifold $M := \mathrm{Reg}\,(X) \setminus \mathrm{Sing}\,(Z_1)$. Locally, at $a \in H$, f can be written, in suitable coordinates $(w_1, \ldots, w_n)$ of M, in the form $f = w_1^m \cdot h$ with $h(a) \neq 0$. This m does not depend on $a \in H$, and, by definition, is called the multiplicity of f along Z_1.

Fix an arbitrary compact set K of X. Define a function $\mu_K : \mathcal{U} \longrightarrow \mathbf{N}$ as follows: For any $f \in \mathcal{U}$ put

$$\mu_K(f) := \begin{cases} \text{the sum of the multiplicities of } f \text{ on those} \\ \text{irreducible components of } Z_f \text{ that meet } K. \end{cases}$$

Here we show

Proposition 1 *The map μ_K is upper semi-continuous.*

Proof. Fix $f \in \mathcal{U}$. We prove the semi-continuity in this function.

We replace K by suitable compact sets K_1 (that avoids the singular locus of Z_f and the singular locus of X) and K_2 (a neighborhood of $K_1 \cap Z_f$ in $\mathrm{Reg}(X)$) such that

$$\begin{cases} \mu_K(f) & = \mu_{K_1}(f) & = \mu_{K_2}(f), \\ \mu_K(g) & \leq \mu_{K_1}(g) & \leq \mu_{K_2}(g) \end{cases}$$

for any g in a neighborhood of f.

To achieve this let $S' := Z_f \cap \mathrm{Sing}(X)$, $S'' = \mathrm{Sing}(Z_f)$ and $S := S' \cup S''$. One has $\dim S \leq n-2$. Now let $x_o \in S \cap K$. There are neighborhoods U of x_o in X, V of $0 \in \mathbf{C}^n$ and a finite holomorphic map $\pi : U \longrightarrow V$, $\pi(x_o) = 0$. Let $(w_1, \ldots, w_n)$ be a system of coordinates of $\mathbf{C}^n$ around 0 (we shrink U and V, if necessary) such that 0 lies isolated in $\pi(S \cap U) \cap \{w_3 = \cdots = w_n = 0\}$. Choose $\varepsilon > 0$; then $\eta > 0$, both small enough such that denoting

$$\begin{cases} W(x_o) & = & \{w \in V\,;\, |w_1(x)|^2 + |w_2(x)|^2 < \varepsilon^2,\ |w_j(x)| < \eta \text{ for } j > 2\}, \\ T(x_o) & = & \{w \in V\,;\, |w_1(x)|^2 + |w_2(x)|^2 = \varepsilon^2, |w_j(x)| \leq \eta \text{ for } j > 2\} \end{cases}$$

we have $\pi(S \cap U) \cap T(x_o) = \emptyset$ and the irreducible components of Z_f that do not meet K, do not meet $\overline{U(x_o)} := \pi^{-1}(W(x_o))$, too. Set $B(x_o) := \pi^{-1}(T(x_o))$. Let $U(x_1), \ldots, U(x_p)$ be a covering of $S \cap K$. We define

$$K_1 := (K \setminus \bigcup_{1 \leq j \leq p} U(x_j)) \cup \bigcup_{1 \leq j \leq p} B(x_j).$$

Notice that any hypersurface of X meeting $U(x_j)$ at point $a \in U(x_j)$ meets also $B(x_j)$ (Otherwise the trace of the image of this hypersurface by π on the set

$$\{w \in V\,;\ |w_1(x)|^2 + |w_2(x)|^2 < \varepsilon^2,\ w_j = \pi_j(a),\ j > 2\}$$

will be a compact analytic set of positive dimension). Thus for any $g \in \mathcal{U}$ we have $\mu_K(g) \leq \mu_{K_1}(g)$. Moreover, by construction $\mu_K(f) \geq \mu_{K_1}(f)$, $K_1 \cap$ Sing $(Z_f) = \emptyset$ and $K_1 \cap Z_f \subset \mathrm{Reg}(X)$.

Now let $X'_1, \ldots, X'_r$ be the connected components of $Z_f \setminus \mathrm{Sing}\,(Z_f)$ that meet K_1 and $L_j \subset$ Reg (X), $1 \leq j \leq r$, mutually disjoint compact neighborhoods of $X'_j \cap K_1$ in X such that $Z_f \cap L_j = X'_j \cap L_j$. Set

$$K_2 := \bigcup_{1 \leq j \leq r} L_j.$$

Thus $\mu_{K_2}(f) = \sum_{j=1}^{r} \mu_{L_j}(f) = \mu_{K_1}(f)$. Since K_2 is a neighborhood of $Z_f \cap K_1$ we get

$$\inf_{x \in K_1 \setminus \overset{\circ}{K}_2} |f(x)| > 0.$$

Hence $Z_g \cap K_1 \subset Z_g \cap K_2$ for g that approaches f. This implies $\mu_{K_1}(g) \leq \mu_{K_2}(g) \leq \sum_{j=1}^{r} \mu_{L_j}(g)$.

It remains to prove the semicontinuity of μ_{L_j} at f. For the sake of simplicity we drop the index j and consider L a compact set of Reg (X) such that $Z_f \cap L = Z' \cap L$ where Z' is a connected component of $Z_f \setminus$ Sing(Z_f). Let $P := \{w \in \mathbf{C}^n;\ |w_j| < 1,\ 1 \leq j \leq n\}$ the unit polydisc. Since $Z' \cap L$ is compact there are finitely many charts $\theta_\nu : U_\nu \longrightarrow P$ defined by $\theta_\nu(x) = (w_1(x), \ldots, w_n(x))$, $x \in U_\nu$ such that U_ν cover $Z' \cap L$ and Z' has the equation $w_1(x) = 0$. We assume also $Z_f \cap U_\nu = Z' \cap U_\nu$. Let $\pi_\nu : U_\nu \longrightarrow X' \cap U_\nu$ be given in local coordinates by $(w_1, \ldots, w_n) \rightsquigarrow (0, w_2, \ldots, w_n)$. Since Z' is connected we may take $\bigcup \pi_\nu(U_\nu)$ connected.

For any couple (ν_1, ν_2) of distinct indices such that $\pi_{\nu_1}(U_{\nu_1}) \cap \pi_{\nu_2}(U_{\nu_2}) \neq \emptyset$ we choose a point $x_{\nu_1\,\nu_2} \in \pi_{\nu_1}(U_{\nu_1}) \cap \pi_{\nu_2}(U_{\nu_2})$. Define for $\varepsilon \in (0,1)$ the sets

$$\begin{aligned} A^\varepsilon_\nu &= \{x \in U_\nu\,;\ |w_j(x)| \leq 1 - \varepsilon, 1 \leq j \leq n\}, \\ B^\varepsilon_\nu &= \{x \in U_\nu\,;\ |w_1(x)| < \varepsilon, |w_j(x)| \leq 1 - \varepsilon, 1 < j \leq n\}, \\ C^\varepsilon_\nu &= A^\varepsilon_\nu \setminus B^\varepsilon_\nu. \end{aligned}$$

Now we consider $\varepsilon > 0$ small enough such that the following conditions hold

(1) $x_{\nu_1\,\nu_2} \in \pi_{\nu_1}(A^\varepsilon_{\nu_1}) \cap \pi_{\nu_2}(A^\varepsilon_{\nu_2})$;

(2) $\pi^{-1}_{\nu_1}(x_{\nu_1\,\nu_2}) \cap B^\varepsilon_{\nu_1} \subset A^\varepsilon_{\nu_2}$;

(3) $A^\varepsilon := \bigcup A^\varepsilon_\nu$ is a compact neighborhood of $X' \cap L$.

¿From (3) and since $Z_f \cap A^\varepsilon = X' \cap A^\varepsilon$ we deduce $\mu_L \leq \mu_{A^\varepsilon}$ in a neighborhood of f in $\mathcal{O}(X)$ and that

$$\mu_L(f) = \mu_{A^\varepsilon}(f) = \text{the multiplicity } m \text{ of } \ f \text{ on } X'.$$

The proof will be concluded if we show that $\mu_{A^\varepsilon}(g) \leq m$ for

$$\sup_{A^\varepsilon} |g - f| < \inf_{C^\varepsilon} |f| \tag{4}$$

where $C^\varepsilon = \cup C^\varepsilon_\nu$ (Note that $Z_f \cap C^\varepsilon = \emptyset$, hence $\inf_{C^\varepsilon} |f| > 0$).

Let Y be an irreducible component of Z_f that meets A^ε, say, for instance $Y \cap A^\varepsilon_\nu \neq \emptyset$. Then inequality (4) implies $Y \cap A^\varepsilon_{\nu_1} = Y \cap B^\varepsilon_{\nu_1}$, thus for all $x \in \pi_{\nu_1}(A^\varepsilon_{\nu_1})$ the analytic set $Y \cap B^\varepsilon_{\nu_1} \cap \pi^{-1}(x)$ of the disc $B^\varepsilon_{\nu_1} \cap \pi^{-1}(x)$ is compact. Consequently the fibres $Y \cap B^\varepsilon_{\nu_1} \cap \pi^{-1}(x)$ are discrete, and, as $\dim Y = \dim X' = n - 1$, $\pi_{\nu_1}(Y \cap B^\varepsilon_{\nu_1})$ is open in $\pi_{\nu_1}(B^\varepsilon_{\nu_1}) = \pi_{\nu_1}(A^\varepsilon_{\nu_1})$. Since $\pi_{\nu_1}(Y \cap B^\varepsilon_{\nu_1}) = \pi_{\nu_1}(Y \cap A^\varepsilon_{\nu_1})$ is compact and non empty, $\pi_{\nu_1}(Y \cap B^\varepsilon_{\nu_1}) = \pi_{\nu_1}(A^\varepsilon_{\nu_1})$ by connectivity of $\pi_{\nu_1}(A^\varepsilon_{\nu_1})$. (1) and (2) imply that for all ν_2 with $\pi_{\nu_1}(A^\varepsilon_{\nu_1}) \cap \pi_{\nu_2}(A^\varepsilon_{\nu_2}) \neq \emptyset$ we have $Y \cap A^\varepsilon_{\nu_2} \neq \emptyset$. Hence $\pi_{\nu_2}(Y \cap B^\varepsilon_{\nu_2}) = \pi_{\nu_2}(A^\varepsilon_{\nu_2})$ by repeating the same procedure as above. Now as $\cup \pi_\nu(A^\varepsilon_\nu)$ is connected, we have $\pi_\nu(Y \cap B^\varepsilon_\nu) = \pi_\nu(A^\varepsilon_\nu)$ for all ν.

Now choose ν_o an arbitrary index and $x_o \in \pi_{x_o}(A^\varepsilon_{\nu_o})$. By the theorem of Rouché and condition (4) the function g has precisely m zeroes (counted with multiplicities) in the disc $B^\varepsilon_{\nu_o} \cap \pi^{-1}_{\nu_o}(x_o)$. If $Y_1, \ldots, Y_k$ are the irreducible components of Z_g that meet A^ε (hence also $B^\varepsilon_{\nu_o} \cap \pi^{-1}_{\nu_o}(x_o)$) and $m_1, \ldots, m_k$ are the multiplicities of g on those components, any point of $Y_j \cap B^\varepsilon_{\nu_o} \cap \pi^{-1}_{\nu_o}(x_o)$, $1 \leq j \leq k$, contributes by at least m_j zeroes. Thus

$$\mu_{A^\varepsilon}(g) = m_1 + \cdots + m_r \leq m.$$

Proposition 2 *Let $f \in \mathcal{U}, K$ a compact set of X and $Z_1 = \overline{Z'_1}, \ldots,\ Z_t = \overline{Z'_t}$ be the irreducible components of Z_f that meet K. Then for any open subset Ω of X that meets each Z_j, the irreducible components of Z_g that meet K meet also Ω as soon as g is sufficiently near f.*

Proof. We keep notations as in the proof of Proposition 1 assuming that g is taken sufficiently near to f. Thus any irreducible component Y of Z_g such that $Y \cap K \neq \emptyset$ cuts L_j for $j = 1$ or $2, \ldots,$ or r. We choose the sets U_ν such that $\Omega \cap \cup \pi_\nu(U_\nu) \neq \emptyset$; this is always possible because $\mathrm{Reg}(X)$ is connected. Finally consider $x_o \in \Omega$ and $\varepsilon > 0$ small enough such that x_o belongs to some $B^\varepsilon_{\nu_o}$ with $\pi^{-1}_{\nu_o}(x_o) \cap B^\varepsilon_{\nu_o} \subset \Omega$. The proof of the proposition follows.

3 The proof of Theorem 2

Let $f_1, \ldots, f_k \in \mathcal{O}(X)$ such that $\{p, q\} = \{x \in X \mid f_1(x) = \cdots = f_k(x) = 0\}$ (In fact we may always take $k = n$, but we do not need this). Let $\pi : X \longrightarrow \mathbf{C}^n$ be an almost proper holomorphic map (see [3]). Thus π is surjective and has discrete fibers. Let

$$S := \mathrm{Sing}(X) \cup \{x \in \mathrm{Reg}(X) \mid \mathrm{rank}_x(\pi) \leq n - 1\}.$$

Therefore A is an analytic subset of X with $\dim(S) \leq n-1$.

At this moment we break the proof into three steps:

Step I *Let $\mathcal{A}$ be the set of all $(g_1, \ldots, g_k) \in \mathcal{O}(X)^k$ such that if we set $F := \sum_{j=1}^k f_j g_j$ then*

$$(\sharp) \qquad \qquad dim\ Z_F \cap S \leq n-2.$$

Then $\mathcal{A}$ is G_δ and dense in $\mathcal{O}(X)^k$.

Indeed, let $\{Y_s\}$ be all irreducible components of S of dimension $n-1$. Fix mutually distint points $y_s \in Y_s$, $y_s \neq p, q$. It is obvious that $\mathcal{A} = \cap \mathcal{A}_s$ where $\mathcal{A}_s := \{\, (g_1, \ldots, g_k) \in \mathcal{O}(X)^k\,;\ \sum_{j=1}^k f_j(y_s) g_j(y_s) \neq 0 \,\}$ are open subsets of $\mathcal{O}(X)^k$. Thus $\mathcal{A}$ is G_δ.

Now we check density. Let $(g_1, \ldots, g_k) \in \mathcal{O}(X)^k$ and for $(a_1, \ldots, a_k) \in \mathbf{C}^k$ consider $F_a := \sum_{j=1}^k f_j(g_j + a_j)$. Correspondingly define affine maps $\lambda_s : \mathbf{C}^k \longrightarrow \mathbf{C}$ by

$$\lambda_s(a) := \sum_{j=1}^k f_j(y_s)(g_j(y_s) + a_j),\ a \in \mathbf{C}^k.$$

Then λ_s is surjective. Consequently E :=the union of all $\lambda_s^{-1}(0)$ has zero Lebesgue measure in $\mathbf{C}^k$. Therefore any $a \in \mathbf{C}^k \setminus E$ fulfills ($\sharp$), whence the density of $\mathcal{A}$.

Step II *Let $\mathcal{B}$ be the set of all $(g_1, \ldots, g_k) \in \mathcal{O}(X)^k$ with the following property: If $F := \sum_{j=1}^k f_j g_j$ then any point $x \in Z_F \cap$ Reg (X), $x \neq p, q$ is regular for F. Then $\mathcal{B}$ is G_δ and dense in $\mathcal{O}(X)^k$.*

Indeed, let $M :=$ Reg $(X) \setminus \{p, q\}$ and $\{L_\nu\}$ a countable covering of M by compact sets contained in local charts. Then $\mathcal{B} = \cap \mathcal{B}_\nu$ where

$$\mathcal{B}_\nu := \{(g_1, \ldots, g_k) \in \mathcal{O}(X)^k\,;\ F := \sum_{j=1}^k f_j g_j \text{ is regular on } Z_F \cap L_\nu\,\}$$

are open subsets of $\mathcal{O}(X)^k$. Hence $\mathcal{B}$ is G_δ.

Note the following simple fact. Let M be a complex manifold and $h_1, \ldots, h_r \in \mathcal{O}(M)$. Set $T = \{t \in \mathbf{C}^r \mid t_1 h_1 + \cdots + t_r h_r \text{ has at least one critical point }\}$. Then T is measurable in $\mathbf{C}^r$ as being the image of an analytic subset of $\mathbf{C}^r \times M$ through the canonical projection $\mathbf{C}^r \times M \longrightarrow \mathbf{C}^r$.

Now in order to verify density it suffices to check that given $(g_1, \ldots, g_k) \in \mathcal{O}(X)^k$, the set of all $a = (a_1, \ldots, a_k) \in \mathbf{C}^k$ such that $F_a := \sum_{j=1}^k f_j(g_j + a_j)$ has a critical point on Reg $(X) \setminus Z_{f_j}$ is of Lebesgue measure zero for any $1 \leq j \leq k$. Indeed fix $a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k$. Then F_a has a critical point on Reg $(X) \subset Z_{f_j}$ if and only if a_j is a critical values of the function

$$-\frac{F_a}{f_j} + a_j = -\frac{1}{f_j} \sum_{s \neq j} f_s(g_s + a_s) - g_j$$

defined on Reg $(X) \setminus Z_{f_j}$. Now the set of all such critical values is negligible (Theorem of Sard) and the conclusion of Step II follows by Fubini.

Fix a point $x_o \in X$ such that $\pi(x_o) = 0$ and let K_ν be the compact connected component of $\pi^{-1}(B_\nu)$ that contains x_o, where B_ν is the closed ball of $\mathbf{C}^n$ with radius $\nu \in \mathbf{N}$. Then $\{K_\nu\}$ is an increasing sequence of compact sets that invades X.

Step III *Let $\mathcal{E}_\nu$ be the set of all $(g_1, \ldots, g_k) \in \mathcal{A} \cap \mathcal{B}$ such that if $F = \sum_{j=1}^k f_j g_j$, then $\mu_{K_\nu}(F) \leq 1$, i.e. at most one irreducible component of Z_F does meet K_ν, component on which F has multiplicity 1. Then $\mathcal{E}_\nu$ is G_δ and dense in $\mathcal{O}(X)^k$.*

That $\mathcal{E}_\nu$ is G_δ follows by standars arguments from Proposition 1.

To show density fix $(g_1, \ldots, g_k) \in \mathcal{O}(X)^k$ and let $a, b \in \mathbf{C}^k$ be small enough (in the given usual euclidean norm on $\mathbf{C}^k$) such that the functions

$$F_a := \sum_{j=1}^k f_j(g_j + a_j),\ F_b := \sum_{j=1}^k b_j(g_j + b_j)$$

have a common zero set $Z_{F_a} \cap Z_{F_b}$ of pure dimension $n-2$, $F_a,\ F_b \in \mathcal{A} \cap \mathcal{B}$. Note that the analytic sets $Z_{F_a} \cap A$, $Z_{F_b} \cap A$ have dimensions $\leq n-2$. Further one choses λ near enough to 1 such that the critical zeros of $F_a + \lambda F_b$ on $\mathrm{Reg}(X)$ are contained in $Z_{F_a} \cap Z_{F_b}$ and $\dim Y \cap A \leq n-2$. Here we put Y the hypersurface defined by $F_a + \lambda F_b = 0$. (Consider λ a regular values of $-F_a/F_b$ on $\mathrm{Reg}(X)\backslash Z_{F_b}$ and $1/\lambda$ regular values of $-F_b/F_a$ on $\mathrm{Reg}(X)\backslash Z_{F_a}$). Set $\Gamma = \pi(S \cup Y)$. Note that Γ is a countable union of locally analytic sets in $\mathbf{C}^n$ of dimensions $\leq n-1$.

Let $Y_1, \ldots, Y_r$ be the different irreducible components of Y meeting K_ν. Choose on each Y_j a point z_j, regular for Y and X, $z_j \notin S \cup Z_{F_b}$ and $\pi(z_j) \notin B_\nu$. Further there are mutually distinct vectors $v_1, \ldots, v_r \in \mathbf{C}^n$ such that denoting by H_j the hyperplane $\langle z - \pi(z_j), v_j\rangle = 0$ in $\mathbf{C}^n$ the following properties hold:

(5) $H_j \cap B_\nu = \emptyset$, $\|v_j - \pi(z_j)\| \leq 1$ (Note that B_ν is a closed convex subset of $\mathbf{C}^n$);

(6) $\pi^{-1}(H_j)$ intersects Y_j transversally at z_j;

(7) $H_j \not\ni \pi(z_s)$, $\forall\ s \neq j$;

(8) The subspaces $H_1 \cap H_j$, $j > 1$, are mutually disjoint and not contained in Γ.

Further, by (8), select for any $j > 1$, $x_j \in \pi^{-1}(H_1 \cap H_j)$ such that

(9) $\pi(x_j) \notin \Gamma \cup \bigcup_{1<s\neq j} H_s$.

Some comments are appropriate here. Roughly speaking the analytic sets $\pi^{-1}(H_j)$ connect the components $Y_1, \ldots, Y_r$. Then by deformation of the set $Y_1 \cup \cdots \cup Y_r \cup \pi^{-1}(H_1) \cup \cdots \cup \pi^{-1}(H_r)$ we shall obtain an irreducible hypersurface. To accomplish this set for any $\varepsilon \in \mathbf{C}$,

$$G_\varepsilon = \frac{1}{2}(F_a + \lambda F_b) \prod_{j=1}^n \left(1 - \frac{\langle \pi,\ v_j\rangle}{\langle \pi(z_j), v_j\rangle}\right) + \varepsilon F_b.$$

We will examine Z_{G_ε} near the points z_j, $1 \leq j \leq r$ and x_j, $1 < j \leq r$. But here $F_b \neq 0$; hence we restrict our consideration to

$$\frac{G_\varepsilon}{F_b} = \frac{1}{2} \cdot \frac{F_a + \lambda F_b}{F_b} \prod_{j=1}^r \left(1 - \frac{\langle \pi,\ v_j\rangle}{\langle \pi(z_j),\ v_j\rangle}\right) + \varepsilon;$$

Now at points z_j, thanks to (6), we may consider complex coordinates $(w_1, \ldots, w_n)$ such that

$$w_1 = 1 - \frac{\langle \pi,\ v_j \rangle}{\langle \pi(z_j),\ v_j \rangle},\ w_2 = \frac{1}{2} \cdot \frac{F_a + \lambda F_b}{F_b} \prod_{\substack{1 \le s \le r \\ s \ne j}} \left(1 - \frac{\langle \pi,\ v_s \rangle}{\langle \pi(z_j),\ v_s \rangle} \right);$$

at x_j, $j > 1$, due to (8) and (9), the coordinates $(w_1, \ldots, w_n)$ may be chosen such that

$$w_1 = \frac{1}{2} \cdot \frac{F_a + \lambda F_b}{F_b} \prod_{1 < s \le r} \left(1 - \frac{\langle \pi,\ v_s \rangle}{\langle \pi(z_s),\ v_s \rangle} \right),\ w_2 = 1 - \frac{\langle \pi,\ v_1 \rangle}{\langle \pi(z_1),\ v_1 \rangle} \cdot$$

Let D_{z_j} (resp. D_{x_j}, $j > 1$) be open neighborhoods of z_j (resp. x_j) such that the coordinates $(w_1, \ldots, w_n)$ realize an isomorphism of D_{z_j} (resp. D_{x_j}) onto the polydisc $P_\delta := \{ w \in \mathbf{C}^n\ ;\ |w_j| < \delta,\ 1 \le j \le n \}$. In D_{z_j} and D_{x_j}, Z_{G_ε} has the equation $w_1 w_2 + \varepsilon = 0$; hence if $0 < |\varepsilon| < \delta^2$, $Z_{G_\varepsilon} \cap D_{z_j}$ and $Z_{G_\varepsilon} \cap D_{x_j}$ are irreducible. On the other hand Z_{G_o} has equation $w_1 w_2 = 0$, where $w_1 = 0$ represents $\pi^{-1}(H_j)$, and $w_2 = 0$ gives the hypersurface Y (in D_{z_j}) or $\pi^{-1}(H_1)$ (in D_{x_j}).

Thus one can choose compact sets L_j of D_{z_j} and M_j of D_{x_j}, $j > 1$ such that

$$Z_{G_o} \cap L_j = \pi^{-1}(H_j) \cap L_j \quad \text{and} \quad \pi^{-1}(H_j) \cap \overset{\circ}{L_j} \ne \emptyset,$$

$$Z_{G_o} \cap M_j = \pi^{-1}(H_1) \cap M_j \quad \text{and} \quad \pi^{-1}(H_1) \cap \overset{\circ}{M_j} \ne \emptyset,$$

and it is clear that the hyperbola $w_1 w_2 + \varepsilon = 0$ cut L_j or M_j, as soon as ε is small enough.

We now use three times Proposition 2 with $f = G_o$, $g = G_\varepsilon$ and K, $\{ Z_1, \ldots, Z_r \}$ and U are respectively replaced by

$$(*_1) \qquad K_\nu,\ \{ Y_1, \ldots, Y_r \},\ \bigcup_{1 \le j \le r} D_{z_j};$$

$$(*_2) \qquad L_j,\ \{ \pi^{-1}(H_j) \},\ D_{x_j} \quad \text{for } j > 1;$$

$$(*_3) \qquad M_j,\ \{ \pi^{-1}(H_1) \},\ D_{z_1} \quad \text{for } j > 1;$$

If $\varepsilon \ne 0$ is small enough and if T is an irreducible component of Z_{G_ε} that meets K_ν we deduce the following

$$(10) \qquad T \text{ meets } \bigcup_{j=1}^{r} D_{z_j}.$$

We shall show that $T \cap D_{z_1} \ne \emptyset$. If $T \cap D_{z_j} \ne \emptyset$ for some $j > 1$, by $(*_2)$

$$T \cap L_j = \{ w_1 w_2 + \varepsilon = 0 \} \cap L_j \ne \emptyset;$$

hence T cuts D_{x_j}. Consequently $T \cap M_j = \{ w_1 w_2 + \varepsilon = 0 \} \cap M_j \ne \emptyset$, and, by $(*_3)$, T meets D_{z_1}.

All these possibilities lead us to $T \cap D_{z_1} \neq \emptyset$. Since $Z_{G_\varepsilon} \cap D_{z_1} = \{w_1 w_2 + \varepsilon = 0\} \cap D_{z_1}$ is irreducible and $G_\varepsilon = (w_1 w_2 + \varepsilon) F_b$ has multiplicity 1, Z_{G_ε} possess at most one irreducible component T meeting K_ν on which G_ε with necessity has multiplicity 1. By definition $G_\varepsilon \in \mathcal{E}_\nu$.

On the other hand write $G_\varepsilon = \sum_{j=1}^{k} f_j g_{j,\varepsilon}$ with

$$g_{j,\varepsilon} = \frac{1}{2}\left[(g_j + a_j) + \lambda(g_j + b_j)\right] \cdot \prod_{s=1}^{k} \left(1 - \frac{\langle \pi,\ v_s \rangle}{\langle \pi(z_j),\ v_s \rangle}\right) + \varepsilon(g_j + b_j)$$

and let a and b tend to 0, λ to 1, $\pi(z_s)$ to ∞ and ε to 0. It follows that, using $\|v_s - \pi(z_s)\| \leq 1$, $\{g_{j,\varepsilon}\}$ converges to g_j, whence the density of $\mathcal{E}_\nu$ in $\mathcal{O}(X)^k$. So the proof of Step III.

Consequently by the theorem of Baire $\cap\ \mathcal{E}_\nu$ is dense and G_δ in $\mathcal{O}(X)^k$, whence the theorem.

4 An application

In this section we deal with a refinement of the usual notion of a holomorphic hull of a compact set in a Stein space, in which the consideration of holomorphic functions is replaced by that of holomorphic maps to $\mathbf{C}^q$, $q \geq 1$.

We recall that the holomorphic hull of a compact set K of a Stein space X equals the intersection of all sets $f^{-1}(\widehat{f(K)})$ as f ranges through the holomorphic functions on X and $\widehat{f(K)}$ is the polynomial hull in $\mathbf{C}$ of $f(K)$, that is obtained by adding to $f(K)$ the relatively compact connected components of $\mathbf{C} \setminus f(K)$, i.e. roughly speaking by filling the holes of $f(K)$.

This fact opens the way to define the holomorphic q-hull of compact sets in Stein spaces. But first, for the sake of simplicity, let us denote for an open subset D of a complex space S by $\theta(D)$ the union of those irreducible components of D (N. B. not the open ones !) that are relatively compact in S. We shall use this notation for open sets of the form $D = S \setminus L$ for some compact set L of S.

Let X be a Stein space, $K \subset X$ a compact set and $q \geq 1$ an integer. We define following Lupacciolu [4] the **q-holomorphic hull** of K by

$$(*) \qquad \widehat{K}_X^{(q)} = \bigcap_{F \in \mathcal{O}^q(X)} F^{-1}[F(K) \ \cup\ \theta(\mathbf{C}^q - F(K))].$$

We list some simple properties. Suppose $n = \dim(X)$. Then

(11) $\widehat{K}_X^{(q)} = K$ if $q \geq n+1$.

(12) $\widehat{K}^{(n)} \subseteq \widehat{K}^{(n-1)} \subseteq \cdots \subseteq \widehat{K}^{(1)} = \widehat{K}$.

(13) If $f \in \mathcal{O}^p(X)$, $p < q$, then $\widehat{K}^{(q)} \cap Z_f \subseteq \widehat{K \cap Z_f}^{(q-p)}$ where $Z_f = \{x \in X\,;\ f(x) = 0\}$.

(14) If $\pi : Y \longrightarrow X$ is a finite surjective morphism, then $\widehat{\pi^{-1}(K)}_Y^{(q)} \subseteq \pi^{-1}(\widehat{K}_X^{(q)})$.

¿From [4] we mention that in definition $(*)$ one can replace $\mathcal{O}(X)^q$ by any other dense subset. For instance if V is a Runge domain of X that contains K then $\widehat{K}_X^{(q)} = \widehat{K}_V^{(q)}$, $\forall\, q \geq 1$.

Here we give the main result of this section.

Theorem 3 *Let X be a purely dimensional Stein space and $K \subset X$ a compact set. Suppose $n = dim(X)$. Then*

$$\widehat{K}_X^{(n)} = K \cup \theta(X \setminus K).$$

Proof. Let $K_1 := K \cup \theta(X \setminus K)$. First we show that $K_1 \subseteq \widehat{K}^{(n)}$. In order to do this let $\pi : X^* \longrightarrow X$ be the normalization map. Put $K^* = \pi^{-1}(K)$. Then $\widehat{K^*}^{(n)} \subseteq \pi^{-1}(\widehat{K}^{(n)})$ and $\pi(K^* \cup \nu(X^* \setminus K^*)) = K_1$. Consequently one has to check the said inclusion in case X is normal. Assume this and let $x_o \in X \setminus \widehat{K}^{(n)}$. Then there is a holomorphic map $F : X \longrightarrow \mathbf{C}^n$ such that $F(x_o)$ lies in the unbounded component of $\mathbf{C}^n \setminus F(K)$. It is a classical fact that there is a n-convex function $\varphi : \mathbf{C}^n \longrightarrow \mathbf{R}$ with $\varphi(F(x_o)) > \|\varphi\|_{F(K)}$.

Now let $\lambda : X \longrightarrow \mathbf{R}$ any smooth strongly plurisubharmonic function (i.e. 1-convex after the terminology of Andreotti and Grauert). Set for any $\varepsilon > 0$, $\psi_\varepsilon : X \longrightarrow \mathbf{R}$ by $\psi_\varepsilon = \varepsilon\lambda + \varphi \circ F$. Then ψ_ε is n-convex and if ε is samll enough $\psi_\varepsilon(x_o) > \|\psi_\varepsilon\|_K$. Since ψ_ε fulfils the maximum principle, $\|\psi_\varepsilon\|_K = \|\psi_\varepsilon\|_{K_1}$; thus $x_o \notin K_1$, whence the desired inclusion. Note that the open sets appearing in $\theta(X \setminus K)$ have the boundary in K as soon as X has no isolated points.

Further we prove the reverse inclusion by induction on n. The case $n = 1$ is treated by Mihalache [5]. So assume that $n > 1$ and the theorem holds for all Stein spaces of pure dimension between 1 and $n - 1$.

Let $a \in X \setminus K_1, Y$ an irreducible component of $X \setminus K$ that contains a (Note that Y is not relatively compact in X) and X_1 the correspondent irreducible component of X that contains Y. Choose $b \in Y \setminus \widehat{K}$ and consider a piecewise C^1 curve $\gamma : [0, 1] \longrightarrow Y$ that joints a with b and $\gamma((0, 1)) \subset \mathrm{Reg}(X)$. By a theorem of Stolzenberg [7], $L = \gamma([0, 1])$ is holomorphically convex in X. Consider Ω a tubular open neighborhood of L that is Runge in X, relatively compact, $\overline{\Omega} \cap K = \emptyset$ and such that $\Omega \cap X_1$ is irreducible. By applying Theorem 2 and Proposition 1 there is a holomorphic map $f_1 \in \mathcal{O}(X_1)$ such that $Z_{f_1} = \{x \in X_1\, ;\, f_1(x) = 0\}$ is an irreducible hypersurface, $a, b \in Z_{f_1}$ and (after shrinking Ω, if necessary) $Z_{f_1} \cap \Omega$ is irreducible. Choose $f \in \mathcal{O}(X)$ that extends f_1 such that $Z_f = \{x \in X\, ;\, f(x) = 0\}$ is purely $n-1$ dimensional. Let V be a relatively compact Runge neighborhood of K that does not contain b. Without any loss of generality we may suppose that $a \in V$.

Now set $X' = V \cap Z_f$, $K' = K \cap Z_f$. Further take $\eta : [0, 1] \longrightarrow Z_f$ a C^1 piecewise curve that connects a with b such that $\eta((0, 1)) \subset \mathrm{Reg}(Z_f) \cap \Omega$. Let $t^* = \sup\{t \in [0, 1]\, ;\, \eta(s) \in X'$ for all $s, 0 \leq s \leq t\}$. Obviously $0 < t^* < 1$. Let S' be the irreducible component of $X' \setminus K'$ that contains $\eta((0, t^*))$. Hence $a \in S$ and S is not relatively compact in X'; thus, by the induction hypothesis $a \notin \widehat{K'}_{X'}^{(n-1)} = \widehat{K \cap Z_f}^{(n-1)}$. Since $a \in Z_f$, by (13), $a \notin \widehat{K}^{(n)}$. This completes the proof of the theorem.

Remark We cannot replace "irreducible component" in the above theorem with "open component", even in the simplest case $n = 1$. Set $X = \mathbf{C} \times \{0\} \cup \{0\} \times \mathbf{C}$ and $K =$

$S^1 \times \{0\}$. Then $\widehat{K} = \Delta \times \{0\}$ where Δ is the closed unit disc in $\mathbf{C}$. However $X \setminus K$ is connected, and, of course, not relatively compact in X. An example with X irreducible is given in [5].

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