

**Gauss–Berezin Integral Operators and Spinors  
over Supergroups  $\mathrm{OSp}(2p—2q)$**

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# Gauss–Berezin integral operators and spinors over supergroups $\mathrm{OSp}(2p|2q)$

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We obtain explicit formulae for the spinor representation of the supergroup  $\mathrm{OSp}(2p|2q)$  and the corresponding super-Olshanski semigroup.

Apparently, first elements of a strange analogy between orthogonal and symplectic spinors<sup>2</sup> were observed by K.O.Friedrichs in early 1950s, see [7].

F.A.Berezin in early 1960s obtained explicit formulae [1] for both representations. In the symplectic case, the representation of  $\mathrm{Sp}(2n, \mathbb{R})$  is realized by certain integral operators of the form

$$Af(z) = \int_{\mathbb{C}^n} \exp\left\{\frac{1}{2}(z \ \bar{u}) \begin{pmatrix} K & L \\ L^t & M \end{pmatrix} \begin{pmatrix} z^t \\ \bar{u}^t \end{pmatrix}\right\} f(u) e^{-|u|^2} du \, d\bar{u} \quad (0.1)$$

in the space of holomorphic functions on  $\mathbb{C}^n$ . Here the sign  $^t$  denotes the transpose;  $z = (z_1 \ \dots \ z_n)$ ,  $u = (u_1 \ \dots \ u_n)$  are vectors-rows and  $K, L, M$  are certain  $n \times n$  matrices (depending on an element  $g \in \mathrm{Sp}(2n, \mathbb{R})$ ).

On the other hand, Berezin obtained formulae for the spinor representation  $\mathrm{O}(2n, \mathbb{R})$ , namely, he wrote certain 'integral operators' of the form

$$Af(z) = \int \exp\left\{\frac{1}{2}(\xi \ \bar{\eta}) \begin{pmatrix} K & L \\ -L^t & M \end{pmatrix} \begin{pmatrix} \xi^t \\ \bar{\eta}^t \end{pmatrix}\right\} f(\xi) e^{\eta \bar{\eta}^t} d\eta \, d\bar{\eta} \quad (0.2)$$

here  $\xi = (\xi_1 \ \dots \ \xi_n)$ ,  $\eta = (\eta_1 \ \dots \ \eta_n)$  are matrices-rows.<sup>3,4</sup>

This strange analogy pushed him to an invention of the 'super-analysis', which mixes even (complex) variables and odd (Grassmann) variables, see [2], see also a more advanced introduction in [9].

The purpose of this paper is to unite formulae (0.1)–(0.2) and to write explicitly the representation of the super(semi)group<sup>5</sup>  $\mathrm{OSp}(2p|2q)$ .

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<sup>2</sup>There is a lot of terms for symplectic spinors, namely, the Weil representation, the Shale–Weil representation, the Segal–Shale–Weil representation, the harmonic representation, and the oscillator representation. For references, see [10]; I make an apology for absence of references in this text; also a historically interesting reference is [4]

<sup>3</sup>Actually, in both cases Berezin considered  $n = \infty$ .

<sup>4</sup>However, such a formula can be written for an open dense subset in  $\mathrm{SO}(2n, \mathbb{R}) \subset \mathrm{O}(2n, \mathbb{R})$ ; this remark produces a two-step definition of a Gauss-Berezin operators below.

<sup>5</sup>Posthumous notes [2] of Berezin published (and completed) by V.P.Palamodov and A.A.Kirillov contain a discussion of spinors over  $\mathfrak{osp}(2p|2q)$  on the level of Lie super-algebra; however, the text of this 'Chapter' is interrupted at this point. See also an another 'posthumous' paper [3].

In Sections 1–2, we define super-analogs of Gaussian integrals and Gaussian integral operators. Sections 3–4 contain preliminaries on supergroups and super-linear relations. In Section 5, we get desired explicit formulae. In the last section we discuss some possible problems arising from our construction.

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## 1 Gauss–Berezin integrals

**1.1. Variables.** We consider 3 types of variables, namely, bosonic variables, fermionic variables, and phantom variables.

a) *Bosonic variables* are the usual real variables. We denote them by  $x_1, x_2, \dots$  or  $y_1, y_2, \dots$

b) *Fermionic variables* are formal variables  $\xi_1, \xi_2, \dots$  (or  $\eta_1, \eta_2, \dots$ ) satisfying the anticommutativity condition

$$\xi_k \xi_l = -\xi_l \xi_k$$

In particular,  $\xi_k^2 = 0$ .

c) *Phantom variables*  $\mathfrak{a}_1, \mathfrak{a}_2, \dots$  are anti-commuting variables,

$$\mathfrak{a}_k \mathfrak{a}_l = -\mathfrak{a}_l \mathfrak{a}_k$$

We define a *phantom algebra*  $\Lambda$  as the algebra of polynomials in the variables  $\mathfrak{a}_j$  (for simplicity, we assume that the number of variables is infinite). We also call elements of  $\Lambda$  by *phantom constants*.

The phantom algebra has a natural  $\mathbb{Z}$ -gradation by a degree of polynomial,

$$\Lambda = \bigoplus_{j=0}^{\infty} \Lambda_j$$

Therefore,  $\Lambda$  admits a  $\mathbb{Z}_2$ -gradation, namely

$$\Lambda_{\text{even}} := \bigoplus \Lambda_{2j}, \quad \Lambda_{\text{odd}} := \bigoplus \Lambda_{2j+1},$$

We define the automorphism  $\mu \mapsto \mu^\sigma$  of  $\Lambda$  by the rule

$$\mu^\sigma = \begin{cases} \mu & \text{if } \mu \text{ is even} \\ -\mu & \text{if } \mu \text{ is odd} \end{cases}$$

(equivalently,  $\mathfrak{a}_j^\sigma = -\mathfrak{a}_j$ ).

The algebra  $\Lambda$  is *super-commutative* in the following sense:

$$\begin{aligned} \mu \in \Lambda, \nu \in \Lambda_{\text{even}} &\implies \mu\nu = \nu\mu, \\ \mu \in \Lambda_{\text{odd}}, \nu \in \Lambda_{\text{odd}} &\implies \mu\nu = -\nu\mu, \end{aligned}$$

Also,

$$\mu \in \Lambda, \nu \in \Lambda_{\text{odd}} \implies \nu\mu = \mu^\sigma \nu, \tag{1.1}$$

Next, let  $\mu = \sum_{j \geq 0} \mu_j \in \Lambda$ , where  $\mu_j \in \Lambda_j$ . We define

$$\pi_{\downarrow}(\mu) = \pi_{\downarrow}\left(\sum_{j \geq 0} \mu_j\right) := \mu_0 \in \mathbb{C}$$

Evidently,

$$\pi_{\downarrow}(\mu_1 \mu_2) = \pi_{\downarrow}(\mu_1) \pi_{\downarrow}(\mu_2)$$

Take  $\varphi \in \Lambda$  such that  $\pi_{\downarrow}(\varphi) = 0$ . Then  $\varphi^N = 0$  for sufficiently large  $N$ . Therefore,

$$(1 + \varphi)^{-1} := \sum_{n \geq 0} (-\varphi)^n$$

Actually, the sum is finite. In particular, if  $\pi_{\downarrow}(\mu) \neq 0$ , then  $\mu$  is invertible.

*Interplay of variables.* We assume that the bosonic variables  $x_l$  commute with the fermionic variables  $\xi_j$  and phantom constants  $\mu \in \Lambda$ . We also assume that  $\xi_j$  and  $\mathbf{a}_l$  anticommute,

$$\xi_j \mathbf{a}_l = -\mathbf{a}_l \xi_j,$$

**1.2. Space of functions.** Fix  $\alpha, \beta \in \mathbb{Z}_+$ . Let  $x_1, \dots, x_\alpha$  be bosonic variables and  $\xi_1, \dots, \xi_\beta$  be fermionic variables.

We define the *Fock-Berezin* space  $\mathbf{F}_{\alpha, \beta}$  as the space of expressions

$$f(x, \xi) := \sum_{0 < i_1 < \dots < i_k \leq \beta} \mu_{i_1 \dots i_k} h_{i_1, \dots, i_k}(x_1, \dots, x_\alpha) \xi_{i_1} \dots \xi_{i_k}, \quad (1.2)$$

where  $\mu$  are phantom constants and  $h$  are functions of a real variable  $x \in \mathbb{R}^\alpha$ . To be definite<sup>6</sup>, we assume that  $h(x)$  are elements of the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$  (i.e., they rapidly decrease with their derivatives).

We say, that a function  $f$  is *even* (respectively *odd*) if it is an even expression in the collection  $\{\xi_i\}, \{\mathbf{a}_k\}$ .

We define the space

$$\pi_{\downarrow}(\mathbf{F}_{\alpha, \beta}) := \mathcal{S}(\mathbb{R}^\alpha) \otimes \mathbb{C}[\xi_1, \dots, \xi_\beta]$$

as the space of expressions (1.2), where  $\mu$  are the usual complex numbers.

We define the canonical map

$$\pi_{\downarrow} : \mathbf{F}_{\alpha, \beta} \rightarrow \pi_{\downarrow}(\mathbf{F}_{\alpha, \beta})$$

in a the natural way, i.e., we replace  $\mu$  by  $\pi_{\downarrow}(\mu)$  in (1.2).

REMARK. The reader can observe, the fermionic variables and phantom variables have equal rights in our definition. However, below their roles are rather different;  $\xi_j$  serve as variables and elements of  $\Lambda$  serve as constants.

**1.3. Intergration.** The symbol

$$\int f(x, \xi) dx = \int f(x, \xi) dx_1 \dots dx_\alpha$$

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<sup>6</sup>and to avoid a discussion of convergence below

denotes the usual integration in the variable  $x$ . Precisely, for  $f$  given by (1.2), we integrate in  $x$  termwise and get a finite sum depending on  $\xi_j, \mathbf{a}_k$ .

The symbol

$$\int f(x, \xi) d\xi = \int f(x, \xi) d\xi_1 \dots d\xi_\beta$$

is the *Berezin integral*. If  $\tau$  is a function independent on  $\xi$ , and  $i_1 < i_2 < \dots < i_k$ , we put

$$\int \tau \cdot \xi_{i_1} \xi_{i_2} \dots \xi_{i_k} d\xi_1 d\xi_2 \dots d\xi_\beta := \begin{cases} \tau & \text{if } \{i_1, i_2, \dots, i_k\} = \{1, 2, \dots, \beta\} \\ 0 & \text{otherwise} \end{cases}$$

**1.4. The Gauss–Berezin integral. Definition.** Consider the expression

$$\begin{aligned} I &= \iint \exp\left\{\frac{1}{2} (x \ \xi) \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \end{pmatrix}\right\} dx d\xi = \\ &= \iint \exp\left\{\frac{1}{2} \sum_{ij} a_{ij} x_i x_j + \sum_{ik} b_{ik} x_i \xi_k + \frac{1}{2} \sum_{kl} c_{kl} \xi_k \xi_l\right\} dx d\xi \quad (1.3) \end{aligned}$$

First, we must explain the notation. The symbol  $^t$  denotes the transposition. The symbols  $x$  and  $\xi$  denote the row-matrices

$$x = (x_1 \ \dots \ x_\alpha) \quad \xi = (\xi_1 \ \dots \ \xi_\beta)$$

and  $x^t, \xi^t$  denote their transposes, i.e., the column-matrices. Further,  $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$  is a block  $(\alpha + \beta) \times (\alpha + \beta)$  matrix composed of phantom constants, moreover

$$\begin{aligned} a_{ij}, c_{kl} &\in \Lambda_{\text{even}}, & b_{ik} &\in \Lambda_{\text{odd}} \\ A &= A^t, & C &= -C^t \end{aligned}$$

For this reason, all the summands in  $\sum \dots$  are even (and the whole integrand is an even function). The conditions of symmetry are the natural conditions for a matrix determining the quadratic form in the variables  $x, \xi$ .

The symbol 'exp' denotes the usual exponent,

$$\exp\{F(x, \xi)\} = \sum_{n=0}^{\infty} \frac{1}{n!} F(x, \xi)^n$$

We also can use the usual identity

$$\exp\{F_1 + F_2\} = \exp\{F_1\} \exp\{F_2\}$$

because all the summands in (1.3) commute.

**Observation 1.1** *The integrand is a convergent series.*

This is completely obvious, however we present explanations. First, the integrand can be written out as

$$\exp\left\{\frac{1}{2}\sum_{ij}a_{ij}x_ix_j\right\}\exp\left\{\sum_{ik}b_{ik}x_i\xi_k\right\}\exp\left\{\frac{1}{2}\sum_{kl}c_{kl}\xi_k\xi_l\right\}\quad (1.4)$$

we applied  $(B\xi^t)^t = -\xi B^t$ , the minus arises because both  $B$ ,  $\xi$  are odd.

Now let us explain why all the factors (1.4) are convergent series. For instance, we write out the first factor

$$\exp\left\{\frac{1}{2}\sum_{ij}a_{ij}x_ix_j\right\} = \exp\left\{\frac{1}{2}\sum_{ij}\pi_{\downarrow}(a_{ij})x_ix_j\right\}\prod_{ij}\exp\left\{\frac{1}{2}(a_{ij}-\pi_{\downarrow}(a_{ij}))x_ix_j\right\}$$

The expression  $\sum\pi_{\downarrow}(a_{ij})x_ix_j$  is a quadratic form of real variables (with complex coefficients), its exponent converges. All the remaining factors  $\exp\{\dots\}$  are finite sums.  $\square$

**Observation 1.2** *The integral converges iff  $\operatorname{Re} A$  is negative definite.*

This is evident. Indeed, the integrand  $\exp\{\dots\}$  is a finite sum of the form

$$\sum_{i_1<\dots<i_k} \sum_{j_1<\dots<j_l} P_{i_1,\dots,i_k;j_1,\dots,j_l}(x) \exp\left\{\frac{1}{2}\sum_{ij}\pi_{\downarrow}(a_{ij})x_ix_j\right\} \xi_{i_1}\dots\xi_{i_k} \mathbf{a}_{j_1}\dots\mathbf{a}_{j_l}$$

where  $P(x)$  are polynomials. Under the condition of  $\operatorname{Re} A < 0$ , the term-wise integration is possible.  $\square$

**1.5. Pfaffian. Definition.** Let  $Q$  be a skew-symmetric  $2m \times 2m$ -matrix  $Q$  composed of even phantom constants. The Pfaffian  $\operatorname{Pfaff}(Q)$  of  $Q$  is defined by

$$\frac{1}{m!}\left(\frac{1}{2}\sum q_{kl}\xi_k\xi_l\right)^m =: \operatorname{Pfaff}(Q) \xi_1\xi_2\dots\xi_{2m}$$

In other words,

$$\operatorname{Pfaff}(Q) = \frac{1}{m!} \int \left(\frac{1}{2}\sum q_{kl}\xi_k\xi_l\right)^m d\xi = \int \exp\left\{\frac{1}{2}\sum q_{kl}\xi_k\xi_l\right\} d\xi$$

Recall that

$$\operatorname{Pfaff}(Q)^2 = \det Q$$

**1.6. Evaluation of the Gauss–Berezin integral.**

**Theorem 1.3** *Let  $\operatorname{Re} A < 0$ . Then*

$$I = \begin{cases} (2\pi)^{\alpha/2} \det(-A)^{-1/2} \operatorname{Pfaff}(C + B^t A^{-1} B), & \beta \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

REMARK. Thus we get a square root of the Berezinian (see [9]) in the right-hand part.  $\square$

PROOF. First, we integrate in  $x$

$$\begin{aligned} \int \exp\left\{\frac{1}{2}xAx^t + xB\xi^t\right\} dx &= \\ &= \int \exp\left\{\frac{1}{2}(x - \xi B^t A^{-1})A(x^t + A^{-1}B\xi^t)\right\} \exp\left\{\frac{1}{2}\xi B^t A^{-1}B\xi^t\right\} dx \end{aligned}$$

We substitute  $y := x - \xi B^t A^{-1}$  and get

$$\exp\left\{\frac{1}{2}\xi B^t A^{-1}B\xi^t\right\} \int \exp\left\{\frac{1}{2}yAy^t\right\} dy$$

However we must justify the substitution. Let  $\Phi$  be a function on  $\mathbb{R}^n$ , let  $\nu \in \Lambda_{\text{even}}$ ,  $\pi_{\downarrow}(\nu) = 0$ . Then

$$\int \Phi(x + \nu) dx = \int \Phi(x) dx$$

Indeed,

$$\Phi(x + \nu) := \sum_{j=0}^{\infty} \frac{1}{j!} \nu^j \frac{d^j}{dx^j} \Phi(x)$$

A termwise integration of this series in  $x$  gives zero for all  $j \neq 0$ .

Integrating

$$\det(-A)^{-1/2} (2\pi)^{\beta/2} \int \exp\left\{\frac{1}{2}\xi D\xi^t\right\} \cdot \exp\left\{\frac{1}{2}\xi B^t A^{-1}B\xi^t\right\} d\xi$$

we get the Pfaffian.  $\square$

**1.7. A more general Gauss–Berezin integral.** Consider an expression

$$\begin{aligned} J &= \iint \exp\left\{\frac{1}{2} \begin{pmatrix} x & \xi \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \end{pmatrix} + xh^t + \xi g^t\right\} dx d\xi = \\ &= \iint \exp\left\{\frac{1}{2} \sum_{ij} a_{ij} x_i x_j + \sum_{ik} b_{ik} x_i \xi_k + \frac{1}{2} \sum_{kl} c_{kl} \xi_k \xi_l + \sum_j h_j x_j - \sum_k g_k \xi_k\right\} dx d\xi \end{aligned} \tag{1.5}$$

here  $A, B, C$  are the same as above and  $h^t, g^t$  are vector-columns,  $h_j \in \Lambda_{\text{even}}$ ,  $g_k \in \Lambda_{\text{odd}}$ .

We evaluate this integral in two ways.

*First way.* Substituting

$$\begin{pmatrix} y \\ \eta \end{pmatrix} = \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}^{-1} \begin{pmatrix} h^t \\ g^t \end{pmatrix}$$

we get

$$\exp\left\{-\frac{1}{2}(h \ g) \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}^{-1} \begin{pmatrix} h^t \\ g^t \end{pmatrix}\right\} \iint \exp\left\{\frac{1}{2}(x \ \xi) \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \end{pmatrix}\right\} dx d\xi$$

Thus we come to Gauss–Berezin integral (1.3) evaluated above.

This way is not perfect, because it uses an inversion of a matrix  $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$ .

**Observation 1.4** *A matrix  $\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix}$  is invertible iff  $A$  and  $C$  are invertible.*

The necessity is evident; to prove the sufficiency, we note that the matrix

$$T := \begin{pmatrix} A^{-1} & 0 \\ 0 & C^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is composed of nilpotent elements of  $\Lambda$ , we write out  $(1+T)^{-1} = 1-T+T^2-\dots$ .  
□

The matrix  $A$  is invertible, because  $\operatorname{Re} A < 0$ . Nevertheless,  $C$  is skew-symmetric; if its size  $\beta$  is odd, then  $C$  is non-invertible, if  $\beta$  is even, then a matrix  $C$  of general position is invertible.

In the latter case, we can take an invertible perturbation  $C(\varepsilon)$  of  $C$ , evaluate the integral, and pass to the limit as  $\varepsilon \rightarrow 0$ . This leads to the expression written in the next calculation.

*Second way.* We integrate in  $x$ ,

$$\begin{aligned} & \int \exp\left\{\frac{1}{2}xAx^t + xB\xi^t + xh^t\right\} dx = \\ & = \exp\left\{-\frac{1}{2}(h-\xi B^t)A^{-1}(h^t+B\xi^t)\right\} \int \exp\left\{\frac{1}{2}(x+h-\xi B^t)A(x^t+h^t+B\xi^t)\right\} dx \end{aligned}$$

Substituting  $y = x + h - \xi B^t$  and integrating in  $y$ , we get

$$(2\pi)^{\alpha/2} \det(-A)^{-1/2} \int \exp\left\{-\frac{1}{2}(h-\xi B^t)A^{-1}(h^t+B\xi^t) + \xi C\xi^t + \xi g^t\right\} d\xi$$

In fact, we come to an integral of the form

$$\int \exp\left\{\frac{1}{2}\xi D\xi^t + \xi r^t\right\} d\xi \tag{1.6}$$

where a matrix  $D$  is composed of even phantom constants and a vector  $r$  is odd.

If  $D$  is invertible, we shift the argument again  $\eta^t := \xi^t + D^{-1}r^t$  and get

$$\exp\left\{\frac{1}{2}rD^{-1}r^t\right\} \int \exp\left\{\frac{1}{2}\eta D\eta\right\} d\eta$$

the last integral factor is a Pfaffian.

For an arbitrary  $D$ , we can write (1.6) explicitly as follows. For any subset  $I : i_1 < \dots < i_k$  in  $\{1, \dots, \beta\}$  we consider the complementary subset  $J : j_1 < \dots < j_{\beta-k}$ . Define the constant  $\sigma(I) = \pm 1$  as follows

$$(\xi_{i_1} \xi_{i_2} \dots \xi_{i_k}) (\xi_{j_1} \xi_{j_2} \dots \xi_{j_{\beta-k}}) = \sigma(I) \xi_1 \xi_2 \dots \xi_\beta$$

Evidently,

$$\int \exp\left\{\frac{1}{2}\xi D \xi^t + \xi r^t\right\} d\xi = \sum_I \sigma(I) \text{Pfaff} \begin{pmatrix} 0 & d_{i_1 i_2} & \dots & d_{i_1 i_k} \\ d_{i_2 i_1} & 0 & \dots & d_{i_2 i_k} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i_k i_1} & d_{i_k i_2} & \dots & 0 \end{pmatrix} r_{j_1} \dots r_{j_{\beta-k}} \quad (1.7)$$

Recall that  $d_{pq} \in \Lambda_{\text{even}}$  and  $r \in \Lambda_{\text{odd}}$ .

REMARK. If  $d_{p,q} \in \mathbb{C}$ , then the last expression can be written as

$$\ell_1(g) \dots \ell_s(g) \exp\left\{\frac{1}{2}\xi H \xi^t\right\}$$

where  $\ell_1, \dots, \ell_s$  are linear expressions in  $\xi$ , (see [10]). However, in our context, this transformation is impossible.  $\square$

## 2 Gauss–Berezin integral operators

**2.1. Gauss–Berezin vectors in a narrow sense.** A Gauss–Berezin vector (in a narrow sense) in  $\mathbf{F}_{\alpha,\beta}$  is a vector of the form

$$\mathbf{b} \begin{bmatrix} A & B \\ -B^t & C \end{bmatrix} = \lambda \exp\left\{\frac{1}{2} \begin{pmatrix} x & \xi \end{pmatrix} \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \end{pmatrix}\right\} \quad (2.1)$$

where  $A$  is symmetric and even,  $\text{Re } A < 0$ ,  $C$  is skew-symmetric and even, and  $B$  is odd;  $\lambda$  is an even invertible phantom constant.

**2.2. Gauss–Berezin integral operators in a narrow sense.** A *Gauss–Berezin integral operator in a narrow sense* is an operator  $\mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\gamma,\delta}$  of the form

$$\begin{aligned} \mathbf{B}f(x, \xi) &= \\ &= \lambda \cdot \iint \exp\left\{\frac{1}{2} \begin{pmatrix} x & \xi & y & \eta \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \\ y^t \\ \eta^t \end{pmatrix}\right\} f(y, \eta) dy d\eta \end{aligned}$$

whose kernel is a Gauss–Berezin vector in  $\mathbf{F}_{\alpha+\gamma,\beta+\delta}$ .

REMARK. On the other hand, a Gauss–Berezin vector can be regarded as a Gauss–Berezin operator  $\mathbf{F}_{0,0} \rightarrow \mathbf{F}_{\alpha,\beta}$ .  $\square$

For Gauss–Berezin operators

$$\mathbf{B}_1 : \mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\alpha',\beta'}, \quad \mathbf{B}_2 : \mathbf{F}_{\alpha',\beta'} \rightarrow \mathbf{F}_{\alpha'',\beta''}$$

we can evaluate their product. Evidently, this is reduced to an evaluation of a certain Gauss–Berezin integral. However, our final Theorem 5.4 avoids this calculation.

Also, it is more-or less obvious, that for even  $\beta'$  a product of Gauss–Berezin operators of general position is a Gauss–Berezin operator. However, an examination of products requires to extend the last definition.

**2.3. General Gauss–Berezin operators.** Next, we define the first order differential operators

$$\mathfrak{D}[\xi_j]f := \left( \xi_j + \frac{\partial}{\partial \xi_j} \right) f$$

If a function  $f$  is independent on  $\xi_j$ , then

$$\mathfrak{D}[\xi_j]f = \xi_j f, \quad \mathfrak{D}[\xi_j]\xi_j f = f$$

Evidently,

$$\mathfrak{D}[\xi_j]^2 = 1, \quad \mathfrak{D}[\xi_i]\mathfrak{D}[\xi_j] = -\mathfrak{D}[\xi_j]\mathfrak{D}[\xi_i], \quad i \neq j$$

A Gauss–Berezin operator  $\mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\gamma,\delta}$  is an operator of the form

$$\lambda \mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_k}] \mathbf{B} \mathfrak{D}[\eta_{m_1}] \dots \mathfrak{D}[\eta_{m_l}] \quad (2.2)$$

where

- $\mathbf{B}$  is a Gauss –Berezin operator in the narrow sense;
- $i_1 < i_2 < \dots < i_k$ ,  $m_1 < m_2 < \dots < m_l$ , and  $k, l \geq 0$ ;
- $\lambda$  is an even invertible phantom constant.

We say that an operator (2.2) is *even* (respectively *odd*) if  $k + l$  is even (respectively odd). Equivalently, an operator is even (odd) iff its kernel is an even (odd) function.

REMARK. We define the set of Gauss–Berezin operators as a union of  $2^{\beta+\delta}$  sets. However these sets are not disjoint. Actually, we get a (super)manifold consisting of two connected components (namely, even and odd operators). Each set (2.2) is open and dense in the corresponding component. This will be obvious below.  $\square$

**Theorem 2.1** *For each Gauss–Berezin operators*

$$\mathbf{B}_1 : \mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\alpha',\beta'}, \quad \mathbf{B}_2 : \mathbf{F}_{\alpha',\beta'} \rightarrow \mathbf{F}_{\alpha'',\beta''}$$

*their product  $\mathbf{B}_2\mathbf{B}_1$  is either a Gauss–Berezin operator or*

$$\pi_{\downarrow}(\mathbf{B}_2\mathbf{B}_1 f) = 0$$

*for all  $f$ .*

It is proved below, moreover we obtain a geometric description of the product rule.

**2.4. Gauss–Berezin vectors.** A *Gauss–Berezin vector* as a vector of the form

$$\mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_r}] \mathbf{b}[S]$$

where  $\mathbf{b}[S]$  is a vector defined by (2.1).

### 3 Super-groups $\text{OSp}(2p|2q)$

**3.1. Modules  $\Lambda^{p|q}$ .** Let

$$\Lambda^{p|q} := \Lambda^p \oplus \Lambda^q$$

be a direct sum of  $(p+q)$  copies of  $\Lambda$ . We regard elements of  $\Lambda^{p|q}$  as vector-rows

$$(v_1, \dots, v_p; w_1, \dots, w_q)$$

We define a structure of  $\Lambda$ -bimodule on  $\Lambda^{p|q}$ . The addition in  $\Lambda^{p|q}$  is natural. The left multiplication by  $\lambda$  is also natural

$$(v_1, \dots, v_p; w_1, \dots, w_q) \rightarrow (\lambda v_1, \dots, \lambda v_p; \lambda w_1, \dots, \lambda w_q)$$

The right multiplications by  $\varkappa \in \Lambda$  is

$$(v_1, \dots, v_p; w_1, \dots, w_q) \mapsto (v_1 \varkappa, \dots, v_p \varkappa; w_1 \varkappa^\sigma, \dots, w_q \varkappa^\sigma)$$

We define the *even part* of  $\Lambda^{p|q}$  as  $(\Lambda_{\text{even}})^p \oplus (\Lambda_{\text{odd}})^q$  and the *odd part* as  $(\Lambda_{\text{odd}})^p \oplus (\Lambda_{\text{even}})^q$ .

**3.2. Matrices.** We denote by  $\text{Mat}(p|q)$  the space of  $(p+q) \times (p+q)$  matrices over  $\Lambda$ , we represent such matrices in the block form

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

We say that  $\mathcal{A}$  is *even* if all the matrix elements of  $A, D$  are even and all the matrix elements of  $B, C$  are odd. A matrix is *odd* if elements of  $A, C$  are odd and elements of  $B, D$  are even.

A matrix  $\mathcal{A}$  acts on the space  $\Lambda^{p|q}$  as

$$v \rightarrow v\mathcal{A}$$

Such transformations are compatible with the left  $\Lambda$ -module structure on  $\Lambda^{p|q}$ , i.e.,

$$(\lambda v)\mathcal{A} = \lambda(v\mathcal{A}) \quad \lambda \in \Lambda, \quad v \in \Lambda^{p|q}$$

However, *even matrices also regard the right  $\Lambda$ -module structure,*

$$(v\lambda)\mathcal{A} = (v\mathcal{A})\lambda \quad \lambda \in \Lambda$$

(we use the rule (1.1)).

The *super-transpose* of  $\mathcal{A}$  is defined as

$$\mathcal{A}^{st} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} := \begin{cases} \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix} & \text{if } \mathcal{A} \text{ is even} \\ \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix} & \text{if } \mathcal{A} \text{ is odd} \end{cases}$$

and

$$(\mathcal{A}_1 + \mathcal{A}_2)^{st} := \mathcal{A}_1^{st} + \mathcal{A}_2^{st}$$

for even  $\mathcal{A}_1$  and odd  $\mathcal{A}_2$ .

The following identity holds

$$(\mathcal{A}\mathcal{B})^{st} = \begin{cases} \mathcal{B}^{st}\mathcal{A}^{st} & \text{if } \mathcal{A} \text{ or } \mathcal{B} \text{ are even} \\ -\mathcal{B}^{st}\mathcal{A}^{st} & \text{if both } \mathcal{A} \text{ or } \mathcal{B} \text{ are odd} \end{cases} \quad (3.1)$$

**3.3. The supergroups  $\mathrm{GL}(p|q)$ .** The supergroup  $\mathrm{GL}(p|q)$  is the group of *even* invertible matrices. The following lemma is trivial.

**Lemma 3.1** *An even matrix  $\mathcal{A} \in \mathrm{Mat}(p|q)$  is invertible*

- a) *iff the matrices  $A, D$  are invertible.*
- b) *iff the matrices  $\pi_{\downarrow}(A), \pi_{\downarrow}(D)$  are invertible,*

Here  $\pi_{\downarrow}(A)$  denotes the matrix composed of elements  $\pi_{\downarrow}(a_{kl})$ .

Also, the map  $\pi_{\downarrow}(\mathcal{A})$  is a well-defined epimorphism

$$\pi_{\downarrow} : \mathrm{GL}(p|q) \rightarrow \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$$

(because  $\pi_{\downarrow}(B), \pi_{\downarrow}(D)$  are zeros).

**3.4. The supergroup  $\mathrm{OSp}$ .** We define an ortho-symplectic form on  $\Lambda^{2p|2q}$  by

$$\mathfrak{s}(u, v) := uJv^{st}$$

where  $J$  is a block  $(p + p + q + q) \times (p + p + q + q)$  matrix

$$J := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The supergroup  $\mathrm{OSp}(2p|2q)$  is the subgroup in  $\mathrm{GL}(2p|2q)$  consisting of matrices  $g$  satisfying

$$\mathfrak{s}(u, v) = \mathfrak{s}(ug, vg)$$

Equivalently,

$$gJg^{st} = J$$

(in this conclusion, we use (3.1)).

For an element  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ ,

$$\pi_{\downarrow}(A) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \pi_{\downarrow}(A)^t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \pi_{\downarrow}(D) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pi_{\downarrow}(D)^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

i.e.,

$$\pi_{\downarrow}(A) \in \mathrm{Sp}(2n, \mathbb{C}), \quad \pi_{\downarrow}(D) \in \mathrm{O}(2n, \mathbb{C})$$

**3.5. The super-semigroup  $\Gamma\mathrm{OSp}(2p|2q)$ .** The *Olshanski semigroup*  $\Gamma\mathrm{Sp}(2p, \mathbb{R})$  is defined as the subsemigroup in  $\mathrm{Sp}(2p, \mathbb{C})$  consisting of complex matrices  $g$  satisfying the condition

$$g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} g^* - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leq 0$$

where  $*$  denotes the adjoint matrix (see [11]).

Equivalently, we define the indefinite Hermitian form in  $\mathbb{C}^{2p}$  given by

$$M(v \oplus w, \tilde{v} \oplus \tilde{w}) = -v\tilde{v}^* + w\tilde{w}^* \quad (3.2)$$

Then  $g \in \Gamma\mathrm{Sp}(2p)$  iff

$$M(ug, ug) \leq M(u, u) \quad \text{for all } u \in \mathbb{C}^{2p}$$

We define the super-semigroup (semi-supergroup)  $\Gamma\mathrm{OSp}(2p|2q)$  as a subsemigroup in  $\mathrm{OSp}(2p|2q)$  consisting of matrices  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  such that

$$\pi_{\downarrow}(A) \in \Gamma\mathrm{Sp}(2p, \mathbb{R})$$

We also define the smaller supersemigroup  $\Gamma^{\circ}\mathrm{OSp}(2p|2q)$  as the subsemigroup of matrices satisfying

$$M(u\pi_{\downarrow}(A), u\pi_{\downarrow}(A)) < M(u, u) \quad \text{for all nonzero } u \in \mathbb{C}^{2p}$$

i.e., we require the strict indefinite contractivity condition.

## 4 Super-Grassmannians and super-linear-relations

Here we describe the geometric object numerating Gauss–Berezin operators, namely the space of positive Lagrangian super-linear relations.

**4.1. Super-Grassmannians.** Let  $u_1, \dots, u_r$  be even vectors and  $v_1, \dots, v_s$  be odd vectors in  $\Lambda^{p|q}$ . We suppose that

- $\pi_{\downarrow}(u_j) \in (\mathbb{C}^p \oplus 0)$  are linear independent
- $\pi_{\downarrow}(v_k) \in (0 \oplus \mathbb{C}^q)$  are linear independent.

A subspace of dimension  $\{r|s\}$  is a left  $\Lambda$ -module generated by such vectors.

We define the (super-)Grassmannian  $\text{Gr}_{p|q}^{r|s}$  as the space of all the subspaces in  $\Lambda^{p|q}$  of dimension  $\{r|s\}$ .

By definition, the map  $\pi_{\downarrow}$  projects  $\text{Gr}_{p|q}^{r|s}$  to the product  $\text{Gr}_p^r \times \text{Gr}_q^s$  of human complex Grassmannians. We denote by  $\pi_{\downarrow}^{\pm}$  the natural projections

$$\pi_{\downarrow}^+ : \text{Gr}_{p|q}^{r|s} \rightarrow \text{Gr}_p^r, \quad \pi_{\downarrow}^- : \text{Gr}_{p|q}^{r|s} \rightarrow \text{Gr}_q^s$$

**4.2. Atlas on the Grassmannians.** Define an atlas on the Grassmannian  $\text{Gr}_{p|q}^{r|s}$  in the usual way. Namely, consider the following complementary subspaces

$$V_+ := (\Lambda^r \oplus 0) \oplus (\Lambda^s \oplus 0) \quad V_- := (0 \oplus \Lambda^{p-r}) \oplus (0 \oplus \Lambda^{q-s})$$

in  $\Lambda^{p|q}$ . Let  $S : V_+ \rightarrow V_-$  be an even operator. Then its graph<sup>7</sup> is an element of the Grassmannian.

Permutating coordinates in  $\Lambda^p$  and  $\Lambda^q$ , we get an atlas that covers the whole Grassmannian  $\text{Gr}_{p|q}^{r|s}$ .

**4.3. Lagrangian Grassmannians.** Now, equip the space  $\Lambda^{2p|2q}$  with an ortho-symplectic form  $\mathfrak{s}$  as above. We say that a subspace  $L$  is *isotropic* if the form  $\mathfrak{s}$  is zero on  $L$ .

A *Lagrangian subspace*  $L$  is an isotropic subspace of the maximal possible dimension, i.e.,  $\dim L = \{p|q\}$ . Actually,  $\pi_{\downarrow}(L) \subset \mathbb{C}^{2p} \oplus \mathbb{C}^{2q}$  has the form  $H \oplus T$ , where

- $H \oplus 0$  is a Lagrangian subspace in  $\mathbb{C}^{2p} \oplus 0$  in the usual sense
- $0 \oplus T \subset 0 \oplus \mathbb{C}^{2q}$  is a maximal isotropic subspace with respect to the symmetric bilinear form.

Consider the complementary Lagrangian subspaces

$$V_+ := (\Lambda^p \oplus 0) \oplus (\Lambda^q \oplus 0) \quad V_- := (0 \oplus \Lambda^p) \oplus (0 \oplus \Lambda^q) \quad (4.1)$$

Consider an even operator  $S : V_+ \rightarrow V_-$ ,

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

**Proposition 4.1** *The graph of  $S$  is a Lagrangian subspace iff*

$$A = A^t, \quad D = -D^t, \quad C + B^t = 0 \quad (4.2)$$

PROOF. We write out a vector  $h \in \Lambda^{2p|2q}$  as

$$h = (u_+, u_-; v_+, v_-) \in \Lambda^p \oplus \Lambda^p \oplus \Lambda^q \oplus \Lambda^q$$

---

<sup>7</sup>i.e., the set of all vectors  $v_- \oplus v_- S \in V_- \oplus V_+$ ,

Then

$$\mathfrak{s}(h', h) = u'_+(u_-)^{st} - u'_-(u_+)^{st} + v'_+(v_-)^{st} + v'_-(v_+)^{st}$$

Let  $h$  be in the graph of  $S$ . Then

$$(u_- \quad v_-) = (u_+ \quad v_+) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (u_+A + v_+C \quad u_+B + v_+D)$$

Therefore

$$\begin{aligned} \mathfrak{s}(h', h) &= u'_+(u_+A + v_+C)^{st} - (u'_+A + v'_+C)u_+^{st} + \\ &\quad + v'_+(u_+B + v_+D)^{st} + (u'_+B + v'_+D)v_+^{st} \end{aligned}$$

We emphasize that the matrices  $A, B, C, D$  are even<sup>8</sup>; for this reason, we write  $(u_+A)^{st} = A^{st}(u_+)^{st}$  etc., see (3.1). We come to

$$\begin{aligned} &u'_+(A^{st}(u_+)^{st} + C^{st}(v_+)^{st}) - (u'_+A + v'_+C)u_+^{st} + \\ &\quad + v'_+(B^{st}(u_+)^{st} + D^{st}(v_+)^{st}) + (u'_+B + v'_+D)v_+^{st} \end{aligned}$$

Next,

$$A^{st} = A^t, \quad B^{st} = -B^t, \quad C^{st} = C^t, \quad D^{st} = D^t$$

Therefore we get

$$(u_+)^{st}(A - A^t)(u_+)^{st} + (v_+)^{st}(D + D^t)(v_+)^{st} + (u_+)^{st}(B^t + C)v'_+ + (v_+)^{st}(B + C^t)u'_+$$

This expression is zero iff conditions (4.2) are satisfied.  $\square$

Thus we get a map on the Lagrangian Grassmannian. It remains to describe an atlas. Denote by  $e_i, e'_i, f_j, f'_j$ , where  $i \leq p, j \leq q$ , the natural basis in  $\Lambda^{2p|2q}$ . Consider subsets  $I \subset \{1, 2, \dots, p\}, J \subset \{1, 2, \dots, q\}$ . We define

$$V_+[I, J] = (\oplus_{i \in I} \Lambda e_i) \oplus (\oplus_{k \notin I} \Lambda e'_k) \oplus (\oplus_{j \in J} \Lambda f_j) \oplus (\oplus_{l \notin J} \Lambda f'_l) \quad (4.3)$$

$$V_-[I, J] = (\oplus_{i \notin I} \Lambda e_i) \oplus (\oplus_{k \in I} \Lambda e'_k) \oplus (\oplus_{j \notin J} \Lambda f_j) \oplus (\oplus_{l \in J} \Lambda f'_l) \quad (4.4)$$

Applying the same constructions as above, we get an atlas on the Lagrangian Grassmannian.

**4.4. Components of Lagrangian Grassmannian.** Consider the space  $\mathbb{C}^{2q}$  endowed with a symmetric bilinear form. The Grassmannian  $\text{Isot}_{2q}^q$  of maximal isotropic subspaces in  $\mathbb{C}^{2q}$  consists of two connected components (see, for instance, [10]). On the other hand, the Lagrangian Grassmannian in the symplectic  $\mathbb{C}^{2p}$  is connected.

Therefore the super-Lagrangian Grassmannian consists of two components.

#### 4.5. Intersections of subspaces.

---

<sup>8</sup>Recall that this means that  $A, D$  are composed of even phantom constants and  $C, D$  of odd phantom constants.

**Lemma 4.2** *Let  $L$  be a  $r|s$ -dimensional subspace in  $\Lambda^{r|s}$ ,  $M$  be a  $\rho|\sigma$ -dimensional subspace. Let the following transversality conditions hold*

$$\pi_{\downarrow}^+(L) + \pi_{\downarrow}^+(M) = \mathbb{C}^p, \quad \pi_{\downarrow}^-(L) + \pi_{\downarrow}^-(M) = \mathbb{C}^q \quad (4.5)$$

*Then  $L \cap M$  is a subspace and its dimension is  $(r + \rho - p)|(s + \sigma - q)$ .*

REMARK. If the transversality conditions are not satisfied, then incidentally an  $L \cap M$  is not a subspace.  $\square$

PROOF. To simplify notation, we assume  $q = 0$ . Choose a basis  $e_i, f_j, h_l \in \mathbb{C}^p$  such that  $\pi_{\downarrow}(L)$  is generated by  $e_i, f_j$  and  $\pi_{\downarrow}(M)$  is generated by  $f_j, h_l$ . Consider a basis in  $L$  of the form

$$v_i = e_i + \sum a_{ii'} e_{i'} + \sum b_{ij} f_j + \sum \mu_{ik} h_k \quad (4.6)$$

$$w_j = f_j + \sum c_{ji} e_i + \sum d_{jj'} f_{j'} + \sum \nu_{jk} h_k \quad (4.7)$$

where all the phantom coefficients in the right-hand side have no free terms,

$$a_{ii'}, b_{ij}, \dots \in \oplus_{j \geq 2} \Lambda_j$$

We emphasize that the actual collection of phantom variables  $\mathfrak{a}_j$  present in these formulae is finite; therefore without loss of generality we can assume that the phantom algebra  $\Lambda$  is a Grassmann algebra of finite number of variables.

Changing the basis in  $L$  as

$$v_i \rightarrow v_i + \sum A_{ii'} v_{i'} + \sum B_{ij} w_j$$

$$w_j \rightarrow w_j + \sum C_{ji} v_i + \sum D_{jj'} w_{j'}$$

where  $A, B, C, D$  are phantom constants having no free terms, we can 'kill' all the coefficients  $a, b, c, d$  in (4.6), (4.7). In fact we consequently put them to  $\oplus_{j \geq 4} \Lambda_j, \oplus_{j \geq 6} \Lambda_j$ , etc.

Thus, we can assume that  $L$  is generated by vectors of form

$$\begin{aligned} v_i &= e_i + \sum \mu_{ik} h_k \\ w_j &= f_j + \sum \nu_{jk} h_k \end{aligned}$$

and  $M$  is generated by

$$\begin{aligned} y_j &= f_j + \sum \varphi_{ji} e_i \\ z_k &= h_k + \sum \psi_{ki} e_i \end{aligned}$$

However,

$$y_j - w_j = \sum \nu_{jk} h_k - \sum \varphi_{ji} e_i$$

can be represented as

$$y_j - w_j = \sum S_{ji}v_i - \sum T_{jk}z_k$$

with coefficients  $S_{ji}, T_{jk}$  having trivial free terms.

Now we again correct the basis in  $L$ ,

$$v_j \rightarrow v_j, \quad w_j \rightarrow w_j - \sum S_{ji}v_i$$

and the basis in  $M$

$$y_j \rightarrow y_j - \sum T_{jk}z_k, \quad z_k \rightarrow z_k$$

Now  $y_j = w_j$  and we get a subspace of the desired dimension in  $L \cap M$ .

Next, our space  $\Lambda^p$  is graded,

$$\Lambda^p = \bigoplus_j (\Lambda_{2j})^p$$

and  $L + M = (\Lambda_{\text{even}})^p$ . Therefore we know the dimension of the intersection  $L \cap M$  over  $\mathbb{C}$  and this dimension is achieved on the subspace presented above.  $\square$

**4.6. Linear relations over  $\mathbb{C}$ . Transversality.** Let  $V, W$  be finite-dimensional complex linear spaces. A *linear relation*  $P : V \rightrightarrows W$  is an arbitrary linear subspace  $P \subset V \oplus W$ .

Let  $P : V \rightrightarrows W, Q : W \rightarrow Y$  be linear relations. The product  $QP : V \rightrightarrows Y$  is a linear relation consisting of  $v \oplus y \in V \oplus Y$  such that there is  $w \in W$  satisfying  $v \oplus w \in P, w \oplus y \in Q$ .

For a linear relation  $P : V \rightrightarrows W$  we define:

- the *kernel* is  $\ker P = P \cap (V \oplus 0)$ ,
- the *image*  $\text{im } P$  is the projection of  $P$  to  $W$
- the *domain*  $\text{dom } P$  is the projection of  $P$  to  $V$
- the *indefiniteness* is  $\text{indef } P := P \cap (0 \oplus W)$

We say that linear relations  $P : V \rightarrow W, Q : W \rightarrow Y$  are *transversal* if

$$\text{im } P + \text{dom } Q = W, \tag{4.8}$$

$$\text{indef } P \cap \ker Q = 0 \tag{4.9}$$

**Theorem 4.3** *If  $P : V \rightrightarrows W, Q : W \rightrightarrows Y$  are transversal, then*

$$\dim QP = \dim Q + \dim P - \dim W$$

**PROOF.** We rephrase the definition of the product  $QP$  as follows. Consider the space  $V \oplus W \oplus W \oplus Y$  and the following subspaces

- $P \oplus Q$
- the subspace  $H$  consisting of vectors  $v \oplus w \oplus w \oplus y$
- the subspace  $T$  consisting of vectors  $0 \oplus w \oplus w \oplus 0$

We project  $(P \oplus Q) \cap H$  to  $V \oplus W$  along  $T$ . The result is  $QP \subset V \oplus W$ . The condition (4.8) means that

$$(P \oplus Q) + H = V \oplus W \oplus W \oplus Y$$

therefore we know the dimension of the intersection  $S := (P \oplus Q) \cap H$ .

The condition (4.9) means that the projection  $H \rightarrow V \oplus W$  is injective on  $S$ .  $\square$

**4.7. Super-linear relations.** We define super-linear relations  $P : \Lambda^{p|q} \rightrightarrows \Lambda^{r|s}$  and their products as above.

Next, for a linear relation we define linear relations

$$\pi_{\downarrow}^+(P) : \mathbb{C}^p \rightrightarrows \mathbb{C}^r, \quad \pi_{\downarrow}^-(P) : \mathbb{C}^q \rightrightarrows \mathbb{C}^s$$

in the natural way, we simply project the Grassmannian in  $\Lambda^{p|q} \oplus \Lambda^{r|s}$  onto the product of the complex Grassmannians.

We say that linear relations  $P : V \rightarrow W$  and  $Q : W \rightarrow Y$  are transversal if  $\pi_{\downarrow}^+(P)$  is transversal to  $\pi_{\downarrow}^+(Q)$  and  $\pi_{\downarrow}^-(P)$  is transversal to  $\pi_{\downarrow}^-(Q)$ .

**Theorem 4.4** *If  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  are transversal super-linear relations, then their product is a super-linear relation and*

$$\dim QP = \dim Q + \dim P - \dim W$$

PROOF. We watch the proof of the previous theorem.  $\square$

**4.8. Lagrangian linear relations.** Consider the spaces  $V = \Lambda^{2p|2q}$ ,  $W = \Lambda^{2r|2s}$  endowed with ortho-symplectic forms  $\mathfrak{s}_V$ ,  $\mathfrak{s}_W$  respectively. Define the form  $\mathfrak{s}^{\ominus}$  on  $V \oplus W$  as

$$\mathfrak{s}^{\ominus}(v \oplus w, v' \oplus w') := \mathfrak{s}_V(v, v') - \mathfrak{s}_W(w, w')$$

A Lagrangian super-linear relation  $P : V \rightrightarrows W$  is a Lagrangian subspace in  $V \oplus W$ .

**Observation 4.5** *Let  $g \in \text{OSp}(2p|2q)$ . Then the graph of  $g$  is a Lagrangian linear relation  $\Lambda^{2p|2q} \rightrightarrows \Lambda^{2p|2q}$ .*

**Theorem 4.6** *Let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be transversal Lagrangian linear relations. Then  $QP : V \rightrightarrows Y$  is Lagrangian.*

PROOF. Let  $v \oplus w, v' \oplus w' \in P$  and  $w \oplus y, w' \oplus y' \in Q$ . By definition,

$$\mathfrak{s}_V(v, v') = \mathfrak{s}_W(w, w') = \mathfrak{s}_Y(y, y')$$

therefore  $QP$  is isotropic. By the virtue of Theorem 4.4, we know  $\dim QP$ .  $\square$

**4.9. Components.** In Subsection 4.4 we observed that the set of Lagrangian super-linear relations  $V \rightrightarrows W$  consists of two components. In the proof of Theorem 5.3 we must distinguish them.

Decompose  $V = V_+ \oplus V_-$ ,  $W = W_+ \oplus W_-$  as above (4.1). We say that the component containing the linear relation

$$(V_+ \oplus W_-) : V \rightrightarrows W$$

is even; the another component is odd.

**4.10. Positivity of Lagrangian relations.** Recall that in Subsection 3.5 we defined an Hermitian form on  $\mathbb{C}^{2q}$ . Now translate this notion to the language of Grassmannians.

First, we say that a Lagrangian subspace  $L \subset \Lambda^{2p|2q}$  is (strictly) positive, if the form  $M$  is (strictly) positive on  $\pi_{\downarrow}^+(L)$ .

Second, consider a Lagrangian linear relation  $\Lambda^{2p|2q} \rightrightarrows \Lambda^{2r|2s}$ . Consider the space  $\mathbb{C}^{2p} \oplus \mathbb{C}^{2r}$  endowed with the form

$$M^{\ominus}(v \oplus w, v' \oplus w') = M(v, v') - M(w, w')$$

We say that  $L$  is (strictly) contractive, if  $M^{\ominus}$  is (strictly) positive on  $\pi_{\downarrow}^+(L)$ .

**Observation 4.7** *Let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be transversal contractive Lagrangian relations. Then  $QP$  is contractive.*

## 5 Correspondence between linear relations and Gauss–Berezin operators

**5.1. Creation–annihilation operators.** Let  $V := \Lambda^{2p|2q}$  be the super-linear space endowed with the standard ortho-symplectic bilinear form. For a vector

$$v \oplus w := v_+ \oplus v_- \oplus w_+ \oplus w_- \in \Lambda^{2p|2q}$$

we define the creation–annihilation operator in the Fock–Berezin space  $\mathbf{F}_{p,q}$  by

$$\widehat{a}(v \oplus w)f(x, \xi) = \left( \sum_i v_+^{(i)} \frac{\partial}{\partial x_i} + \sum_i v_-^{(i)} x_i + \sum_j w_+^{(j)} \frac{\partial}{\partial \xi_j} + \sum_j w_-^{(j)} \xi_j \right) f(x, \xi)$$

### 5.2. Annihilators of Gaussian vectors.

**Theorem 5.1** *a) For a Gauss–Berezin vector  $\mathbf{b}$  consider the set  $L$  of all the vectors  $v \oplus w \in \Lambda^{2p|2q}$  such that*

$$\widehat{a}(v \oplus w)\mathbf{b} = 0$$

*Then  $L$  is a positive Lagrangian subspace in  $\Lambda^{2p|2q}$ .*

*b) Moreover the map  $\mathbf{b} \mapsto L$  is a bijection*

$$\left\{ \begin{array}{l} \text{The set of all Gauss–Berezin vectors} \\ \text{defined to within an invertible scalar} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{The positive} \\ \text{Lagrangian Grassmannian} \end{array} \right\}$$

PROOF. First, let  $\mathbf{b}(x, \xi)$  have the standard form (2.1). We write out

$$\begin{aligned} \widehat{a}(v \oplus w)\mathbf{b}(x, \xi) &= \left( \sum_i v_+^{(i)} \frac{\partial}{\partial x_i} + \sum_i v_-^{(i)} x_i + \sum_j w_+^{(j)} \frac{\partial}{\partial \xi_j} + \sum_j w_-^{(j)} \xi_j \right) \times \\ &\quad \times \exp \left\{ \frac{1}{2} (x \quad \xi) \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \begin{pmatrix} x^t \\ \xi^t \end{pmatrix} \right\} = \\ &= \left( v_+(Ax^t + B\xi^t) + v_-x^t + w_+(-B^tx + C\xi^t) + w_-\xi^t \right) \cdot \mathbf{b}(x, \xi) = \\ &= \left( (v_+A - w_+B^t + v_-)x^t + (v_+B + w_+D + w_-)\xi^t \right) \cdot \mathbf{b}(x, \xi) \end{aligned}$$

This is zero iff

$$\begin{cases} v_- = -(v_+A - w_+B^t) \\ w_- = -(v_+B + w_+D) \end{cases}$$

However, this system of equation determines a Lagrangian subspace. The positivity is equivalent to  $\operatorname{Re} A < 0$  (see, for instance [11]).

Next, consider an arbitrary Gauss–Berezin vector

$$\mathbf{b}(x, \xi) = \mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_1}] \mathbf{b}[S] \quad (5.1)$$

where  $\mathbf{b}[S]$  is a standard Gauss–Berezin vector. We have

$$\widehat{a}(v_+ \oplus v_- \oplus w_+ \oplus w_-) \mathfrak{D}[\xi_1] = \mathfrak{D}[\xi_1] \widehat{a}(v_+^\sigma \oplus v_-^\sigma \oplus z_+^\sigma \oplus z_-^\sigma) \quad (5.2)$$

where

$$\begin{aligned} z_+ &:= - \begin{pmatrix} w_-^{(1)} & w_+^{(2)} & w_+^{(3)} & \dots \end{pmatrix} \\ z_- &:= - \begin{pmatrix} w_+^{(1)} & w_-^{(2)} & w_-^{(3)} & \dots \end{pmatrix} \end{aligned}$$

We again come to equations defining a Lagrangian subspace written in the another coordinate system, see (4.3)–(4.4).  $\square$

**Theorem 5.2** *For a positive Lagrangian subspace  $L \subset \Lambda^{2p|2q}$  consider the system of equations*

$$\widehat{a}(v \oplus w)f(x, \xi) = 0 \quad \text{for all } v \oplus w \in L \quad (5.3)$$

*for a function  $f(x, \xi)$ . All its solutions are of the form  $\lambda \mathbf{b}(x, \xi)$ , where  $\mathbf{b}(x, \xi)$  is a Gauss–Berezin vector and  $\lambda$  is a phantom constant.*

PROOF. It suffices to prove the statement for  $L$  being in the principal map. Represent

$$f(x, \xi) = \mathbf{b}(x, \xi) \cdot \varphi(x, \xi)$$

where  $\mathbf{b}(x, \xi)$  is the Gaussian solution of the system (evidently, the function  $\mathbf{b}(x, \xi)$  is well-defined). By the Leibnitz rule,

$$\begin{aligned} 0 &= \widehat{a}(v \oplus w)(\mathbf{b}(x, \xi)\varphi(x, \xi)) = \\ &= \left( \widehat{a}(v \oplus w)\mathbf{b}(x, \xi) \right) \cdot \varphi(x, \xi) + \mathbf{b}(x, \xi) \cdot \left( \sum_j v_+^{(j)} \frac{\partial}{\partial x_i} + \sum_j w_+^{(j)} \frac{\partial}{\partial \xi_i} \right) \varphi(x, \xi) \end{aligned}$$

The first summand is zero by the definition of  $\mathbf{b}(x, \xi)$ . Since  $v_+, w_+$  are arbitrary, we get

$$\frac{\partial}{\partial x_i} \varphi(x, \xi) = 0, \quad \frac{\partial}{\partial \xi_i} \varphi(x, \xi) = 0$$

Therefore,  $\varphi(x, \xi)$  is a phantom constant.  $\square$

**5.3. Gauss–Berezin operators and linear relations.** Let  $V = \Lambda^{2p|2q}$ ,  $\widetilde{V} = \Lambda^{2r|2s}$  be spaces endowed with ortho-symplectic forms.

**Theorem 5.3** *a) For each contractive Lagrangian linear relation  $P : V \rightrightarrows \widetilde{V}$  there exists a unique to within a scalar operator*

$$\mathbf{B}(P) : \mathbf{F}_{p,q} \rightarrow \mathbf{F}_{r,s}$$

such that

1) *The following condition is satisfied*

$$\widehat{a}(\widetilde{v})\mathbf{B}(P) = \mathbf{B}(P)\widehat{a}(v) \quad \text{for all } v \oplus \widetilde{v} \in P$$

2)  *$\mathbf{B}(P)$  has even kernel if  $P$  is even and odd kernel if  $P$  is odd.*

*b) Moreover, the operator  $\mathbf{B}(P)$  is a Gauss–Berezin operator and all the Gaussian–Berezin operators arise in this context.*

**PROOF.** We must write out the differential equations for the kernel  $K(x, \xi, y, \eta)$  of the operator  $\mathbf{B}(P)$ . Denote

$$v = v_+ \oplus v_- \oplus w_+ \oplus w_-, \quad \widetilde{v} = \widetilde{v}_+ \oplus \widetilde{v}_- \oplus \widetilde{w}_+ \oplus \widetilde{w}_-$$

Then

$$\widehat{a}(v) \int K(x, \xi, y, \eta) f(y, \eta) dy d\eta = \int K(x, \xi, y, \eta) \widehat{a}(\widetilde{v}) f(y, \eta) dy d\eta$$

Let  $P$  be even. Integrating by parts in the right-hand side, we get:

$$\begin{aligned} &\left( \sum_i \widetilde{v}_+^{(i)} \frac{\partial}{\partial x_i} + \sum_i \widetilde{v}_-^{(i)} x_i + \sum_j \widetilde{w}_+^{(j)} \frac{\partial}{\partial \xi_j} + \sum_j \widetilde{w}_-^{(j)} \xi_j \right) K(x, \xi, y, \eta) = \\ &= \left( - \sum_i v_+^{(i)} \frac{\partial}{\partial y_i} + \sum_i v_-^{(i)} y_i - \sum_j w_+^{(j)} \frac{\partial}{\partial \eta_j} + \sum_j w_-^{(j)} \eta_j \right) K(x, \xi, y, \eta) \end{aligned}$$

This system of equations has the form (5.3) and determines a Gaussian vector. Nevertheless, in the odd case we get

$$\begin{aligned} & \left( \sum_i \tilde{v}_+^{(i)} \frac{\partial}{\partial x_i} + \sum_i \tilde{v}_-^{(i)} x_i + \sum_j \tilde{w}_+^{(j)} \frac{\partial}{\partial \xi_j} + \sum_j \tilde{w}_-^{(j)} \xi_j \right) K(x, \xi, y, \eta) = \\ & = \left( - \sum_i (v_+^{(i)})^\sigma \frac{\partial}{\partial y_i} + \sum_i (v_-^{(i)})^\sigma y_i + \sum_j (w_+^{(j)})^\sigma \frac{\partial}{\partial \eta_j} + \sum_j (w_-^{(j)})^\sigma \eta_j \right) K(x, \xi, y, \eta) \end{aligned}$$

and this is not a system of the desired form.

However, the change of coordinates (5.1) on the set of Gauss–Berezin vectors and the corresponding change of coordinates on the set of Gauss–Berezin operators have a minor dissimilarity. We represent an odd operator  $\mathbf{B}(P)$  as

$$\mathbf{B}(P) = \int L(x, \xi; y, \eta) \mathfrak{D}(\eta_1) f(y, \eta) dy d\eta$$

Writing the equation

$$\widehat{a}(\tilde{v}) \int L(x, \xi, y, \eta) \mathfrak{D}(\eta_1) f(y, \eta) dy d\eta = \int L(x, \xi, y, \eta) \widehat{a}(v) \mathfrak{D}(\eta_1) f(y, \eta) dy d\eta$$

and transposing  $\widehat{a}(v)$  and  $\mathfrak{D}(\eta_1)$  by (5.2), we come to desired equations without  $\sigma$ . Therefore,  $L(x, \xi, y, \eta)$  is a Gauss–Berezin vector  $\in \mathbf{F}_{\alpha+\gamma, \beta+\delta}$  and  $\mathbf{B}(P)$  is a Gauss–Berezin operator.  $\square$

#### 5.4. Products of Gauss–Berezin operators.

**Theorem 5.4** *Let  $P : V \rightrightarrows W$ ,  $Q : W \rightrightarrows Y$  be strictly contractive Lagrangian relations. Assume that  $P, Q$  are transversal. Then*

$$\mathbf{B}(P)\mathbf{B}(Q) = \lambda \mathbf{B}(QP)$$

where  $\lambda = \lambda(P, Q)$  is an even invertible phantom constant.

PROOF. Let  $v \oplus w \in P$ ,  $w \oplus y \in Q$ . Then

$$\mathbf{B}(Q)\mathbf{B}(P)\widehat{a}(v) = \mathbf{B}(Q)\widehat{a}(w)\mathbf{B}(P) = \widehat{a}(y)\mathbf{B}(Q)\mathbf{B}(P)$$

On the other hand,

$$\widehat{a}(y)\mathbf{B}(QP) = \mathbf{B}(QP)\widehat{a}(v)$$

Since these relations define a unique operator, we get  $\mathbf{B}(QP) = \lambda \mathbf{B}(P)\mathbf{B}(Q)$ . However,

$$\pi_{\downarrow}(\mathbf{B}(P)\mathbf{B}(Q)) \neq 0$$

is non-vanishing (see [10]).  $\square$

**Corollary 5.5** *We get a projective representation of the Olshanski super-semi-group  $\Gamma^\circ \text{OSp}(2p|2q)$ .*

## 6 Final remarks

**6.1. A modification of definition of the Gauss–Berezin operators. I.**  
 First, we define the dual Fock space  $\mathbf{F}_{\alpha,\beta}^\square$ , whose elements are sums (1.2), where  $h(x)$  are tempered distributions on  $\mathbb{R}^\alpha$ . We define Gauss–Berezin distributions as expressions

$$\mathbf{b}(x, \xi) = \lambda \mathfrak{D}[\xi_{i_1}] \dots \mathfrak{D}[\xi_{i_k}] \exp\left\{\frac{1}{2} (x \ \xi) \begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} (x \ \xi)\right\} \cdot \delta_H(x)$$

where  $\operatorname{Re} A \leq 0$ ,  $H$  is a subspace in  $\mathbb{R}^\alpha$ , and  $\delta_H$  is the delta-function of the subspace  $H$ .

Now can define slightly more general Gauss–Berezin operators  $\mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\gamma,\delta}^\square$

$$\mathbf{B}f(x, \xi) = \int \mathbf{b}(x, \xi, y, \eta) f(y, \eta) dy d\eta$$

where  $\mathbf{b}(x, \xi, y, \eta)$  is a Gauss–Berezin distribution.

**Theorem 6.1** *The set of Gauss–Berezin operators  $\mathbf{F}_{\alpha,\beta} \rightarrow \mathbf{F}_{\gamma,\delta}^\square$  defined to within a scalar even factor is in one-to-one correspondence with the set of positive Lagrangian linear relations  $\Lambda^{2\alpha|2\beta} \rightrightarrows \Lambda^{2\gamma|2\delta}$ .*

This is a variant of Theorem 5.3. We omit a proof.

**Corollary 6.2** *The representation of  $\Gamma^\circ \operatorname{OSp}(2p|2q)$  constructed above extends to  $\Gamma \operatorname{OSp}(2p|2q)$*

**6.2. Modification of the definition of Gauss–Berezin operators. II.**  
 There arises the following question:

— *Is it possible to describe explicitly the semigroup in  $\mathbf{F}_{p,q}$  generated by all the Gauss–Berezin operators?*

**6.3. Howe duality.** In a certain sense, the orthogonal and symplectic spinors are universal objects in the representation theory. In particular, One can produce numerous representations from spinors by the Howe duality. However, the Howe duality exists for supergroups, see [5], [6], [8].

Therefore, there arise two questions.

— *Is it possible to produce infinite-dimensional ‘unitary representations’ of supergroups from our construction?*

— *We extend the spinor representation to the set of linear relations. For what representations of supergroups such extensions are possible?*

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