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Based on the Smolyanov–Weizsäcker Approach**

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# Approximation of the heat kernel on a Riemannian manifold based on the Smolyanov–Weizsäcker approach

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## Abstract

Let  $M$  be a compact Riemannian manifold without boundary isometrically embedded into  $\mathbb{R}^m$ ,  $\mathbb{W}_{M,t}^x$  be the distribution of a Brownian bridge starting at  $x \in M$  and returning to  $M$  at time  $t$ . Let  $Q_t : C(M) \rightarrow C(M)$ ,  $(Q_t f)(x) = \int_{C([0,1],\mathbb{R}^m)} f(\omega(t)) \mathbb{W}_{M,t}^x(d\omega)$ , and let  $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_n = t\}$  be a partition of  $[0, t]$ . It was shown in [2] that

$$Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} f \rightarrow e^{-t \frac{\Delta_M}{2}} f, \quad \text{as } |\mathcal{P}| \rightarrow 0, \quad (1)$$

in  $C(M)$ . Taking into consideration integral representations:

$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} f)(x) = \int_M q_{\mathcal{P}}(x, y) f(y) \lambda_M(dy)$  and  $(e^{-t \frac{\Delta_M}{2}} f)(x) = \int_M h(x, y, t) f(y) \lambda_M(dy)$ , where  $\lambda_M$  is the volume measure on  $M$ ,  $h(x, y, t)$  is the heat kernel on  $M$ , one interprets relation (1) as a weak convergence in  $C(M)$  of the integral kernels:

$$q_{\mathcal{P}}(x, y) \rightarrow h(x, y, t). \quad (2)$$

The present paper improves the result of [2], and shows that convergence in (2) is uniform on  $M \times M$ .

*Keywords:* Gaussian integrals on compact Riemannian manifolds, heat kernel, Smolyanov–Weizsäcker approach, Smolyanov–Weizsäcker surface measures

## 1 Introduction

Let  $M$  be a compact Riemannian manifold without boundary isometrically embedded into  $\mathbb{R}^m$ ,  $\dim M = d$ . Define

$$q(x, y, t) = \frac{e^{-\frac{|x-y|^2}{2t}}}{\int_M e^{-\frac{|x-\bar{y}|^2}{2t}} \lambda_M(d\bar{y})}$$

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where  $\lambda_M$  is the volume measure on  $M$ . It was shown in [2] that the following limit exists relative to the family of bounded continuous functions, and defines a probability measure on  $C([0, 1], \mathbb{R}^m)$ :

$$\int_{C([0,1],\mathbb{R}^m)} f(\omega) \mathbb{W}_{M,t}^x(d\omega) = \lim_{\varepsilon \rightarrow 0} \frac{\int_{\pi_t^{-1}(U_\varepsilon(M))} f(\omega) \mathbb{W}^x(d\omega)}{\mathbb{W}^x(\pi_t^{-1}(U_\varepsilon(M)))}$$

where  $x \in M$ ,  $t \in [0, 1]$ ,  $\mathbb{W}^x$  is the Wiener measure on  $C([0, 1], \mathbb{R}^m)$ ,  $U_\varepsilon(M)$  is the  $\varepsilon$ -neighborhood of  $M$ ,  $\pi_t$  is the evaluation mapping  $C([0, 1], \mathbb{R}^m) \rightarrow \mathbb{R}^m$ ,  $\varphi \mapsto \varphi(t)$ . The measure  $\mathbb{W}_{M,t}^x$  is the distribution of a Brownian motion on  $\mathbb{R}^m$  conditioned to return to  $M$  at time  $t$  (Brownian bridge). We introduce operators  $Q_t$  as defined in [2]. If  $f$  is a cylinder function satisfying the relation  $f(\omega) = f(\pi_t^{-1}(\omega(t)))$ , and  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is such that  $f(\omega) = g(\omega(t))$ , then

$$(Q_t g)(x) = \int_M g(y) q(x, y, t) \lambda_M(dy) = \int_{C([0,1],\mathbb{R}^m)} f(\omega) \mathbb{W}_{M,t}^x(d\omega). \quad (3)$$

Let  $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_n = t\}$  be a partition of the interval  $[0, t]$ , and let

$$q_{\mathcal{P}}(x, y) = \int_M dx_1 q(x, x_1, t_1) \int_M dx_2 q(x_1, x_2, t_2 - t_1) \cdots \int_M dx_{n-1} q(x_{n-2}, x_{n-1}, t_{n-1} - t_{n-2}) q(x_{n-1}, y, t_n - t_{n-1}). \quad (4)$$

Taking into account the representation (3), we obtain:

$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} g)(x) = \int_M q_{\mathcal{P}}(x, y) g(y) \lambda_M(dy).$$

Let  $h(x, y, t)$ ,  $x, y \in M$ ,  $t \in \mathbb{R}$ , denote the heat kernel on the manifold  $M$ . We have

$$(e^{-t \frac{\Delta_M}{2}} g)(x) = \int_M h(x, y, t) g(y) \lambda_M(dy).$$

The paper [2] states that

$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} g)(x) \rightarrow (e^{-t \frac{\Delta_M}{2}} g)(x)$$

uniformly in  $x \in M$ . Theorem 1 below improves this result of [2].

## 2 Main Theorem

**THEOREM 1.** *Let the partition  $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_n = t\}$  satisfy the following condition: there exists an integer  $k$  such that  $\min\{t_i - t_{i-1}\} > |\mathcal{P}|^k$ , where  $|\mathcal{P}|$  denotes the mesh of  $\mathcal{P}$ . Then, for all  $t \in [0, 1]$ ,*

$$\lim_{|\mathcal{P}| \rightarrow 0} q_{\mathcal{P}}(x, y) = h(x, y, t)$$

*uniformly in  $x, y \in M$ .*

For the proof of the theorem we will need a few lemmas. For  $x, y \in M$ ,  $t \in [0, 1]$ , we define

$$p(x, y, t) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2t}},$$

$$\mathcal{E}(x, y, t) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{d(x,y)^2}{2t}},$$

where  $d(x, y)$  is the geodesic distance between  $x$  and  $y$ .

LEMMA 1. *There exist bounded functions  $\Theta_1, \Theta_2, \Theta_3, \Theta_4, \Theta_5 : M \times M \times [0, 1] \rightarrow \mathbb{R}$  such that*

$$q(x, y, t) = p(x, y, t)(1 + \Theta_1(x, y, t) t), \quad (5)$$

and for all  $x, y \in M$  satisfying  $|x - y| < t^\alpha$ , where  $\frac{1}{4} < \alpha < \frac{1}{2}$ , the following relations hold:

$$p(x, y, t) = \mathcal{E}(x, y, t)(1 + \Theta_2(x, y, t) t^{4\alpha-1}), \quad (6)$$

$$\mathcal{E}(x, y, t) = h(x, y, t)(1 + \Theta_3(x, y, t) t^{2\alpha}), \quad (7)$$

$$q(x, y, t) = h(x, y, t)(1 + \Theta_4(x, y, t) t^{4\alpha-1}), \quad (8)$$

$$h(x, y, t) = p(x, y, t)(1 + \Theta_5(x, y, t) t^{4\alpha-1}). \quad (9)$$

*Proof.* The proof of relation (5) follows from the asymptotic expansion [2]:

$$\frac{1}{(2\pi t)^{\frac{d}{2}}} \int_M e^{-\frac{|x-y|^2}{2t}} \lambda_M(dy) = 1 - t \left( \frac{1}{6} \text{scal}(x) + \frac{1}{16} \Delta_M \Delta_M |x - \cdot|^2|_x \right) + tR(t, x),$$

where  $|R(t, y)| < Kt^{1/2}$ ,  $K$  is a constant, and  $\text{scal}(y)$  is the scalar curvature at the point  $y$ . To prove (6), notice that

$$|x - y|^2 = d(x, y)^2 + \theta(x, y)d(x, y)^4,$$

where  $\theta$  is bounded on  $M \times M$ . Applying the Taylor expansion to  $e^{-\frac{\theta(x,y)d(x,y)^4}{2t}}$ , we can easily see the existence of a bounded function  $\Theta_2 : M \times M \times [0, 1] \rightarrow \mathbb{R}$  such that

$$e^{-\frac{\theta(x,y)d(x,y)^4}{2t}} = 1 + \Theta_2(x, y, t) t^{4\alpha-1}$$

for  $x, y \in M$  satisfying  $|x - y| < t^\alpha$ . This proves relation (6). Relation (7) follows from the following representation of  $h(x, y, t)$  for  $y$  in a neighborhood of  $x$  [1]:

$$h(x, y, t) = \mathcal{E}(x, y, t) \left( \sum_{i=0}^k u_i(x, y) t^i + O(t^{k+1}) \right)$$

where  $u_i : M \times M \rightarrow \mathbb{R}$  are continuous,  $u_0(x, x) = 1$ , and  $\nabla_M u_0(x, x) = 0$ . Applying Taylor expansion to  $u_0(x, y)$  for  $y \in M$  satisfying  $|x - y| < t^\alpha$ , we obtain (7). Relation (8) is a consequence of (5), (6), and (7) if we notice that for  $\frac{1}{4} < \alpha < \frac{1}{2}$ ,  $4\alpha - 1 < 2\alpha$ . Relation (9) is an immediate corollary of (8).  $\square$

LEMMA 2. Let  $\frac{1}{4} < \alpha < \frac{1}{2}$ . Then, there exist a bounded functions  $R : M \times M \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ , and  $\theta : M \times M \times [0, 1] \times [0, 1] \rightarrow M$  such that for all  $x, z \in M$ , and  $t_1, t_2 \in [0, 1]$ ,

$$\int_M q(x, y, t_1)h(y, z, t_2)\lambda_M(dy) = h(x, z, t_1 + t_2)(1 + \Theta_4(x, \theta(x, z, t_1, t_2), t_1)t_1^{4\alpha-1}) + \frac{R(x, z, t_1, t_2)}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}}.$$

*Proof.* Let  $U_{t_1}(x) = \{y \in M : |y - x| < t_1^\alpha\}$ . Then

$$\begin{aligned} \int_{M \setminus U_{t_1}(x)} p(x, y, t_1)h(y, z, t_2)\lambda_M(dy) &< \frac{1}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}}, \\ \int_{M \setminus U_{t_1}(x)} h(x, y, t_1)h(y, z, t_2)\lambda_M(dy) \\ &= \int_{M \setminus U_{t_1}(x)} p(x, y, t_1)h(y, z, t_2)(1 + \Theta_5(x, y, t_1)t_1^{4\alpha-1})\lambda_M(dy) < \frac{K_1}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}} \end{aligned} \quad (10)$$

where  $K_1$  is a constant independent of  $x, z, t_1$ , and  $t_2$ . Inequality (10) and relation (5) imply the existence of a constant  $K_2$  such that

$$\int_{M \setminus U_{t_1}(x)} q(x, y, t_1)h(y, z, t_2)\lambda_M(dy) < \frac{K_2}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}}. \quad (11)$$

Further, using relation (8) of Lemma 1 and the two inequalities above, we obtain:

$$\begin{aligned} &\int_{U_{t_1}(x)} q(x, y, t_1)h(y, z, t_2)\lambda_M(dy) \\ &= \int_{U_{t_1}(x)} h(x, y, t_1)h(y, z, t_2)\lambda_M(dy) \\ &\quad + t_1^{4\alpha-1} \int_{U_{t_1}(x)} h(x, y, t_1)h(y, z, t_2)\Theta_4(x, y, t_1)\lambda_M(dy) \\ &= \int_M h(x, y, t_1)h(y, z, t_2)\lambda_M(dy) (1 + t_1^{4\alpha-1}\Theta_4(x, \theta(x, z, t_1, t_2), t_1)) \\ &\quad + \frac{\bar{R}(x, z, t_1, t_2)}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}}, \end{aligned}$$

where  $\bar{R} : M \times M \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  is bounded. Applying inequality (11), we obtain:

$$\int_M q(x, y, t_1)h(y, z, t_2)\lambda_M(dy) = h(x, z, t_1 + t_2)(1 + t_1^{4\alpha-1}\Theta_4(x, \theta(x, z, t_1, t_2), t_1)) + \frac{R(x, z, t_1, t_2)}{(2\pi t_1)^{\frac{d}{2}}} e^{-\frac{1}{2t_1^{1-2\alpha}}},$$

where, again,  $R : M \times M \times [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  is bounded by  $K = K_1 + K_2$ . This proves the lemma.  $\square$

We will again need the operators  $Q_s$  below, and so we recall their definition:

$$Q_s : C(M) \rightarrow C(M), f \mapsto \int_M q(\cdot, y, s) f(y) \lambda_M(dy).$$

LEMMA 3. *Let  $\mathcal{P}$  be a partition of  $[0, t]$  as above, and let  $\tau = t_n - t_{n-1}$ , the length of the last partition interval, be such that  $\tau^{d+9} > |\mathcal{P} \setminus \{t_n\}|$ . Then, as the mesh of  $\mathcal{P}$  tends to zero,*

$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} p(\cdot, y, \tau))(x) \rightarrow h(x, y, t), \quad (12)$$

$$(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} h(\cdot, y, \tau))(x) \rightarrow h(x, y, t), \quad (13)$$

uniformly in  $x, y \in M$ .

*Proof.* Let  $y$  be fixed. From the paper [2], we have the following inequality:

$$\|(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} - e^{-\frac{t-\tau}{2}\Delta_M}) p(\cdot, y, \tau)\| \leq K t \|p(\cdot, y, \tau)\|_4 \sqrt{|\mathcal{P} \setminus \{t_n\}|},$$

where the norm  $\|\cdot\|_4$  is described in [2]. Note that

$$\|p(\cdot, y, \tau)\|_4 < \frac{\bar{K}}{\tau^{\frac{d}{2}+4}},$$

where  $\bar{K}$  is a constant. Next, since we assumed that  $|\mathcal{P} \setminus \{t_n\}| < \tau^{d+9}$ , we obtain:

$$\|(Q_{t_1-t_0} \cdots Q_{t_n-t_{n-1}} - e^{-\frac{t-\tau}{2}\Delta_M}) p(\cdot, y, \tau)\| < \tilde{K} \sqrt{\tau} \rightarrow 0, \quad |\mathcal{P}| \rightarrow 0$$

where  $\tilde{K}$  is a constant. Further, note that

$$(e^{-\frac{t-\tau}{2}\Delta_M}) p(\cdot, y, \tau)(x) = \int_M h(x, z, t - \tau) p(z, y, \tau) \lambda_M(dz).$$

Now, in the last integral, we apply the asymptotic expansion [2] relative to the small parameter  $\tau$  to the function  $h(x, z, t - \tau)$ . We obtain:

$$\begin{aligned} & \int_M h(x, z, t - \tau) p(z, y, \tau) \lambda_M(dz) \\ &= h(x, y, t - \tau) - \frac{\tau}{2} \Delta_M h(x, y, t - \tau) \\ & \quad + \tau h(x, y, t - \tau) \left( \frac{1}{6} \text{scal}(x) + \frac{1}{16} \Delta_M \Delta_M |x - \cdot|^2|_x \right) + \tau R(t\tau, x). \end{aligned}$$

Clearly, as  $\tau \rightarrow 0$ ,

$$\int_M h(x, z, t - \tau) p(z, y, \tau) \lambda_M(dz) \rightarrow h(x, y, t)$$

uniformly in  $x, y \in M$ . This proves (12). Relation (9) shows that (13) also holds.  $\square$

LEMMA 4. Let  $\lambda_i \in \mathbb{R}$  be such that  $\sum_{i=1}^k \lambda_i = \tau$ , and let  $\tau^p < K \max\{\lambda_i\}$ ,  $p > 1$ ,  $K$  a constant. Further, assume that there exists an integer  $q > 1$  such that  $\min\{\lambda_i\} > (\max\{\lambda_i\})^q$ . Then there exists a sufficiently small number  $x > 0$  such that  $\sum_{i=1}^k \lambda_i^{1-x} \rightarrow 0$ , as  $\max\{\lambda_i\} \rightarrow 0$ .

*Proof.* We have

$$\lambda_1^{1-x} + \lambda_2^{1-x} + \cdots + \lambda_k^{1-x} \leq \frac{\tau}{(\max\{\lambda_i\})^{qx}} \leq K \tau^{1-pqx}.$$

Choosing  $x < \frac{1}{pq}$  proves the lemma.  $\square$

*Proof of Theorem 1.* We have

$$q_{\mathcal{P}}(x, y) = (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}} q(\cdot, y, t_n - t_{n-1}))(x).$$

Applying relation (8), we obtain

$$\begin{aligned} q_{\mathcal{P}}(x, y) = & \\ & \left(1 + (t_n - t_{n-1}) \Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})\right) (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}} h(\cdot, y, t_n - t_{n-1}))(x), \end{aligned} \quad (14)$$

where  $x_{\mathcal{P}}^{(n)} \in M$  is a point on  $M$  depending on all points of the partition  $\mathcal{P}$ . Continuing transformations of the last term in (14), we obtain:

$$\begin{aligned} & (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}} h(\cdot, y, t_n - t_{n-1}))(x) = \\ & (Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}} \int_M q(\cdot, y_{n-1}, t_{n-1} - t_{n-2}) h(y_{n-1}, y, t_n - t_{n-1}) \lambda_M(dy_{n-1}))(x). \end{aligned}$$

Applying Lemma 2, we obtain

$$\begin{aligned} & (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}} h(\cdot, y, t_n - t_{n-1}))(x) \\ & = (Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}} h(\cdot, y, t_{n-2} - t_n) \\ & \quad \times (1 + (t_{n-1} - t_{n-2})^{4\alpha-1} \Theta_4(\cdot, y_{t_{n-1}t_{n-2}}^{(n-1)}, t_{n-1} - t_{n-2}))(x) \\ & + \frac{R_{n-2}(x, y, \mathcal{P})}{(2\pi(t_{n-1} - t_{n-2}))^{\frac{d}{2}}} e^{-\frac{1}{2(t_{n-1}-t_{n-2})^{1-2\alpha}}}, \end{aligned}$$

where  $R_{n-2}(x, y, \mathcal{P}) = (Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}} R(\cdot, y, t_{n-1}, t_{n-2}))(x)$ , where the function  $R(\cdot, \cdot, \cdot, \cdot)$  is as described in Lemma 2. The function  $R_{n-2}$  is obviously bounded by the same constant  $K$  as the function  $R$ . Finally, applying the mean value theorem to the function  $\Theta_4$ , we obtain

$$\begin{aligned} & (Q_{t_1-t_0} \cdots Q_{t_{n-1}-t_{n-2}} h(\cdot, y, t_n - t_{n-1}))(x) \\ & = (1 + (t_{n-1} - t_{n-2})^{4\alpha-1} \Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})) \\ & \quad \times (Q_{t_1-t_0} \cdots Q_{t_{n-2}-t_{n-3}} h(\cdot, y, t_{n-2} - t_n))(x) \\ & + \frac{R_{n-2}(x, y, \mathcal{P})}{(2\pi(t_{n-1} - t_{n-2}))^{\frac{d}{2}}} e^{-\frac{1}{2(t_{n-1}-t_{n-2})^{1-2\alpha}}}, \end{aligned} \quad (15)$$

where  $x_{\mathcal{P}}^{(n-1)}$  and  $y_{\mathcal{P}}^{(n-1)}$  are points on the manifold  $M$ . Let  $N$  be the smallest number satisfying  $\tau = t_n - t_{n-N} > |\mathcal{P}|^{\frac{1}{d+9}}$ . Also, this implies that  $\tau - (t_{n-N+1} - t_{n-N}) < |\mathcal{P}|^{\frac{1}{d+9}}$ , and hence,  $\tau < |\mathcal{P}|^{\frac{1}{d+9}} + |\mathcal{P}| < 2|\mathcal{P}|^{\frac{1}{d+9}}$ . Repeating the argument used in (15)  $N - 2$  times, we obtain

$$\begin{aligned} q_{\mathcal{P}}(x, y) = & \\ & (1 + (t_n - t_{n-1}) \Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})) \\ & \times (1 + (t_{n-1} - t_{n-2})^{4\alpha-1} \Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})) \cdots \\ & \times (1 + (t_{n-N+1} - t_{n-N})^{4\alpha-1} \Theta_4(x_{\mathcal{P}}^{(n-N+1)}, y_{\mathcal{P}}^{(n-N+1)}, t_{n-N+1} - t_{n-N})) \\ & \times (Q_{t_1-t_0} \cdots Q_{t_{n-N}-t_{n-N-1}} h(\cdot, y, t_n - t_{n-N}))(x) \\ & + \sum_{k=2}^N \frac{R_{n-k}(x, y, \mathcal{P}) \prod_{j=1}^{k-2} (1 + (t_{n-j} - t_{n-j-1})^{4\alpha-1})}{(2\pi(t_{n-k+1} - t_{n-k}))^{\frac{d}{2}}} e^{-\frac{1}{2(t_{n-k+1} - t_{n-k})^{1-2\alpha}}}, \end{aligned}$$

where all functions  $R_{n-k}$  are bounded by the same constant. Now we just have to prove that as  $|\mathcal{P}| \rightarrow 0$ ,

$$(Q_{t_1-t_0} \cdots Q_{t_{n-N}-t_{n-N-1}} h(\cdot, y, t_n - t_{n-N}))(x) \rightarrow h(x, y, t), \quad (16)$$

$$\begin{aligned} & (1 + (t_n - t_{n-1}) \Theta_4(x_{\mathcal{P}}^{(n)}, y, t_n - t_{n-1})) \\ & \times (1 + (t_{n-1} - t_{n-2})^{4\alpha-1} \Theta_4(x_{\mathcal{P}}^{(n-1)}, y_{\mathcal{P}}^{(n-1)}, t_{n-1} - t_{n-2})) \cdots \\ & \times (1 + (t_{n-N+1} - t_{n-N})^{4\alpha-1} \Theta_4(x_{\mathcal{P}}^{(n-N+1)}, y_{\mathcal{P}}^{(n-N+1)}, t_{n-N+1} - t_{n-N})) \rightarrow 1, \end{aligned} \quad (17)$$

$$\sum_{k=2}^N \frac{R_{n-k}(x, y, \mathcal{P}) \prod_{j=1}^{k-2} (1 + (t_{n-j} - t_{n-j-1})^{4\alpha-1})}{(2\pi(t_{n-k+1} - t_{n-k}))^{\frac{d}{2}}} e^{-\frac{1}{2(t_{n-k+1} - t_{n-k})^{1-2\alpha}}} \rightarrow 0 \quad (18)$$

uniformly in  $x, y \in M$ . Note that  $|\mathcal{P}|^{\frac{1}{d+9}} < t_n - t_{n-N} < 2|\mathcal{P}|^{\frac{1}{d+9}}$ . By Lemma 3,

$$(Q_{t_1-t_0} \cdots Q_{t_{n-N}-t_{n-N-1}} h(\cdot, y, \tau))(x) \rightarrow h(x, y, t),$$

and the convergence is uniform in  $x, y \in M$ . Further, for simplicity introduce the notation  $\tau_i = t_n - t_{n-i}$ , for  $i = 1, \dots, N$ , and  $\Theta^{(i)} = \Theta_4(x_{\mathcal{P}}^{n-i+1}, y_{\mathcal{P}}^{n-i+1}, t_{n-i+1} - t_{n-i})$ . Relation (17) holds if and only if

$$\sum_{i=1}^N \log(1 + \tau_i^{4\alpha-1} \Theta^{(i)}) \rightarrow 0, \quad \text{as } |\mathcal{P}| \rightarrow 0.$$

To prove this, we use the inequality

$$\log(1 + \tau_i^{4\alpha-1} \Theta^{(i)}) < \tau_i^{4\alpha-1} \Theta^{(i)}.$$

To treat negative numbers  $\tau_i^{4\alpha-1}\Theta^{(i)}$ , we consider the mesh of  $\mathcal{P}$  small enough, so that  $|\tau_i^{4\alpha-1}\Theta^{(i)}| < C|\mathcal{P}|^{4\alpha-1} < \varepsilon$ , where  $C$  is a constant, and  $\varepsilon$  is sufficiently small, so that inequality

$$\frac{1}{2}\tau_i^{4\alpha-1}\Theta^{(i)} < \log(1 + \tau_i^{4\alpha-1}\Theta^{(i)})$$

holds. Considering both cases of a positive and a negative value of  $\tau_i^{4\alpha-1}\Theta^{(i)}$ , we write down this inequality in the form

$$\frac{1}{2}\min\{0, \tau_i^{4\alpha-1}\Theta^{(i)}\} < \log(1 + \tau_i^{4\alpha-1}\Theta^{(i)}) < \tau_i^{4\alpha-1}\Theta^{(i)}.$$

From this and from the fact that all  $\Theta^{(i)}$  are bounded by the same constant  $C$ , it follows that the uniform convergence in  $x, y \in M$  in (17) will hold if

$$\sum_{i=1}^N \tau_i^{4\alpha-1} \rightarrow 0, \quad \text{as } |\mathcal{P}| \rightarrow 0.$$

This will follow from Lemma 4 if we choose  $\alpha < \frac{1}{2}$  sufficiently close to  $\frac{1}{2}$ . Thus, (17) is proved. Relation (18) is obvious if we notice that all functions  $R_k$  are bounded by the same constant, and the products by which  $R_k$  are being multiplied converge uniformly to 1. Hence, we have proved that

$$\lim_{|\mathcal{P}| \rightarrow 0} q_{\mathcal{P}}(x, y) = h(x, y, t)$$

uniformly in  $x, y \in M$ . The theorem is proved.  $\square$

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